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On Optimal Communication Rules for Certain Types of Teams[1]

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1. Introduction:

J. Marschak has recently formulated the general problem of finding optimal rules of communication between members of an organization or team (cf. [1] and [2]). One such problem arises when there are several observers reporting to a central office, which then acts on the basis of these reports. If communication is costly it may not always be worthwhile for the observers to report the information they possess, and the organization faces the problem of finding the best rule for determining when communication shall take place. This paper presents a few preliminary results of an effort to treat this problem in a general fashion.

A closely related problem is encountered in Marschak's model of an arbitrage firm. The latest and most complete discussion of this is by Beckmann [3]. Earlier discussions are found in [1] and [2].

Approaching the problem gradually, I shall first discuss the special case in which there is only one observer (section 2). We then

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turn (in section 3) to the case of two observers, which contains, I believe, most of the conceptual difficulties present with any number of observers.

2. One Observer

Let us consider then a team, consisting of an actor A, who must periodically take some action a, and an observer 0 who periodically observes the value of a random variable X, which varies independently from period to period. After seeing x the observer may, if he likes, communicate its value to the actor but this costs the team an amount c. At the end of each period the team receives the payoff u(a, x). The team wants to know how to maximize the expected net payoff.

A procedure for the team consists of three things:

1) A function p(x) which equals 1 or 0 according as 0 does or does not report the value of x to A.

2) A function a(x) which tells us what action is taken by A if x is reported

3) An action a_o taken by A if he receives no report.

If x has the distribution F(x) (x is a real number), then the procedure $[p(x), a(x), a_o]$ results in the expected net payoff:

(1)
$$U = \int \left\{ [u(a(x), x) - c]p(x) + [u(a_o, x)][1 - p(x)] \right\} dF(x)$$
$$= \int \left\{ [u(a(x), x) - c - u(a_o, x)]p(x) + [u(a_o, x)] \right\} dF(x)$$

From this we see immediately that:

i) a(x) must be that a which maximizes u(a, x)

ii) For any fixed a_o, the optimal $\hat{p}(x)$ must be 0 or 1 according as its coefficient

$$u(a(x), x) - c - u(a_o, x)$$

is positive or negative.[2]

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2. In fact we can see that this would be true even if we had decided to let p(x) denote the probability that x is reported.

These two points suggest defining the <u>regret</u>

(2) $$r(a_o, x) \equiv \max_a u(a, x) - u(a_o, x)$$

for we can simply say that:

<u>I</u> <u>For any</u> a_o, <u>the best p(x) is</u> 1 <u>or</u> 0 <u>according as</u> $r(a_o, x)$ <u>is greater or less than c.</u>

It would greatly facilitate the actual computation of the best procedure if we could be sure that the set N of x's for which p(x) = 0 (i. e., the x's which are not reported) is something simple, for example, an interval. For this question the structure of $r(a_o, x)$ is the important thing. For example, the following two statements are trivial consequences of statement (I).

(<u>IIa</u>) If, for every a, r(a, x) is monotone in x, then the set N of x's for which no report is made is a half-line (possibly empty).

(<u>IIb</u>) If, for every a, there is an x(a) such that for $x \leq x(a)$ r(a, x) is non increasing in x, and for $x \geq x(a)$ r(a, x) is non decreasing in x, then N will be an interval (possibly infinite or degenerate).

In more special circumstances, more can be said about the solution. An interesting special case is the following:

<u>III</u> Theorem. Suppose that the actions a are also real numbers and that $r(a, x) = w(|a - x|)$ is an increasing function of $|a - x|$ alone. Suppose also that the distribution of X has a density function f(x) which is unimodal and symmetric about m (the mean of x).

Then the solution is described by:

i) $a_o = m$

ii) N is the interval $[m - k, m + k]$ where k is determined by $w(k) = c$

<u>proof:</u> We first note that statement (IIb) applies here and in fact

for any a_c, $N = [a_o - k, a_o + k]$. Secondly, maximizing the expected payoff is equivalent to maximizing:

$$V(a_o) \equiv \int [c - r(a_o, x)][1 - p(x)]\, dF(x)$$

$$= \int_{a_o-k}^{a_o+k} [c - w(|x - a_o|)]\, f(x)\, dx$$

[Just add $c - \int \max_a u(a, x)\, dF(x)$ to U in (1)].

let $t \equiv x - a_o$, $g(t) = f(t + m)$. Then:

$$V(a_o) = \int_{-k}^{k} [c - w(|t|)]\, g(t + a_o - m)\, dt$$

Both $[c - w(|x|)$ and $g(x)$ are positive, unimodal, and symmetric about zero. Hence $V(a_o)$ is maximized when $a_o = m$.

3. Two Observers:

We encounter our first real "organizational" problem as we turn to the case of two observers and one actor. Each observer O_i sees a random variable X_i and may or may not report it to the actor A, who takes an action a. The payoff is $u(a, x_1, x_2)$ and the cost of a single communication is c. Somewhat as in the previous section, a procedure for the team as a whole consists of:

1) Two functions $p_i(x_i)$ which are 1 or 0 according as x_i is or is not reported to A.

2) A function $a_{12}(x_1, x_2)$ which tells A what action to take if both x_1 and x_2 are reported.

3) Two functions $a_i(x_i)$ which give A's action if only x_i is reported.

4) An action a_o which A takes in the absence of any report.

The expected payoff is:

$$U = \int \Big\{ [u(a_{12}(x_1, x_2), x_1, x_2) - 2c]\, p_1(x_1)\, p_2(x_2)$$
$$+ [u(a_1(x_1), x_1, x_2) - c]\, p_1(x_1)[1 - p_2(x_2)]$$
$$+ [U(a_2(x_2), x_1, x_2) - c][1 - p_1(x_1)]\, p_2(x_2)$$
$$+ u(a_o, x_1, x_2)[1-p_1(x_1)][1 - p_2(x_2)] \Big\} \, dF(x)$$

Again we make the obvious remark that $a_{12}(x_1, x_2)$ is that a which maximizes $u(a, x_1, x_2)$. Unfortunately, the other considerations of the previous section do not seem to have any helpful carry-over to this case. The main new complications are the presence of the functions $a_i(x_i)$ and the possible dependence of the random variables x_i.

Thus far I have only been able to deal with the following special case, which, however, still contains a few elements of generality: Suppose a can take on only two values, which we will call act and do nothing, with corresponding payoff functions $u(x_1, x_2)$ and zero.

IV Theorem[3,4] If: (i) $u(x_1, x_2) = v_1(x_1) + v_2(x_2)$ where each v_i is differentiable and increasing

(ii) The X_i are independent with density functions $f_i(x_i)$

Then: Within the class of procedures for which the set of points of discontinuity of each function p_i is discrete, any optimal procedure has the property that the set of x_i such that $p_i(x_i) = 0$ is an interval

3. The author is indebted to M. Beckmann for helpful discussions concerning the proof of this theorem. Any errors are mine, of course.

4. It should be pointed out that the statement of the theorem (incorrectly) quoted in J. Marschaks's, "Basic Problems in the Economic Theory of Teams," C. C. D. P. Economics No. 2051, page 40, is stronger than the present one, and has not been proved.

(possibly infinite or degenerate).

proof: As a first step we calculate the optimal $a_1(x_1)$, $a_2(x_2)$ and a_o for any fixed functions p_1 and p_2.

If A knows the value of x_1 but not of x_2, then he will act only if

(4) $$w_1(x_1) \equiv \int u(x_1, x_2)(1 - p_2(x_2))\, fx(x_2)\, dx_2 > 0$$

Let $q_1(x_1)$ be the characteristic function of the set of x_1's for which (4) holds. Later we will need the fact that, because of the monotonicity of u,

(5a) $$q_1(x_1) = \begin{cases} 1 & \text{if } x_1 > x_1' \\ 0 & \text{if } x_1 < x_1' \end{cases}$$

where x_1' is defined by

$$\begin{cases} w_1(x_1') = 0 \text{ if this has a root} \\ x_1' = +\infty \text{ if } w_1(x_1) < 0 \text{ for all } x_1 \\ x_1' = -\infty \text{ if } w_1(x_1) > 0 \text{ for all } x_1 \end{cases}$$

Similarly, in the case in which x_2 but not x_1 is reported we have the corresponding $w_2(x_2)$, $q_2(x_2)$ and x_2'.

If neither x_1 nor x_2 is reported, A will act only if

(6) $$n \equiv \iint u(x_1, x_2)(1 - p_1(x_1))(1 - p_2(x_2)) f_1 f_2\, dx_1 dx_2 > 0$$

Let $q_o = 1$ or 0 according as $n > 0$ or $n \leq 0$.

Finally, if both x_1 and x_2 are reported, A will act only if $u(x_1, x_2) > 0$ and we can let q_{12} be the characteristic function of that set of points (x_1, x_2).

The functions p_1, p_2, q_{12}, q_1, q_2 and q_o determine a policy which we will label $P(p_1, p_2)$. Recalling that a cost c is paid if a variable x_i

is reported, we are now in a position to write down the expected net payoff U corresponding to $P(p_1, p_2)$:[5]

(7) $$U = \iint u\left\{p_1p_2q_{12} + p_1(1 - p_2)q_1 + (1 - p_1)p_2q_2 + (1 - p_1)(1 - p_2)q_o\right\} f_1f_2dx_1dx_2 - c\left[\int p_1f_1dx_1 + \int p_2f_2dx_2\right]$$

Heuristically speaking, we are going to examine the policy $P(p_1, p_2)$ with an eye to improving it by changing, say, the function p_2 in the neighborhood of some point $\bar{x}_2$, and calculate the "local" advantage of reporting $\bar{x}_2$ as against not reporting it. We shall see that this "local advantage" is at first a decreasing, and then an increasing function of $\bar{x}_2$, as $\bar{x}_2$ increases, which will imply that any optimal policy must have the property that if a low value of x_2 is reported (i. e., lower than some value ξ) any lower value will also be reported, while if a high value is reported (higher than ξ) so will all higher values.

Turning to the actual proof, consider a point $\bar{x}_2$ in an interval I on which p_2 is constant, and consider a smaller interval I_ε of length ε which is contained in I and which contains $\bar{x}_2$. Modify p_2 by changing it on the whole interval I_ε, that is $\Delta p_2(x_2) = 1$ on I_ε (or $\Delta p_2(x_2) = -1$ on I_ε) and $= 0$ elsewhere. The corresponding change in U is:

$$\Delta U = \iint u\left\{p_1(\Delta p_2)q_{12} + p_1[-(\Delta p_2)q_1 + (1 - p_2)\Delta q_1 - (\Delta p_2)(\Delta q_1)] + (1 - p_1)(\Delta p_2)q_2 - (1 - p_1)(\Delta p_2)q_o\right\} f_1f_2dx_1dx_2 - c\int(\Delta p_2)f_2dx_2$$

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5. Whenever possible the variables of integration will be omitted, e. g., $\int p_2(x_2)\, f_2(x_2)dx_2$ will be written $\int p_2f_2dx_2$.

(8)
$$U = \iint u \Big\{ p_1(q_{12} - q_1)(\Delta p_2) + (1 - p_1)(q_2 - q_o)(\Delta p_2) + p_1(1 - p_2)(\Delta q_1) - p_1(\Delta p_2)(\Delta q_1) \Big\} f_1 f_2 dx_1 dx_2 - c \int (\Delta p_2) f_2 dx_2$$

We are assuming here that ε is small enough so that the value of q_o is not altered. In that case q_1 is the only other part of the policy $P(p_1, p_2)$ which depends upon p_2.

Concentrating upon the first term within the braces in (8), applying the theorem of mean value for integrals gives

$$\iint (\Delta p_2) f_2 \, u \, p_1(q_{12} - q_1) f_1 \, dx_1 \, dx_2$$
$$= \int [\int (\Delta p_2) f_2 dx_2][u(x_1, \xi) p_1(x_1)(q_{12}(x_1, \xi) - q_1(x_1)) f_1(x_1)] dx_1$$
$$= [\int \Delta p_2 f_2 dx_2][\int u(x_1, \xi) p_1 \, (q_{12}(x_1, \xi) - q_1) f_1 \, dx_1]$$

where ξ is in I_ε. If in a similar manner we treat all the terms of (8) except the 3rd term within the braces, we obtain

(9)
$$\frac{\Delta U}{\int (\Delta p_2) f_2 dx_2} = \int u(x_1, \xi) p_1 \, (q_{12}(x_1, \xi) - q_1) \, f_1 \, dx_1 + \int u(x_1, \eta)(1 - p_1)(q_2(\eta) - q_o) \, f_1 \, dx_1 - (\Delta q_1) \, u(x_1, \xi) p_1 \, f_1 \, dx_1 + \frac{\iint u \, p_1 \, (1 - p_2)(\Delta q_1) \, f_1 f_2 \, dx_1 \, dx_2}{\int \Delta p_2 \, f_2 \, dx_2} - c$$

where ξ, η, ξ in I_ε.

Again using the mean value theorem, for the fourth term of (9) we have:

$$\iint u\, p_1(1 - p_2)\, f_2(\Delta q_1) f_1\, dx_1\, dx_2 =$$

$$\int u(\eta, x_2)\, p_1(\eta)(1 - p_2) f_2\, dx_2 \int (\Delta q_1)\, f_1\, dx_1$$

where η is between x_1' and $x_1' + \Delta x_1$. (cf. equation (5)).

The following limit statements are immediate:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int (\Delta p_2) f_2 dx_2 = f_2(\bar{x}_2)\, \Delta p_2(\bar{x}_2)$$

$$\lim_{\varepsilon \to 0} \eta = x_1'$$

$$\lim_{\varepsilon \to 0} \int u(\eta, x_2) p_1(\eta)(1 - p_2) f_2 dx_2 = 0$$

Furthermore

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int (\Delta q_1) f_1 dx_1 = \lim \frac{1}{\varepsilon} \int_{x_1'}^{x_1' + \Delta x_1'} f_1 dx_1 = (\lim \frac{\Delta x_1'}{\varepsilon})\, f_1(x_1')$$

which is finite almost everywhere.

Hence the third term of (9) approaches zero as $\varepsilon \to 0$

Looking at the other terms of (9) we have

$$\lim_{\varepsilon \to 0} \xi = \lim \eta = \lim \zeta = \bar{x}_2$$

$$\lim \int (\Delta q_1)\, u(x_1, \xi) p_1\, f_1 dx_1 = 0$$

If we take the liberty of using the notation $\dfrac{dU}{dp_2(\bar{x}_2)} \equiv \lim\limits_{\varepsilon \to 0} \dfrac{\Delta U}{\int (\Delta p_2) f_2 dx_2}$

and put together all these last remarks we have:

$$\text{(11)} \qquad \frac{dU}{dp_2(\bar{x}_2)} = \int u(x_1, \bar{x}_2)\, p_1(q_{12}(x_1, \bar{x}_2) - q_1)\, f_1 dx_1$$

$$+ \int u(x_1\, \bar{x}_2)(1 - p_1)(q_2(\bar{x}_2) - q_o)\, f_1 dx_1$$

$$= c$$

$$= T_1(\bar{x}_2) + T_2(\bar{x}_2) - c$$

[Note that $\frac{dU}{dp(\bar{x}_2)}$ is independent of the way in which the intervals I_ε were selected except for points $\bar{x}_2$ which are points of discontinuity for p_2, and these latter form a set of measure zero.]

If $\frac{dU}{dp_2(\bar{x}_2)} > 0$ and $p_2(x_2) = 0$ in the neighborhood of $\bar{x}_2$, then the policy $P(p_1, p_2)$ can be improved by changing p_2 in some (sufficiently small) interval I_ε around $\bar{x}_2$. The same is true if $\frac{dU}{dp_2(\bar{x}_2)} < 0$ and $p_2(x_2) = 1$ in the neighborhood of $\bar{x}_2$. It is in this sense that $\frac{dU}{dp_2(\bar{x}_2)}$ measures the "local advantage" of reporting $\bar{x}_2$ over not reporting $\bar{x}_2$. We shall now show that there is a point ξ such that for $\bar{x}_2 < \xi$, $\frac{dU}{dp_2(\bar{x}_2)}$ is a decreasing function of $\bar{x}_2$ and is increasing for $\bar{x}_2 > \xi$. From these last two facts, the conclusion of the theorem follows immediately.

To complete the proof, then, we consider, in turn, the functions T_1 and T_2 in equation (11). We first define $\rho_1(x_2)$ and $\rho_2(x)$ by:

$$(12)\quad \begin{cases} u(\rho_1(x_2), x_2) = 0 & \text{if there is such a root} \\ \rho_1(x_2) = +\infty & \text{if } u(x_1, x_2) < 0 \text{ for all } x_1 \\ \rho_1(x_2) = -\infty & \text{if } u(x_1, x_2) > 0 \text{ for all } x_1 \\ \text{and symmetrically for } \rho_2(x_1) & \end{cases}$$

T_1 can be rewritten, using (12) and (5):

$$T_1 = \int_{\rho_1(x_2)}^{x_1'} u(x_1, \bar{x}_2)\, p_1\, f_1\, dx_1$$

If $\rho_1(\bar{x}_2) < x_1'$, then T_1 increases with $\bar{x}_2$. This follows from the facts

(i) u increases with $\bar{x}_2$

(ii) $\rho_1(\bar{x}_2)$ increases with $\bar{x}_2$

(iii) $u(x_1, \bar{x}_2) \geqslant 0$ for $x_1 \geqslant \rho_1(\bar{x}_2)$

Similarly, if $\rho_1(\bar{x}_2) > x_1'$, then T_1 is a decreasing function of $\bar{x}_2$.

Since $\rho_1(\bar{x}_2) \geqslant x_1'$ if and only if $\bar{x}_2 \leqslant \rho_2(x_1')$ we have

$$(13) \qquad \begin{cases} T_1 \text{ decreases as } \bar{x}_2 \text{ increases for } \bar{x}_2 < \rho_2(x_1') \\ T_1 \text{ increases as } \bar{x}_2 \text{ increases for } \bar{x}_2 > \rho_2(x_1') \end{cases}$$

Turning to T_2 recall now that:

$$q_2(x_2) = \begin{cases} 1 & \text{if } x_2 > x_2' \\ 0 & \text{if } x_2 \leqslant x_2' \end{cases}$$

where x_2' is defined by

$$\begin{cases} w_2(x_2') \equiv \int u(x_1, x_2')(1 - p_1) f_1 dx_1 = 0 & \text{if this has a root} \\ x_2' = +\infty & \text{if } w_2(x_2) < 0 \text{ for all } x_2 \\ x_2' = -\infty & \text{if } w_2(x_2) < 0 \text{ for all } x_2 \end{cases}$$

There are two cases, according as $q_o = 1$ or 0 (see equation (6).

<u>If</u> $q_o = 1$:

$$(14a) \qquad \begin{cases} T_2 \text{ decreases as } \bar{x}_2 \text{ increases, for } \bar{x}_2 < x_2' \\ T_2 = 0, \qquad \text{for } \bar{x}_2 > x_2 \end{cases}$$

ı the other hand, **if** $q_o = \underline{1}$:

$$(14b) \qquad \begin{cases} T_2 = 0, & \text{for } \bar{x}_2 < x_2' \\ T_2 \text{ increases as } x_2 \text{ increases,} & \text{for } \bar{x}_2 > x_2' \end{cases}$$

Comparing (14) with (13) will reveal that our proof is complete if we can demonstrate the following

Lemma. $x_2' \leq \rho_2(x_1')$ if and only if $n \geq 0$ (i. e. if and only if $q_o = \underline{1}$).

proof: Recall that $u(x_1, x_2) = v_1(x_1) + v_2(x_2)$ (where each v_i is an increasing function of x_i)

From the definition of x_2':

$$\int [v_1(x_1) + v_2(x_2')]\, p_1(x_1)\, f_1(x_1)\, dx_1 = 0$$

$$\int v_1(x_1) p_1 f_1 dx_1 + v_2(x_2') \int p_1 f_1 dx_1 = 0$$

$$(15) \qquad v_2(x_2') = - \frac{\int v_1(x_1) p_1 f_1 dx_1}{\int p_1 f_1 dx_1}$$

Similarly

$$v_1(x_1') = - \frac{\int v_2(x_2) p_2 f_2 dx_2}{\int p_2 f_2 dx_2}$$

and by the definition of the ρ function

$$v_1(x_1') + v_2(\rho_2(x_1')) = 0$$

$$(16) \qquad v_2(\rho_2(x_1')) = \frac{\int v_2(x_2) p_2 f_2 dx_2}{\int p_2 f_2 dx_2}$$

Since $v_2(x_2') \leq v_2(\rho_2(x_1'))$ if and only if $x_2' \leq \rho_2(x_1')$, it follows from equations (15) and (16) that $x_2' \leq \rho_2(x_1')$ if and only if

$$-\frac{\int v_1(x_1)p_1f_1dx_1}{\int p_1f_1dx_1} \leq \frac{\int v_2(x_2)p_2f_2dx_2}{\int p_2f_2dx_2}$$

or

$$\int v_2(x_2)p_2f_2dx_2 \int p_1f_1dx_1 + \int v_1(x_1)p_1f_1dx_1 \int p_2f_2dx_2 \geqslant 0$$

or

$$\int\int u(x_1, x_2)\; p_1p_2f_1f_2dx_1dx_2 \geqslant 0.$$

Q. E. D.

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