On Marschak's Model of an Arbitrage Firm[1]

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1. In order to demonstrate his approach to communication rules in organizations, Marschak has constructed a model of an arbitrage[2] firm in which the problem of optimum communication rules appears in its simplest form.

2. Consider a firm which consists of two partners: one of them keeps himself informed of the market price of the firm's (unique) product; the other, of the market price of the (unique) raw material used by the firm. Each of these prices has a rectangular probability distribution, between a known maximum and a known minimum.

Each partner can commit the firm for an integral number of commodity units, not to exceed a certain limit. The firm sells the amount of product ordered and at the same time buys the corresponding quantity of raw material. In the light of his market observation each partner can either (1) do nothing unless contacted by the other partner, (2) telephone the other partner at an arranged time, or (3) decide to commit the firms to an amount within his limits, if no phone call is received. In case (2), the partners exchange

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1. Paper presented before the East Lansing Meeting of the Econometric Society, September 1952. This discussion paper supercedes Economics 2041.
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2. J. Marschak and D. Waterman: "On Optimum Communication Rules for Teams," Cowles Commission Discussion Paper, Economics No. 2029.

information and either commit the firm to the purchase and sale (up to a certain limit) of an integral number of units, or do nothing—depending on whether the price of the product exceeds or does not exceed the cost of raw material. A "rule" prescribes to each of the partners when to take actions (1), (2), (3).

Problem: find the rule that will maximize the expected profit.

3. This rule will, of course, depend on the cost of phoning. It is safe to assert that if these costs are high, phoning will never be advisable. If phoning costs are zero, one optimal solution is to always telephone, since one can never lose, and can sometimes win by doing so. If phoning costs are moderate the rule one would expect to be best is, proceeding from less favorable to more favorable information: do nothing, phone, and make increasingly larger commitments. Closer examination disproves some parts of this guess. Commitments ought to be made for the maximum amount or not at all. We shall also see that the intervals of action in the order of increasingly favorable information follow a subsequence of the following sequence: phone, do nothing, phone, commit, phone.

4. Let x_1 = price of product

$1 - x_2$ = cost of raw material plus (fixed) cost of labor, etc., per unit of finished product.

For the sake of mathematical simplicity we assume that the two prices, per unit of finished product, have the same range. In this case, the units and the origin can be chosen so as to make each price vary between 0 and 1. Let n_i be the number of units to which partner i commits the firm (i=1, 2), a the limit of individual commitments and b that of joint commitment. Assume the cost of phoning to be d. The profits are $\max(x_1 + x_2 - 1, 0) \cdot b - d$ if any one of the two partners phones and $(x_1 + x_2 - 1)(n_1 + n_2)$ if they commit for amounts n_1, n_2 respectively.

5. In order to obtain a continuous problem (which is easier from a mathematical point of view) suppose that the desired rule prescribes the probabilities of the various actions, rather than specifying behavior with certainty. Let

$q_i(n_i, x_i)$	be the prescribed probability that partner i commits to n_i units
$p_i(x_i)$	the prescribed probability that partner i phones .

The expected profit is now

$$(1) \quad u = \int_0^1 dx_1 \int_0^1 dx_2 \Big\{ (x_1 + x_2 - 1)\cdot[(1 - p_2)\cdot \sum_{n=1}^{a} n\cdot q_1(n,x_1) + (1 - p_1)\cdot \sum_{n=1}^{a} n\cdot q_2(n,x_2)] + [\max (x_1 + x_2 - 1, 0)\cdot b - d]\cdot[1 - (1 - p_1)\cdot(1 - p_2)] \Big\}$$

6. Integration with respect to x_2 (say) will result in

$$(2) \quad u = \int_0^1 dx_1 [\lambda_1(x_1)\cdot p_1(x_1) + \mu_1(x_1) \sum_{n=1}^{a} n\cdot q_1(n,x_1) + \nu_1(x_1)]$$

where λ_1, μ_1, ν_1 are coefficients which do not depend on $p_1(x_1)$ or the $q_1(n,x_1)$.

The problem may now be formulated as maximizing u of (2) with respect to the (measurable) functions $p_1(x_1)$, $q_1(n,x_1)$, n=1, ..., a; on the interval $0 \leq x_1 \leq 1$ subject to the constraints

$$0 \leq p_1(x_1) \qquad\qquad 0 \leq q_1(n,x_1)$$

$$p_1(x_1) + \sum_n q_1(n,x_1) \leq 1 \quad .$$

In order to maximize the integral (2), the following choices are necessary, except on a set of x_1 of measure zero. On the sets of x_1 where both coefficients $\lambda_1(x_1)$ and $\mu_1(x_1)$ are negative, keep $p_1(x_1)$ and all $q_1(n,x_1)$ zero, on the sets of x_1 where one of the coefficients is positive, shift all the available weight (=1) into the term whose coefficient is the largest of the numbers $\lambda_1(x_1), \mu_1(x_1), 2\cdot\mu_1(x_1), \ldots a\mu_1(x_1)$, i.e., $\mathrm{Max}(\lambda_1(x_1), a\cdot\mu_1(x_1))$. To summarize

Table 1

If		Choose		
$\lambda_1(x_1)$	$a\cdot\mu_1(x_1)$	$p_1(x_1)$	$q_1(n,x_1)$ $n < a$	$q_1(a,x_1)$
< 0		0		
	< 0		0	0
$= 0$	$= 0$	Arbitrary		
> 0	$< \lambda_1(x_1)$	1	0	0
> 0	$= \ _1(x_1)$	ρ	0	$1-\rho$
		where $0 \leqq \rho \leqq 1$		
$< a\cdot\mu_1$	> 0	0	0	1

Since the functions $\lambda_1(x_1)$ and $a\cdot\mu_1(x_1)$ are continuous, the sets $\{x_1 : p_1(x_1) = 0\}$, $\{x_1 : q_1(a,x_1) = 0\}$ are finite unions of intervals. We may fix therefore arbitrarily the values of $p_1(x_1)$, $q_1(a,x_1)$ at the boundaries of these sets and obtain the simplified schema:

$$p_1(x_1) = \begin{cases} 1 & \text{if} \quad \lambda_1(x_1) \geqq \mathrm{Max}[0, a\cdot\mu_1(x_1)] \\ 0 & \text{otherwise.} \end{cases}$$

$$q_1(a,x_1) = \begin{cases} 1 & \text{if} \quad a\cdot\mu_1(x_1) > \mathrm{Max}[0, \lambda_1(x_1)] \\ 0 & \text{otherwise} \end{cases}$$

$$q_1(n,x_1) \equiv 0 \qquad n < a.$$

It is seen that the coefficient $\nu_1(x_1)$ is irrelevant for the determination of the $p_1(x_1)$, $q_1(n,x_1)$. The last equation states that commitments are made for the maximum amount or not at all, a fact that must be attributed to the linearity of this model.

7. By differentiation with respect to x_1 of

$$\lambda_1(x_1) = \int_0^1 dx_2 \Big\{(x_1 + x_2 - 1)q_2(a,x_2)\, a + (1 - p_2(x_2))\,[d - b \max(x_1 + x_2 - 1, 0)]\Big\}$$

and

$$\mu_1(x_1) = \int_0^1 dx_2\, (1 - p_2(x_2))\,(x_1 + x_2 - 1)$$

under the integral, it is easily found that μ_1 is a linear function with a nonnegative slope coefficient of x_1 and that λ_1 is a convex function of x_1.

The convex curve $y = \lambda_1(x_1)$ intersects the two lines $y = a\cdot\mu_1(x_1)$ and $y = 0$ in at most two points each. Since these are the points that separate the subsets of the unit interval that correspond to different reactions, there exists a solution which contains 5 or fewer such subsets. The following may be said about the order of these intervals. Since $\mu_1(x_1)$ has a nonnegative slope it follows that the interval "do nothing" must be to the left of the interval "commit." The solutions are therefore all to be found among the subsequences of the sequence phone, do nothing, phone, commit, phone. This result is valid with respect to both partners.

8. Further studies have been made to ascertain for each partner the lengths and positions of the (at most) five intervals which must obey the sequence just stated. This work was done by Waterman, Faxen, Herstein and Beckmann at the Cowles Commission and by Professor Dvoretzky of the University of Jerusalem as a consultant of the Cowles Commission. The following definitive

results have been obtained by Messrs. Kiefer and Orey of Cornell University, subject to the assumption that the lengths of corresponding intervals be the same for both partners (Dvoretzky has shown that this symmetry with respect to lengths implies also symmetry with respect to the separation points).

Assume that the limit for joint commitments is twice that for individual commitments of each partner

$$b = 2a$$

and denote the cost of communication per unit of commitment by $\frac{c}{2} = \frac{d}{b}$ or $c = \frac{d}{a}$. The solution of Kiefer and Orey can then be described as follows. For given $c \geq 0$ there are two alternative rules of behavior, the same to be chosen for each partner, however.

1. Alternative (Figure 1)

if $0 \leq x_i \leq f(c)$	do nothing
$f(c) \leq x_i \leq g(c)$	phone
$g(c) \leq x_i \leq 1$	commit for amount a

2. Alternative

if $0 \leq x_i \leq 1 - g(c)$	do nothing
$1 - g(c) \leq x_i \leq 1 - f(c)$	phone
$1 - f(c) \leq x_i \leq 1$	commit for amount a.

Here f(c), g(c) denote the following functions

$$f(c) = \begin{cases} 0 & \text{for} \quad 0 \leq c \leq c^* \\ \frac{1}{2} & \quad c^* < c \end{cases}$$

where c^* is defined as $\frac{1}{6} s^*$,

s^* being the solution of

$$2 s^4 + 139 s^3 - 2016 s^2 + 2592 s - 864 = 0$$

which satisfies $\frac{3}{4} < s^* < 1$

$$g(c) = \begin{cases} \text{any value of } \frac{1}{2} \leq g \leq 1 & \text{for } c = 0 \\ \frac{1}{2} - \frac{c}{8} - \frac{1}{8}\sqrt{c^2 + 8c} & \text{for } 0 < c \leq c^* \\ \frac{1}{2} & \text{for } c^* < c \end{cases}$$

The parabolic section of $g(c)$ touches the axis $c = 0$ at its apex.

The two alternative solutions are in a certain symmetry relation: either may be obtained from the other by replacing the first function $(f(c), 1 - g(c))$ by the difference of 1 and the second function $(1 - g(c), f(c))$.

A remarkable property of this solution is that it requires, for each given cost of communication, only 2 intervals of response to observation x_1. In other words phoning figures as a substitute for one of the other actions. This is plausible in view of the simple (linear) character of our profit function and because for large enough c the solution must consist of two intervals "do nothing" and "commit." Note that all of the possible patterns represent subsequences of the sequence "<u>do nothing</u>, <u>phone</u>, <u>commit</u>.

An interesting feature is the discontinuity with respect to c of the interval lengths in Figure 1. This is indicative of the fact that the solution represents a boundary maximum rather than an interior maximum, which would depend on its parameters in a continuous fashion.

It is not difficult to show that of all patterns consisting of two intervals for each given c, including asymmetric ones, the Orey-Kiefer solution is optimal. First note that for large c the optimal pattern must certainly be of this form. Next that for sufficiently small c the only cases which need be investigated are those where at least one of the two partners must be assigned a phoning interval. This reduces the number of possible configurations to $6 \cdot 6 - 2 \cdot 2 = 32$ ones of which 4 are symmetric in the sense of having the same intervals (if not interval lengths) in the same order for each partner. Of these 4, 2 correspond to the Kiefer-Orey solution, which is obtained by simple maximization given the pattern. The remaining two symmetric patterns q, p; q, p and p,n; p,n are easily shown to be inferior. The remaining 28 configurations, which by interchanging the partners are reduced to 14 patterns, can be eliminated with little trouble, some of these by just looking at them. This establishes the optimality of the cited solution with respect to all 2-interval patterns.

It seems that the conditions which will produce 3 and more intervals in a solution are as yet but insufficiently understood.

In conclusion I would like to say that this model is of course much too simplified to be useful for immediate application. But it may suggest the lines along which one has to proceed in order to obtain practially significant conclusions.

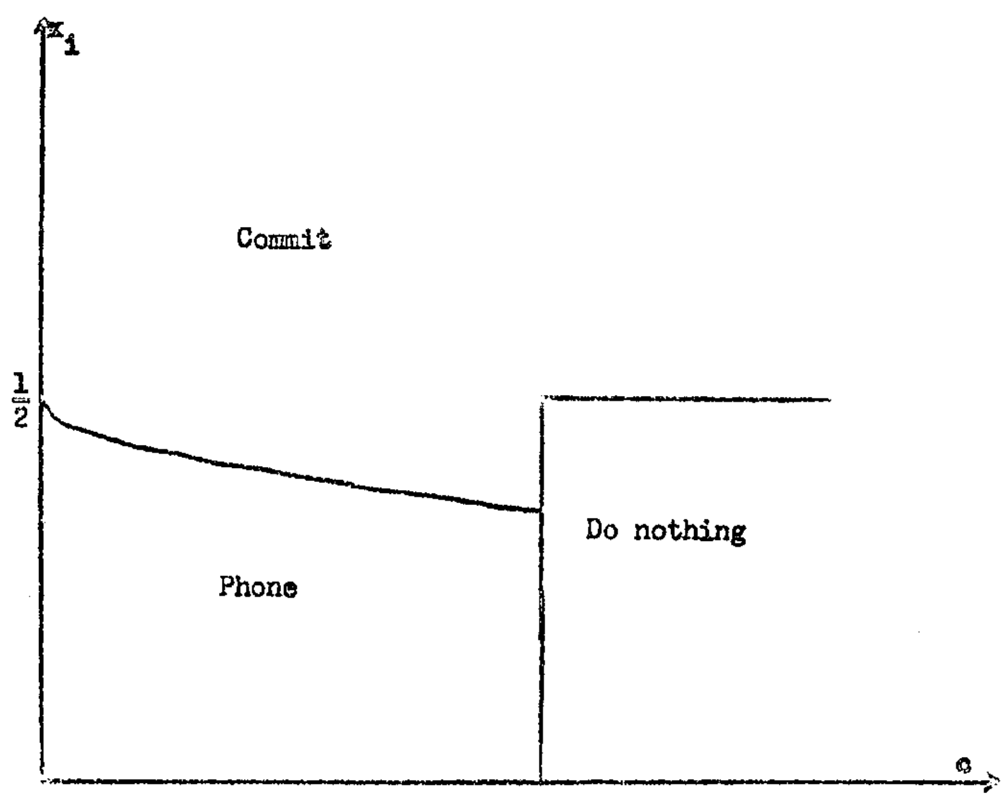

Figure 1.