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Notes on Two Approaches to the Production Rate Problem[1]

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In a factory producing a single commodity, what is the rule for determining the optimal rate of production: $P(t)$? Modigliani and Hohn [1], and Simon [2] have proposed solutions to the problem which, superficially, appear quite different. In Morin's revision [3] of the former's treatment, the methods of the calculus of variations are used; in the latter treatment, Laplace transform methods are applied to the problem. In spite of these apparent differences, the problems and their treatments are closely related.

In both cases the problem is, for a specified future pattern of sales $S(t)$, to minimize a certain functional which is the integral over time of the expenditure rate, $C(t)$. The rate of expenditure is a function of the size of inventory held, $I(t)$, the rate of production, $P(t)$, and certain derivatives of these. Since the inventory is the integral of the difference between rate of production and rate of sales, the latter being given, the expenditure function can be regarded as a function of the inventory and its derivatives:

$$(1) \qquad F = \int_{t_0}^{t_1} C\left\{I, \mathring{I}, \mathring{\mathring{I}}, \ldots\right\} dt$$

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Conditions that a Linear Decision Rule Represent the Solution of a Variational Problem.

Laplace transform methods apply to linear differential equations with constant coefficients. Under what conditions can the optimal production paths be represented by such equations? Since the calculus of variation methods do not involve such a limitation, we will begin by applying the calculus of variations to a rather broad class of cost functions to see under what conditions they lead to differential equations of the kind specified. Let the problem be to minimize:

$$(2) \qquad F = \int_{t_o}^{t_1} \phi \left\{ y(t), y_{(1)}(t), \ldots, y_{(n)}(t) \right\} dt$$

where $y_{(k)}(t) = \frac{d^k}{dt^k} y(t)$, and where appropriate side conditions are imposed to make the problem determinate. We assume only that ϕ is a polynomial in $y(t)$ and its first n derivatives, so that we may write:

$$(3) \qquad \phi(t) = C + \sum_{i=0}^{n} b_i y_{(i)} + \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} y_{(i)} y_{(j)} + \Psi(t)$$

where $\Psi(t)$ is a polynomial composed of terms of third and higher degree in $y(t)$ and its derivatives.

To obtain a necessary condition for a minimum, we form Euler's equation:

$$(4) \qquad \phi_o - \frac{d}{dt} \phi_1 + \ldots + (-1)^n \frac{d^n}{dt^n} \phi_n = 0$$

where $\phi_k = \frac{\partial \phi}{\partial y_{(k)}}$.

Equation (4) is a differential equation of order not greater than 2n in $y(t)$. We now make the assumption:

(A). Equation (4) is a homogenous linear differential equation with constant coefficients.

From (3), we obtain:

$$(5) \qquad \phi_k = b_k + \sum_{i=0}^{n} a_{ik}\, y_{(i)} + \sum_{j=0}^{n} a_{kj}\, y_{(j)} + \Psi_k(t) \qquad (k = 0, \ldots, n)$$

$$(6) \qquad \frac{d^k}{dt^k}\phi_k = \sum_{i=0}^{n} a_{ik}\, y_{(i+k)} + \sum_{j=0}^{n} a_{kj}\, y_{(j+k)} + \frac{d^k}{dt^k}\Psi_k(t) \qquad (k = 1, \ldots, n) .$$

Substituting (5) and (6) in (4), we have, under assumption (A):

$$(7) \qquad b_o + \sum_{i=0}^{n} a_{io}\, y_{(i)} + \sum_{j=0}^{n} a_{oj}\, y_{(j)} + \Psi_o(t)$$

$$+ \sum_{k=1}^{n} (-1)^n \left\{ \sum_{i=0}^{m} a_{ik}\, y_{(i+k)} + \sum_{j=0}^{n} a_{kj}\, y_{(j+k)} + \frac{d^k}{dt^k}\Psi_k(t) \right\}$$

$$= \alpha_o y + \alpha_1 y_{(1)} + \ldots. + \alpha_{2n} y_{(2n)} = 0 .$$

Rearranging the terms of (7), we obtain:

$$(8) \qquad b_o + \sum_{k=0}^{n} (-1)^k \left\{ \sum_{i=0}^{n} (a_{ik}\, y_{(i+k)} + a_{ki}\, y_{(i+k)} \right\}$$

$$+ \Psi_o(t) + \sum_{k=1}^{n} (-1)^n \frac{d^k}{dt^k}\Psi_k(t) = \sum_{g=0}^{2n} \alpha_g\, y_{(g)} = 0 .$$

Now, since all the terms of Ψ are of third or higher degree in y and its derivatives, all the nonvanishing terms of Ψ_k will be of second or higher degree, as will also all the terms of $\frac{d^k}{dt^k}\Psi_k$. Hence, the linearity of the differential equation implies that:

$$(9) \qquad \Psi_o(t) + \sum_{k=1}^{n} (-1)^n \frac{d^k}{dt^k}\Psi_k(t) \equiv 0$$

and its homogeneity implies

$$(10) \qquad b_o = 0$$

A necessary and sufficient condition for (9) [see Courant-Hilbert, vol. I, p. 167] is that:

$$(11) \qquad \int_{t_o}^{t} \Psi(y, \ldots, y_{(n)}) = \xi(y, \ldots, y_{(n-1)}) \quad ,$$

i.e., that Ψ is independent of the path and hence does not affect the optimum.

We have proved the

Theorem: A necessary and sufficient condition that the extremal corresponding to (2) satisfies assumption (A) is that:

$$(12) \qquad F = \int_{t_o}^{t} [c + \sum_{i=1}^{n} b_i y_{(i)} + \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} y_{(i)} y_{(j)}] \, dt + \xi(y, \ldots, y_{(n-1)}) \; .$$

Implications of the Conditions.

1. The first implication we draw from the theorem is that in order for Laplace transformation methods to be strictly applicable, the cost function to be minimized must have the quadratic form exhibited in (12).

2. From (8) we can deduce that in $\sum_{g=0}^{2n} \alpha_g y_{(g)}$, all α_g must vanish for odd g.

Proof: Since all summations in (8) run from zero to n, for each term $(-1)^k (a_{ik} + a_{ki}) y_{(i+k)}$ we have a corresponding term $(-1)^i (a_{ki} + a_{ik}) y_{(k+i)}$. If (i+k) is an odd number these terms will have opposite sign and, being otherwise identical, will vanish.

3. Now form the characteristic equation for (8):

$$(13) \qquad \sum_{k=0}^{n} \alpha_{2k} p^{2k} = 0$$

Because this is an even function, if p_1 is a root, $-p_1$ is also a root. Hence, either (1) all the roots are pure imaginaries, or (2) there is at least one root with positive real part. If we interpret the system described by the Euler equation as a linear "filter" or decision rule, the behavior produced by application of the rule will be dynamically unstable.

In treating the problem of the optimal production rate by Laplace transform methods, Simon [2, pp. 262-264] also obtained a dynamically unstable decision rule. This was no coincidence, for the problem he solved is precisely equivalent to the variational problem we have been considering. Parseval's theorem [Titchmarsh, p. 438] provides the basis for the equivalence. Let $x(t)$ be a function possessing a Laplace transform, and let $y(p)$ be that transform. Then Parseval's theorem states that, under appropriate conditions,

$$(14) \qquad \int_{-\infty}^{\infty} [x(t)]^2 \, dt = \int_{-\infty}^{\infty} [y(p)]^2 \, dp$$

Given a set, S, of functions $\{x_i(t)\}$, and the corresponding set, T, of transforms $\{y_i(p)\}$, it is clear that if x_k is the member of S for which the left-hand side of (14) is a minimum, then y_k, the transform of x_k, is the member of T for which the right-hand side of (14) is a minimum. It does not matter whether we find x_k directly by application of the calculus of variations, or whether, by ordinary methods of minimization, we find y_k.[2/]

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2. Professor A. Charnes pointed out to me the role of Parseval's theorem in this equivalence. The method applied in [2] is essentially the method of Ritz for the "direct" solution of variational problems [Courant-Hilbert, I: p.150].

Application to the Production Rate Problem.

(In what follows we assume that the relevant cost functions are quadratic--that is, satisfy equation (12).)

The problem in [1] was to determine the optimal production rate, given the path of future sales. (We will call this the case of certainty.) The problem in [2] was to determine the optimal production rate when the path of future sales was unknown. (We will call this the case of uncertainty.) We have seen that, for a given quadratic cost function, both problems lead to the same differential equation. This equation states necessary conditions that an optimal production path must satisfy.

If the Euler-Lagrange equation is of order k, it leaves unspecified k initial conditions. Hence there corresponds to the Euler-Lagrange equation a k parameter family of paths, one of which is presumably the optimum optimorum. Some of the additional relations may be obtained from side conditions that the solution must satisfy--for example, a specified initial inventory and initial production rate. Suppose there are r of these side conditions. Then (k-r) additional relations must be found in order to determine the production rate uniquely. In the case of certainty, there is no difficulty in finding these additional relations, but in general they depend upon the whole future sales path. Hence, in the case of uncertainty, we are left without sufficient means for determining the optimum optimorum.

If the Euler-Lagrange equation were dynamically stable, this would not cause serious trouble. We could simply select a sufficient number of additional arbitrary initial conditions to give a unique solution, and start the system off. Because of the dynamic instability, however, the initial transients thus introduced will not be damped out. The resulting path will be optimal only in the sense that it will give the lowest cost of all paths satisfying the same initial conditions--but this will in general (because of the instability) not be in the neighborhood of the optimum optimorum.

Since, in fact, future sales are never known with certainty, the Modigliani-Hohn procedure is never strictly applicable. Hence, in actual situations, it would appear that we would have to proceed in one of two directions.

I. <u>Forecasting Method</u>. We make a forecast of future sales for the relevant time period, and behave as though these were the actual future sales. We determine in this way the optimal production for the first period. At the end of the first period we will have a new forecast and a new optimum, which provides the plan for the second period, and so on. Two observations are in order:

(1) It is not clear in what sense a path so determined will be optimal ex post, simply because each segment of it is optimal ex ante.

(2) It is still necessary to show that the path so determined would not suffer from dynamic instability.

II. <u>Feedback Method</u>. We restrict the class of admissible decision rules to those that are dynamically stable, and find a rule belonging to that class which is optimal in some sense that does not depend on the forecast of sales (e.g., optimal for response to a unit pulse of sales).

Of course, what we have called the forecasting method involves feedback, and what we have called the feedback method involves at least implicit short-run forecasting. That is to say, whichever alternative we take, we are led in the direction of combining forecasting methods with dynamically stable feedback to correct forecasting errors. What seems to me hopeful about this combined approach is that it still leaves room for genuine uncertainty and does not require, for its application, assumptions about the probability distribution of estimates of future sales. Finally, it proceeds on the very human assumption that, whatever procedure is adopted, errors of forecast, planning, and implementation of plans will be made, and that these errors must be corrected without introducing into the system sources of dynamic instability.

REFERENCES

[1] Modigliani, F. and F. Hohn, "Solution of Certain Problems of Production Planning Over Time," Cowles Commission Discussion Paper, Economics No. 2038.

[2] Simon, H. A., "On the Application of Servomechanism Theory in the Study of Production Control," Econometrica, Vol. 20, April 1952, pp. 247-268.

[3] Morin, F., "Note on an Inventory Problem Discussed by Modigliani and Hohn," Cowles Commission Discussion Paper, May 27, 1952, Economics No. 2042.

[4] Courant and Hilbert, Mathematische Physik.

[5] Titchmarsh, The Theory of Functions.