A Lagrangean Multiplier Rule in Linear Activity

Analysis and Some of Its Applications

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1. The standard theory of price and its linear descendant, activity analysis, cover much of the same ground, economically speaking. The assumption of linearity, which at first sight appears restrictive enough, is in fact of no great importance since for all practical purposes it may be replaced by that of piece-wise linearity, as will become clearer later on, (an approximation which is certainly as good as that by continuously differentiable functions). On the other hand, activity analysis is well adapted to economic problems insofar as its variables are constrained to be non-negative, a characteristic feature of many economic magnitudes. The significance of this fact is perhaps brought out best by way of example. Consider the case of the consumer's equilibrium [Hicks, pp. 8-9]. Let

\[ x_r \] denote amounts of commodities consumed

\[ p_r \] commodity prices

\[ m \] the consumer's income

\[ u(x_1, \ldots, x_n) \] his utility function

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Then the problem is that of maximizing the function \( u(x_1, \ldots, x_n) \) with respect to the quantities \( x_r \) subject to the budget constraint \( \sum_r p_r x_r = M \). The Lagrangean multiplier rule reduces this problem to that of extremizing \( u(x_1, \ldots, x_n) + \lambda (M - \sum_r p_r x_r) \). The equilibrium conditions are then

\[
(1) \quad \frac{\partial u}{\partial x_r} = \lambda p_r
\]

where the Lagrangean multiplier \( \lambda \) corresponds to "what is known in Marshall's theory as marginal utility of money".

One may ask with [Houthakker] what happens to these equations and the Engel curves derived from them, if one admits the fact that all commodity quantities have to be non-negative. Of course one would expect that the "\( = \)" sign in (1) should be replaced by a "\( \leq \)" sign in all cases where a quantity turns out to be zero. Actually, this simple assertion whose validity hinges on the concavity of \( u(x_1, \ldots, x_n) \) is quite tedious to prove. This problem leads straight into the controversial field of non-linear activity analysis [Kuhn-Tucker], [Slater].

It will be seen later on that if \( u(x_1, \ldots, x_n) \) is assumed concave and piecewise linear, a linear activity analysis problem results. It is natural however to ask whether this problem can be solved in the same way as before, that is, by means of a suitable Lagrangean Multiplier Rule (LMR). As it turns out, linear activity analysis indeed provides such a LMR. Its principle results, except for computational technique are in fact all embodied in such a LMR. Since on the other hand economists have been using a similar tool in traditional price theory, it seems worth while to develop the LMR of activity analysis explicitly and to demonstrate some of its applications to price theory and to some standard problems of activity analysis itself.
Section 2 states the LMR (Theorem 1) and proves it by way of related lemma in [Gale-Kuhn-Tucker]. As a corollary, the minimax property of Lagrangean Multipliers is obtained (Theorem 2).

Section 3 and 4 sketch applications to the Hitchcock-Koopmans transportation problem [Hitchcock], [Koopmans] and the gasoline mixture problem [Charnes-Cooper-Mellon]. The problem raised in this introductory section is studied further in section 5. Sections 6 and 7 are concerned with applications of the LMR to the theory of the firm.

2. The Lagrangean Multiplier Rule of Linear Activity Analysis states necessary and sufficient conditions for the solution of the fundamental problem of maximizing, in the Pareto sense, a linear vector function subject to linear inequalities as constraints.

Theorem 1

Necessary and sufficient for \( x = \bar{x} \) to yield a Pareto maximum of \( Rx \) subject to the constraints \( x \geq 0 \), \( Ax = c \) is the existence of parameter vectors

\[ \lambda > 0, \mu \geq 0 \]

such that component-wise

\[ \mu_i = 0 \quad \text{if} \quad (A \bar{x})_i > c_i \]

and such that

\[ \lambda'Rx + \mu'(Ax - c) \]

takes an unconstrained minimum for \( x = \bar{x} \) over the positive orthant \( x \geq 0 \).

Proof:

\( \bar{x} = (\bar{x}_j) \) is a maximum point of the kind in question if and only if the form

\[ -\sum_j b_{ij} \Delta x_j \]

constrained by \( \sum_j a_{kj} \Delta x_j \geq 0 \) for all \( k \) with \( \sum_j a_{kj} \bar{x}_j = c_k \)

and by \( \Delta x_j \geq 0 \) for all \( j \) with \( \bar{x}_j = 0 \), has the Pareto maximum zero. This condition is equivalent to the following one due to [Gale, Kuhn, and Tucker] which is based on a familiar inequality of Minkowski's:
\[ \sum_{k} a_{ki} x_{i} + \sum_{j} b_{ij} x_{j} \] for some set of \( \lambda_{i} > 0 \),

\[ \lambda_{k} x_{i} = 0, \quad \psi_{j} x_{j} = 0 \] where \( k \) denotes those indices \( k \) for which

\[ \sum_{j} a_{kj} x_{j} = c_{k} \] and \( j \) the \( j \) for which \( \overline{x}_{j} = 0 \). In other words the necessary and sufficient conditions for a maximum are that

\[ \begin{align*}
2 \frac{\partial}{\partial \overline{x}_{j}} & \left[ \sum_{i} \lambda_{i} b_{ij} \overline{x}_{j} + \sum_{k} \lambda_{k} \left( \sum_{j} a_{kj} x_{j} - c_{k} \right) \right] \\
(2) \quad \sum_{i} b_{ij} \overline{x}_{j} & + \sum_{k} \lambda_{k} \left( \sum_{j} a_{kj} x_{j} - c_{k} \right) \\
(3) \quad \lambda_{k} \left( \sum_{j} a_{kj} x_{j} - c_{k} \right) & = 0
\end{align*} \]

But the left hand expression of (2) equals

\[ \frac{\partial}{\partial \overline{x}_{j}} \left[ \sum_{i} \lambda_{i} b_{ij} \overline{x}_{j} + \sum_{k} \lambda_{k} \left( \sum_{j} a_{kj} x_{j} - c_{k} \right) \right] \]

Hence (2) is equivalent to the condition that the bilinear form

\[ L = \sum_{ij} \lambda_{i} b_{ij} \overline{x}_{j} + \sum_{k} \lambda_{k} \left( \sum_{j} a_{kj} x_{j} - c_{k} \right) \]

has a maximum point \( \overline{x}_{j} \) over the positive orthant. This completes the proof.

The assertion of the LMR may be expressed in a perhaps more concise form revealing the maximum-minimum property of Lagrange multipliers [Courant-Hilbert, pp. 142-145]. This formulation also points to the possibility of calculating activity analysis problems by the same methods as developed for solving 2-person zero-sum games [Dentzig].

**Theorem 2**

Let \( \overline{x} \) be a solution of problem 1, \( \lambda \) the associated Lagrange parameter, and

\[ L(\lambda, x) = \lambda^{'} Bx + \lambda^{'} (Ax - c) \]

Then

\[ \begin{align*}
(4) \quad L(\lambda, x) & \leq L(\lambda, \overline{x}) \leq L(\lambda, x) \\
\text{or in other words} & \\
(5) \quad L(\lambda, \overline{x}) & = \operatorname{Max} \operatorname{Min} L(\lambda, x) = \operatorname{Min} \operatorname{Max} L(\lambda, x)
\end{align*} \]

\[ x \geq 0 \quad \lambda \geq 0 \quad x \geq 0 \]
Proof:

The first inequality may also be written

\[ \sum_{j} b_{ij} \lambda_i + \sum_{k} a_{kj} \mu_k \leq 0 \]

for all \( x_j > 0 \). If \( \bar{x}_j > 0 \) it is necessary and sufficient for this that

\[ \sum_{i} b_{ij} \lambda_i + \sum_{k} a_{kj} \mu_k = 0. \]

If \( \bar{x}_j = 0 \) it is necessary and sufficient that

\[ \sum_{i} b_{ij} \lambda_i + \sum_{k} a_{kj} \mu_k \leq 0. \]

With regard to the second inequality it follows similarly that

\[ \sum_{j} a_{kj} \bar{x}_j - c_k \begin{cases} \geq 0 & \text{if } \mu_k \geq 0 \\ = 0 & \text{if } \mu_k = 0 \\ > 0 & \text{if } \mu_k > 0. \end{cases} \]

To prove the first assertion (5), note that we have shown (first inequality (1))

\[ (6) \quad L(\bar{x}, \bar{\mu}) = \max_{x \geq 0} L(\mu, x). \]

Let us define a function \( \bar{\mu}(x) \)

\[ \bar{\mu}(x) \begin{cases} = 0 & \text{if } \sum_{i} a_{ij} x_j - c_i \geq 0 \\ < 0 & \text{if } \sum_{i} a_{ij} x_j - c_i < 0. \end{cases} \]

It is then seen that

\[ L(\bar{\mu}(x), x) = \min_{\mu \geq 0} L(\mu, x) \]

Since \( \bar{\mu} \) satisfies the conditions for \( \bar{\mu}(\bar{x}) \) we have by (6)

\[ L(\bar{\mu}, \bar{x}) \leq \max_{x \geq 0} \min_{\mu \geq 0} L(\mu, x) \]

Similarly, it is proved that \( L(\bar{\mu}, \bar{x}) \leq \min_{\mu \geq 0} \max_{x \geq 0} L(\mu, x) \). But since always

\[ \max \min = \min \max \] (von Neumann and Morgenstern p. 95) this proves (5).

Theorem 2 is of course well known [Dantzig], [Gale, Kuhn, Tucker].

Theorem 1 is not stated in the literature explicitly but is clearly implied by the lemma used in the proof [Gale, Kuhn, Tucker, lemma 3, p. 318] and [Koopmans, theorem 5.4.4, p. 62]. The advantage of the present formulation
over alternative statements is debatable. It seems however that a La- 
grangean Multiplier Rule is handy to the economist since it yields readily
the efficiency prices he is looking for. Koopmans' theory of activity
analysis while essentially serving the same purpose is constructed on the
basis of certain technology axioms and on an assumption about the signs of
the components of \( c \) (in theorem 1) which in practical applications can be
 tedious to check or difficult to satisfy. The multiplier rules as developed
in [Kuhn, Tucker] and [Slater] for non-linear problems require additional
assumptions, or where not, are subject to controversy. Finally the simplex
method approach of [Dantzig],[Dorfman] puts emphasis on the computational
side of the problem, which appears to be more complex than the basic
economic aspects are.

It must be noted that the two theorems do not settle the question of
the existence of (finite) problem solutions. In Koopmans' approach too the
existence is postulated rather than demonstrated. But in most economic
applications this question is relatively unimportant since a solution is
known to exist for a priori reasons.

3. To give a demonstration of possible uses of theorem 1, we shall dis-
cuss first two well-known problems of applied activity analysis and next
three simple applications to price theory. It will be noted that these are
cases of single-valued rather than Pareto optima.

The Hitchcock-Koopmans transportation problem. Let there be given a trans-
portation program \((q_i)\) specifying for each location \(i\) of a given set the net
amount of commodity to be received per unit of time. We assume (differing here
from Koopmans) that shipments have to be made within a given network of
routes \(ij\) connecting all of these points. Denote by \(x_{ij}\) the commodity flow
from \( i \) to \( j \) (per unit of time) and by \( k_{ij} \) the unit cost of transportation from \( i \) to \( j \). The problem is to find the

\[
\min \sum_{i,j} k_{ij} x_{ij}
\]

subject to

\[
\sum_{j} (x_{ij} - x_{ji}) + q_i = 0 \quad \text{for all } i
\]

where the summations are understood to extend only over routes of the network. Since \( \sum_i q_i = 0 \) we may replace the \( < \) in \( \text{by } = \). With the IMR we obtain immediately the efficiency condition:

\[
\frac{\partial}{\partial x_{ij}} \left[ \sum_{i,j} k_{ij} x_{ij} - \lambda_i \left( \sum_{j} (x_{ij} - x_{ji}) + q_i \right) \right] \left\{ \begin{array}{l} \geq \lambda_j \geq 0 \text{ if } x_{ij} \geq 0 \\
= \lambda_j = 0 \text{ if } x_{ij} = 0
\end{array} \right\}
\]

or

(7) \( k_{ij} \left\{ \begin{array}{l} \lambda_j = \lambda_i \text{ if } x_{ij} \geq 0 \\
\geq \lambda_i \geq 0 \text{ if } x_{ij} = 0
\end{array} \right\}
\)

(7) is the solution (2.13) of [Koopmans, Reiter] which implies that the efficient flow graph is a tree containing no closed circuits except possibly "neutral" ones. The \( \lambda_i \) are efficiency prices [Koopmans, p. 96] associated with the locations of the commodity (in Koopmans case the commodity is "empty ship capacity"). Equations (7) therefore express profitability conditions for shipments in terms of efficiency prices.

4. A Mixture Problem [Coffres-Cooper-McLenn]. Let there be available quantities \( y_s \) (\( s=1, \ldots, m \)) of materials (fuels, nuts) which can be mixed into products \( r (r=1, \ldots, n) \) (gasoline, nut mix) such that \( a_{rs} \) units of \( s \) enter into one unit of \( r \). If \( p_r, p_s \) are the market prices of materials and products, what is the most profitable use of the given quantities \( y_s \)? Suppose that \( x_r \)
units of \( r \) are produced and \( x_s \) units of \( s \) are sold. This yields a profit of

\[
\sum_{r} x_r p_r + \sum_{s} x_s p_s^s
\]

The constraints on the \( x_r, x_s \) are

(8) \( x_r \geq 0 \)

(9) \( x_s \geq 0 \)

(10) \( x_s + \sum_r a_{rs} x_r \leq y_s \).

The conditions for

\[
\begin{align*}
\max & \quad \sum_r x_r p_r + \sum_s x_s p_s + \sum_s \lambda_s (y_s - x_s - \sum_r a_{rs} x_r) \\
x_r & \geq 0, \quad x_s \geq 0
\end{align*}
\]

are

(11) \( p_r - \sum_s \lambda_s a_{rs} \left\{ \begin{array}{l}
\geq x_r \\
= 0
\end{array} \right. \quad \text{if } \lambda_r \left\{ \begin{array}{l}
\geq x_r \\
= 0
\end{array} \right. 0
\]

(12) \( p_s - \lambda_s \left\{ \begin{array}{l}
\geq x_s \\
= 0
\end{array} \right. \quad \text{if } \lambda_s \left\{ \begin{array}{l}
\geq x_s \\
= 0
\end{array} \right. 0
\]

The parameters \( \lambda_s \) satisfy a constraint

(13) \( \lambda_s \left\{ \begin{array}{l}
\geq x_s \\
= 0
\end{array} \right. \quad \text{if } x_s + \sum_r a_{rs} x_r \left\{ \begin{array}{l}
\geq y_s \\
< y_s
\end{array} \right. \)

From (12) it follows that \( \lambda_s = 0 \) is impossible and hence from (13) that the

"=\" sign is attained in all constraints (10). The \( \lambda_s \) are the internal prices

of materials \( s \), and (11) and (12) state the usual profitability conditions.

5. We return now to the earlier problem of finding the equilibrium

conditions of a household with respect to all commodities whether consumed or

not. The piecewise linear utility function shall be defined as the inter-

section of halfspaces

\[
y \leq \sum_r a_r (i_1, \ldots, i_n) x_r + b(i_1, \ldots, i_n)
\]
Here the $i = (i_1, \ldots, i_n)$ denote lattice points with integers as coordinates.

We postulate that these half-spaces be so chosen as to make each point defined by
\[ x_r = i_r \quad r = 1, \ldots, n \]
\[ y = \sum_{r} a_r (i) x_r + b(i) \]
a vertex of the polyhedron. This is possible in a convex body. With each vertex $i$ then is associated one facet whose equation is
\[ y = \sum_{r} a_r (i) x_r + b(i) \]

For each point of the utility function one or several equations (14) hold according as the point is in the interior or at the boundary of a facet. Equilibrium is achieved when $y$ is maximized with respect to the $x_r$ subject to the conditions
\[ y \leq \sum_{r} a_r (i) x_r + b(i) \]
for all $i$,
\[ x_r \geq 0 \quad \text{and} \]
\[ \sum_{r} x_r p_r \leq M. \]

In terms of the LMR we have the problem of finding
\[ \text{Max} \left\{ y + \sum_{i} \lambda (i) \left( \sum_{r} a_r (i) x_r + b(i) - y \right) + M \left( M - \sum_{r} p_r x_r \right) \right\} \]
with some set of non-negative $\lambda (i), M$ such that
\[ \lambda (i) = 0 \quad \text{if} \ \sum_{r} a_r (i) x_r + b(i) \geq y \]
\[ M = 0 \quad \text{if} \ \sum_{r} x_r p_r \leq M. \]

The solutions $\bar{y}, \bar{x}_r$ satisfy necessary and, in connection with the constraints (15), (16), (17) sufficient conditions, obtained by differentiating with respect to $y$ and $x_r$ of (18):
\[ 1 = \sum_{i} \lambda (i) \]
\[ \lambda_i = 0 \]
$\sum \lambda(i) \left( \frac{a_r(i)}{x_r(i)} \right) \mu_p r \quad \text{if} \quad \sum x_r(i) \neq 0$

Case I  If the "=" sign in constraint (15) is taken on for one facet \( i \) only, then (20) takes the form

$\sum a_r(i) \lambda(i) \left( \frac{a_r(i)}{x_r(i)} \right) \mu_p r \quad \text{for} \quad \sum x_r(i) \neq 0$

Hence the marginal rates of substitution of commodities consumed in positive quantities, $\frac{a_s}{a_r}$, must equal the price ratio $\frac{p_s}{p_r}$.

Case II  Otherwise $\lambda(i) > 0$ for several \( i \). By convexity these facets \( i \) must be adjacent. One has

$\sum a_r(i) \lambda(i) \left( \frac{a_r(i)}{x_r(i)} \right) \mu_p r \quad \text{for} \quad \sum x_r(i) \neq 0$.

Here the left side represents a weighted average, with weights $\lambda(i) \neq 0$ and $\sum \lambda(i) = 1$ (equation (19)) of the marginal utilities for the facets $i$ on whose common boundary the equilibrium point is located. The fact to be emphasized is that marginal inequalities hold with respect to all commodities consumed in zero amounts. We note in passing that a utility function of this type can give rise to piecewise linear Engel curves similar to those discussed by [Houthakker]. These Engel curves can be derived from a finite number of equilibrium points, each of which can be computed by some of the familiar methods of linear activity analysis. A drawback of this approach is that one can no longer predict the effect of datum variations in as simple a fashion as by means of the Slutsky equations.

If the changes are small enough to leave the quantity vector on the same facet, the effects of income changes are indeterminate and of price changes are trivial (movement to the boundary). But beyond a given facet there is no simple prediction.
6. In the same way as demonstrated for the elementary theory of consumption, it would be possible to derive the characteristic marginal equations and inequalities that govern the equilibrium of a firm confronted with a family of piecewise linear and convex production functions. It will be more interesting instead to analyse the short run allocation problem [Dorffman] in more concrete terms, admitting imperfections of the labor and loanable funds markets, and assuming a fixed stock of capital equipment. We single out for separate treatment the problem of optimum inventory policy.

Assume that the firm can choose between \( n \) different processes, so defined that they may run concurrently and independently of each other. Let the price differential between finished product and raw material inputs for each process be given. Assume that for each process a fixed input ratio holds with respect to the various types of labor and capital and with respects to funds, independently of the process level. The problem is that of allocating resources so as to maximize profits. The following tables summarize data and notations.

<table>
<thead>
<tr>
<th>resource</th>
<th>availability</th>
<th>amounts used</th>
<th>money costs per unit</th>
<th>index range</th>
</tr>
</thead>
<tbody>
<tr>
<td>labor</td>
<td>( \ell_s )</td>
<td>( x_s )</td>
<td>( t_s )</td>
<td>( S=1,\ldots,S_L )</td>
</tr>
<tr>
<td>newly hired</td>
<td>unlimited</td>
<td>( y_s )</td>
<td>( w_s )</td>
<td></td>
</tr>
<tr>
<td>Capital equipment</td>
<td>( c_s )</td>
<td></td>
<td>0</td>
<td>( S=1,\ldots,S_C )</td>
</tr>
<tr>
<td>funds</td>
<td>( m_s )</td>
<td>( z_s )</td>
<td>( i_s )</td>
<td>( S=1,\ldots,S_F )</td>
</tr>
<tr>
<td>Materials</td>
<td>unlimited</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>process level</th>
<th>input of labor</th>
<th>of capital</th>
<th>of funds</th>
<th>profit on materials</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_r )</td>
<td>( L_{rs} )</td>
<td>( C_{rs} )</td>
<td>( R_r )</td>
<td>( F_r )</td>
</tr>
</tbody>
</table>
Problem: Find conditions under which the maximum profit

\[
\text{Max } \left( \sum_{r} u_r p - \sum_{s} x_s t - \sum_{s} y_s w - \sum_{s} z_s i \right) \\
\]  

is obtained by the firm subject to the limits existing with respect to the following factors

labor employed:

\[ \sum_{r} u_r L_{rs} \leq x_s + y_s \]

capital capacity:

\[ \sum_{r} u_r C_{rs} \leq c_s \]

funds borrowed:

\[ \sum_{r} u_r L_{rs} \leq z_s \]

old labor force:

\[ x_s \leq \ell_s \]

borrowing limits:

\[ z_s \leq m_s \]

and subject to the non-negativity constraints

\[ u_r \geq 0, \quad x_s \geq 0, \quad y_s \geq 0, \quad z_s \geq 0 \]

Solution of this problem with the \text{LIR} reveals the existence of efficiency prices for each type of labor (\( \lambda_s \)), capital equipment (\( \mu_s \)), and for funds (\( \psi \)) not discriminating between various sources. A line of production is carried on at a positive or zero level respectively, if the (marginal or average) cost calculated in terms of these efficiency prices is equal or greater than equal to marginal returns from the product.

\[
(22) \quad \nabla_r \left\{ \begin{array}{l} x_s \geq 0 \\ \text{if } p_r - \sum_{s} \lambda_s = \sum_{s} \mu_s - \sum_{s} \psi \end{array} \right\} \geq 0
\]

Efficiency prices and external prices may be different: the efficiency price of labor is equal to the marginal wage rate; and the efficiency price of funds is equal to the marginal \textit{rate of interest plus possibly} a premium in case the total limit of borrowing is reached. Finally there is an imputation of efficiency rent to the standing equipment. Again rent is zero where capacity
cannot be fully utilized. To summarize, the following are necessary and sufficient conditions for maximization of profits:

1. The profitability principle: activities resulting in negative profit rates are to be dropped.

2. The marginal cost principle: efficiency prices of factors obtained in the market are equal to the market price of the last (most expensive) units purchased.

3. The rent principle: resources available to the firm are evaluated at efficiency rents that absorb any positive profit. Rents are zero if there remains unused capacity.

The reader will find no difficulty in extending the analysis to more general situations, e.g., limitations on material availabilities or piecewise linear, but convex input relations. In the latter case, not all profit is absorbed by efficiency rents so that an intra-marginal profit emerges. The model then approaches the familiar theory of the firm. A distinctive feature is again the presence of marginal inequalities with respect to all quantities which enter at zero levels, and also the fact that these conditions can be proved strictly sufficient for an efficient allocation.

In the same manner it is possible to analyse the equilibria of other complexes, for instance of production at various locations and between different countries engaged in international trade. The results of such analyses are not very surprising. A less obvious situation seems to arise only with regard to the time dimension of production. We shall present here a very simplified problem that of optimum inventory accumulation under perfect certainty.⁴/⁴

We assume that
the cost of production per unit of output is given as a function of discrete time —

\[ c(t) \]

the production capacity of the plant is fixed —

\[ b \]

the price of the firm's product is known for all future —

\[ p(t) \]

no upper limit exists on inventories but the lower limit is zero;

\[ i \]

the cost of holding a unit of inventory stock over the unit of time (including interest charges) —

\[ a \]

the initial inventory is given —

Our unknowns are:

- sales
  \[ q(t) \]

- the level of output
  \[ x(t) \]

- and inventories
  \[ s(t) \]

Let \( T \) be a finite horizon. The problem is to find

\[
(23) \quad \max_{q_i, x_i, s_i, t=0} \sum_{t=0}^{T} \left[ p(t) q(t) - c(t) x(t) - s(t) i \right]
\]

subject to

\[
\begin{align*}
q(t) &\geq 0 \\
x(t) &\geq 0 \\
 s(t) &\geq 0 \\
x(t) &\leq b \\
s(t + 1) & = s(t) - q(t) + x(t) \\
s(0) & = 0
\end{align*}
\]

The L.H.R. reduces this to finding the

\[
(24) \quad \max \sum_{x \geq 0, q \geq 0, s \geq 0} \sum_{t=0}^{T} \left\{ p(t) q(t) - c(t) x(t) - s(t) i - s(t) \lambda(t) \right\} \\
- \lambda(t) \left[ s(t + 1) - s(t) + q(t) - x(t) \right] - \mu(t) [x(t) - b]
\]
with suitable non-negative multipliers $\lambda(t), \mu(t)$ which vanish when the "s" is not attained in the corresponding constraints.

Differentiation yields

\begin{align*}
(25) \quad p(t) &= \lambda(t) \left\{ \begin{array}{l} = \\ \geq \\ \leq \\ > \\ < \\ = \\ 
\end{array} \right\} \quad 0 \quad \text{if} \quad q(t) \left\{ \begin{array}{l} = \\ \geq \\ \leq \\ > \\ < \\ = \\ 
\end{array} \right\} \quad 0 \\
(26) \quad -c(t) + \lambda(t) &= \mu(t) \left\{ \begin{array}{l} = \\ \geq \\ \leq \\ > \\ < \\ = \\ 
\end{array} \right\} \quad 0 \quad \text{if} \quad x(t) \left\{ \begin{array}{l} = \\ \geq \\ \leq \\ > \\ < \\ = \\ 
\end{array} \right\} \quad 0 \\
(27) \quad i + \lambda(s) &= \lambda(s-1) \left\{ \begin{array}{l} = \\ \geq \\ \leq \\ > \\ < \\ = \\ 
\end{array} \right\} \quad 0 \quad \text{if} \quad s(t) \left\{ \begin{array}{l} = \\ \geq \\ \leq \\ > \\ < \\ = \\ 
\end{array} \right\} \quad 0.
\end{align*}

$\lambda(t)$ is an internal efficiency price of the commodity at time $t$, which never falls below the market price (25). It increases over time by the amount of storage cost, so long as positive inventories are being held, but never by more than this. The (marginal) costs of production are equal to the internal price if the former is raised by the scarcity rent $\mu(t)$ on productive capacity. It follows incidentally that if production is efficient at all, it is also efficient at the maximum output level, and that it is efficient exclusively at the maximum level if costs are positively below the internal efficiency price (the future sales price, discounted by the costs of storage). The time-relating condition (27) is similar in structure to the spatial efficiency condition (7) of section 3.

An important difference appears however in the boundary or initial conditions. In a closed space economy the "equation of continuity" serves as a boundary condition forcing total origins to equal total terminations of commodity flows, and no extra condition is required to make the system determinate. In the present case, we have an initial condition $s(0) = a$ and obtain from (23) directly the terminal condition $s(T) = 0$, since otherwise the right hand sum would not be maximal. However there is no reason to expect that the solutions converge if the horizon $T$ goes to infinity. Thus if the axis of time is considered open
at its future end, an indeterminate situation arises. This remains true also if profits are evaluated at a discount and total profits have a finite value. This seems to constitute a remarkable contrast between allocation in space and in time.

The purpose of this paper has been to show the pertinence of activity analysis to problems usually treated in a different way in the context of the general theory of price. To bridge the differences in method, recourse has been had to a reformulation of principal results of activity analysis in terms of a "Lagrangean Multiplier Rule" and thereby to calculate methods that seem more familiar to students of economic theory.
Appendix

Necessary conditions for the maximum of a (piece-wise continuously differentiable) function subject to (piece-wise continuously differentiable) inequalities as constraints. Let \( f(x) \) be the (scalar) maximand, \( x \) the (vector) variable and \( g(x) \geq 0 \) the (vector) inequality as constraint.

**Lemma:**

\( x \) is a maximum point only if there exists some non-negative parameter (vector) \( \lambda = (\lambda_k) \) such that

\[
\text{grad} \ [f + \lambda \ g] = 0
\]

and

\[
\lambda_k = 0 \quad \text{if} \quad g_k(x) > 0.
\]

**Proof:**

Clearly it is necessary for a constrained maximum, that it cannot be improved by small changes satisfying the constraints:

\( \text{df} \leq 0 \) for all \( \text{dx} \) subject to \( \text{dg}_k(x) = 0 \) for all \( k \) with \( g_k(x) = 0 \). By a well-known lemma of Kinkowski [Gale-Kuhn-Tucker] a necessary and sufficient condition for this linear inequality subject to linear constraints to be satisfied is the existence of a non-negative vector \( \mu = (\mu_k) \) such that

\[
-\frac{\partial f}{\partial x_k} = \sum_{k'} \mu_k \frac{\partial g_{k'}}{\partial x_k}
\]

where \( k' \) runs over all \( k \) for which \( g_k(x) = 0 \). Putting \( \lambda_k = \begin{cases} \mu_k' & \text{if} \quad k = k' \\ 0 & \text{otherwise} \end{cases} \), yields at once the assertion.

**Corollary:**

A necessary condition for \( x \) to be a maximum point of \( f(x) \) subject to the constraints

\[
g(x) = 0
\]
\[
h(x) \geq 0
\]
\[
x \geq 0
\]
is the existence of parameter vectors \( \lambda, \mu \) with \( \lambda' \left[ \begin{array}{c} x \\ \mu \end{array} \right] = 0 \) if
\[
h_k(x) \left[ \begin{array}{c} x \\ \mu \end{array} \right] = 0 \text{ such that } \frac{\partial}{\partial x_i} \left[ f(x) + \lambda' g(x) + \mu' h(x) \right] \left[ \begin{array}{c} x \\ \mu \end{array} \right] = 0 \text{ if } x_i \left[ \begin{array}{c} x \\ \mu \end{array} \right] = 0.
\]

The proof is straightforward with the lemma applied to constraints
\[
g(x) \geq 0
\]
\[
-g(x) \geq 0
\]
\[
h(x) \geq 0
\]
\[
x \geq 0.
\]

The (weak) LMR stated in the corollary contains the mathematical formulation of the principle of marginal inequalities and equalities that supplants the usual marginal relations if the economic variables are non-negative.
1. It might seem to follow from this proof that $\mu$ or $\pi$ are arbitrary except for the stipulation that some components be positive. However the zero conditions on the $\mu(x)$ and $\pi(x)$ simultaneously, force a set of well-defined value(s) (the saddle-point(s)) to be attained.

2. Samuelson's Le Chatelier Principle makes possible some qualitative statements, however, if the duality principle is employed. [Samuelson].

3. Cf. for instance [Hart] for the behavior of the firm under capital rationing.

4. This is a further study of an inventory problem analysed in somewhat different form by [Fisch - Lodigliani] [Morin].

5. Any problem that yields almost periodical functions $s(t)$ as solutions can serve as an example of non-convergence.

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