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A Lagrangean Multiplier Rule in Linear Activity Analysis and Some of its Applications*

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1. The standard theory of price and its linear descendant, activity analysis, cover much of the same ground, economically speaking. The assumption of linearity, which at first sight appears restrictive enough, is in fact of no great importance since for all practical purposes it may be replaced by that of piece-wise linearity, as will become clearer later on, (an approximation which is certainly as good as that by continuously differentiable functions). On the other hand, activity analysis is well adapted to economic problems insofar as its variables are constrained to be non-negative, a characteristic feature of many economic magnitudes. The significance of this fact is perhaps brought out best by way of example. Consider the case of the consumer's equilibrium [Hicks, pp. 8-9]. Let

x_r	denote amounts of commodities consumed
p_r	commodity prices
M	the consumers income
$u(x_1, \ldots x_n)$	his utility function

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Then the problem is that of maximizing the function $u(x_1,\dots x_n)$ with respect to the quantities x_r subject to the budget constraint $\sum_r p_r x_r = M$. The Lagrangean multiplier rule reduces this problem to that of extremizing $u(x_1,\dots x_n) + \mu(M - \sum_r p_r x_r)$. The equilibrium conditions are then

$$(1) \qquad \frac{\partial u}{\partial x_r} = \mu p_r$$

where the Lagrangean multiplier μ corresponds to "what is known in Marshall's theory as marginal utility of money".

One may ask with [Houthakker] what happens to these equations and the Engel curves derived from them, if one admits the fact that all commodity quantities have to be non-negative. Of course one would expect that the "=" sign in (1) should be replaced by a "$\leq$" sign in all cases where a quantity turns out to be zero. Actually, this simple assertion whose validity hinges on the concavity of $u(x_1,\dots x_r)$ is quite tedious to prove. This problem leads straight into the controversial field of non-linear activity analysis [Kuhn-Tucker], [Slater].

It will be seen later on that if $u(x_1,\dots x_n)$ is assumed concave and piecewise linear, a linear activity analysis problem results. It is natural however to ask whether this problem can be solved in the same way as before, that is, by means of a suitable Lagrangean Multiplier Rule (LMR). As it turns out, linear activity analysis indeed provides such a LMR. Its principle results, except for computational technique are in fact all embodied in such a LMR. Since on the other hand economists have been using a similar tool in traditional price theory, it seems worth while to develop the LMR of activity analysis explicitly and to demonstrate some of its applications to price theory and to some standard problems of activity analysis itself.

Section 2 states the LMR (theorem 1) and proves it by way of related lemma in [Gale-Kuhn-Tucker]. As a corollary, the minimax property of Lagrangean Multipliers is obtained (theorem 2).

Section 3 and 4 sketch applications to the Hitchcock-Koopmans transporation problem [Hitchcock],[Koopmans] and the gasoline mixture problem [Charnes-Cooper-Mellon]. The problem raised in this introductory section is studied further in section 5. Sections 6 and 7 are concerned with applications of the LMR to the theory of the firm.

2. The Lagrangean Multiplier Rule of Linear Activity Analysis states necessary and sufficient conditions for the solution of the fundamental problem of maximizing, in the Pareto sense, a linear vector function subject to linear inequalities as constraints.

Theorem 1

Necessary and sufficient for $x = \bar{x}$ to yield a Pareto maximum of Bx subject to the constraints $x \geqq 0$, $Ax \geqq c$ is the existence of parameter vectors $\lambda > 0$, $\bar{\mu} \geqq 0$ such that component-wise

$$\bar{\mu}_i = 0 \quad \text{if} \quad (A \bar{x})_i > c_i$$

and such that

$$\lambda' Bx + \bar{\mu}'(Ax - c)$$

takes on an unconstrained minimum for $x = \bar{x}$ over the positive orthant $x \geqq 0$.

Proof:

$\bar{x} = (\bar{x}_j)$ is a maximum point of the kind in question if and only if the form $- \sum_j b_{ij} \Delta x_j$ constrained by $\sum_j a_{kj} \Delta x_j \geqq 0$ for all k with $\sum_j a_{kj} \bar{x}_j = c_k$ and by $\Delta x_j \geqq 0$ for all j with $\bar{x}_j = 0$, has the Pareto maximum zero. This condition is equivalent to the following one due to [Gale, Kuhn, and Tucker] which is based on a familiar inequality of Minkowski's:

$$\sum_{k'} a_{k'j'} \lambda_{k'} + \nu_{j'} \leqq -\sum_i b_{ij} \mu_i \quad \text{for some set of } \lambda_i > 0,$$

$\mu_{k'} \geqq 0$, $\nu_{j'} \geqq 0$ where k' denotes those indices k for which $\sum_j a_{kj} \bar{x}_j = c_k$ and j' the j for which $\bar{x}_j = 0$. In other words the necessary and sufficient conditions for a maximum are that

(2) $\quad \sum_i b_{ij} \lambda_i + \sum_k a_{kj} \mu_k \begin{Bmatrix} = \\ \leqq \end{Bmatrix} 0 \quad \text{if } \bar{x}_j \begin{Bmatrix} > \\ = \end{Bmatrix} 0$ and

(3) $\quad \mu_k \begin{Bmatrix} = \\ \geqq \end{Bmatrix} 0 \quad \text{if } \sum_j a_{kj} \bar{x}_j \begin{Bmatrix} > \\ = \end{Bmatrix} c_k$

But the left hand expression of (2) equals

$$\frac{\partial}{\partial x_j} \left[\sum_i \mu_i b_{ij} x_j + \sum_k \lambda_k \left(\sum_j a_{kj} - c_k \right) \right].$$

Hence (2) is equivalent to the condition that the bilinear form

$$L \equiv \sum_{ij} \mu_i b_{ij} x_j + \sum_k \lambda_k \left(\sum_j a_{kj} - c_k \right)$$

has a maximum point $\bar{x}_j$ over the positive orthant. This completes the proof.

The assertion of the LMR may be expressed in a perhaps more concise form revealing the maximum-minimum property of Lagrangean multipliers [Courant-Hilbert, pp. 142-145]. This formulation also points to the possibility of calculating activity analysis problems by the same methods as developed for solving 2-person zero-sum games [Dantzig].

<u>Theorem 2</u>

Let $\bar{x}$ be a solution of problem 1, $\bar{\mu}$ the associated Lagrangean parameter, and, $\quad L(\mu, x) = \lambda' Bx + \mu'(Ax-c)$. Then

(4) $\quad L(\bar{\mu}, x) \leqq L(\bar{\mu}, \bar{x}) \leqq L(\mu, \bar{x})$

or in other words

(5) $$L(\bar{\mu}, \bar{x}) = \max_{x \geqq 0} \min_{\mu \geqq 0} L(\mu, x) = \min_{\mu \geqq 0} \max_{x \geqq 0} L(\mu, x)$$

Proof:

The first inequality may also be written

$$\sum_j \left[\sum_i b_{ij} \lambda_i + \sum_k a_{kj} \bar{\mu}_k \right] (x_j - \bar{x}_j) \leqq 0$$

for all $x_j \geqq 0$. If $\bar{x}_j > 0$ it is necessary and sufficient for this that

$$\sum_i b_{ij} \lambda_i + \sum_k a_{kj} \bar{\mu}_k = 0.$$

If $\bar{x}_j = 0$ it is necessary and sufficient that

$$\sum_i b_{ij} \lambda_i + \sum_k a_{kj} \bar{\mu}_k \leqq 0.$$

With regard to the second inequality it follows similarly that

$$\sum_j a_{kj} \bar{x}_j - c_k \begin{Bmatrix} \geqq \\ = \end{Bmatrix} 0 \qquad \text{if } \bar{\mu}_k \begin{Bmatrix} = \\ > \end{Bmatrix} 0.$$

To prove the first assertion (5), note that we have shown (first inequality (4))

(6) $L(\bar{x}, \bar{\mu}) = \underset{x \geqq 0}{\text{Max}}\ L(\bar{\mu}, x)$. Let us define a function $\bar{\mu}(x)$

$$\bar{\mu}(x) \begin{Bmatrix} = \\ \geqq \end{Bmatrix} 0 \qquad \text{if} \quad \sum_j a_{ij} x_j - c_i \begin{Bmatrix} > \\ = \end{Bmatrix} 0.$$

It is then seen that

$$L(\bar{\mu}(x), x) = \underset{\mu \geqq 0}{\text{Min}}\ L(\mu, x)$$

Since $\bar{\mu}$ satisfies the conditions for $\bar{\mu}(\bar{x})$ we have by (6)

$$L(\bar{\mu}, \bar{x}) \leqq \underset{x \geqq 0}{\text{Max}}\ \underset{\mu \geqq 0}{\text{Min}}\ L(\mu, x)$$

Similarly, it is proved that $L(\bar{\mu}, \bar{x}) \geqq \underset{\mu \geqq 0}{\text{Min}}\ \underset{x \geqq 0}{\text{Max}}\ L(\mu, x)$. But since always Max Min = Min Max [von Neumann and Morgenstern p. 95] this proves (5).1/

Theorem 2 is of course well known [Dantzig], [Gale, Kuhn, Tucker].

Theorem 1 is not stated in the literature explicitly but is clearly implied by the lemma used in the proof [Gale, Kuhn, Tucker, lemma 3, p. 318] and [Koopmans, theorem 5.4.1, p. 82]. The advantage of the present formulation over alternative statements is debatable. It seems however that a Lagrangean Multiplier Rule is handy to the economist since it yields readily the efficiency prices he is looking for. Koopmans' theory of activity analysis while essentially serving the same purpose is constructed on the basis of certain technology axioms and on an assumption about the signs of the components of c (in theorem 1) which in practical applications can be tedious to check or difficult to satisfy. The multiplier rules as developed in [Kuhn, Tucker] and [Slater] for non-linear problems require additional assumptions, or where not, are subject to controversy. Finally the simplex method approach of [Dantzig], [Dorfman] puts emphasis on the computational side of the problem, which appears to be more complex than the basic economic aspects are.

It must be noted that the two theorems do not settle the question of the existence of (finite) problem solutions. In Koopmans' approach too the existence is postulated rather than demonstrated. But in most economic applications this question is relatively unimportant since a solution is known to exist for a priori reasons.

3. To give a demonstration of possible uses of theorem 1, we shall discuss first two well-known problems of applied activity analysis and next three simple applications to price theory. It will be noted that these are cases of single-valued rather than Pareto optima.

The Hitchcock-Koopmans transportation problem. Let there be given a transportation program (q_i) specifying for each location i of a given set the net amount of commodity to be received per unit of time. We assume (differing here from Koopmans) that shipments have to be made within a given network of routes ij connecting all of these points. Denote by x_{ij} the commodity flow from i to j (per unit of time) and by k_{ij} the unit cost of transportation from i to j. The problem is to find the

$$\underset{x_{ij}\geq 0}{\mathrm{Min}} \sum_{ij} k_{ij}\, x_{ij}$$

subject to

$$\sum_j (x_{ij} - x_{ji}) + q_i = 0 \qquad \text{for all } i$$

where the summations are understood to extend only over routes of the network. Since $\sum_i q_i = 0$ we may replace the = in the constraint by $\leq$. With the IMR we obtain immediately the efficiency condition:

$$\frac{\partial}{\partial x_{ij}} \left[-\sum_{ij} k_{ij}\, x_{ij} - \lambda_i \left(\sum_j (x_{ij} - x_{ji}) + q_i\right)\right] \begin{Bmatrix} = \\ \leq \end{Bmatrix} 0 \text{ if } \bar{x}_{ij} \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

or

$$(7) \quad k_{ij} \begin{Bmatrix} = \\ \geq \end{Bmatrix} \lambda_j - \lambda_i \text{ if } \bar{x}_{ij} \begin{Bmatrix} > \\ = \end{Bmatrix} 0.$$

(7) is the solution (2.13) of [Koopmans, Reiter] which implies that the efficient flow graph is a tree containing no closed circuits except possibly "neutral" ones. The λ_i are efficiency prices [Koopmans, p. 96] associated with the locations of the commodity (in Koopmans case the commodity is "empty ship capacity"). Equations (7) therefore express profitability conditions for shipments in terms of efficiency prices.

4. <u>A Mixture Problem</u> [Charnes-Cooper-Mellon]. Let there be available quantities y_s (s=1, ..., m) of materials (fuels, nuts) which can be mixed into products r(r=1, ..., n) (gasoline, nut mix) such that a_{rs} units of s enter into one unit of r. If p_r, p_s are the market prices of materials and products, what is the most profitable use of the given quantities y_s? Suppose that x_r

units of r are produced and x_s units of s are sold. This yields a profit of

$$\sum_r x_r p_r + \sum_s x_s p_s .$$

The constraints on the x_r, x_s are

(8) $x_r \geqq 0$

(9) $x_s \geqq 0$

(10) $x_s + \sum_r a_{rs} x_r \leqq y_s$.

The conditions for

$$\underset{x_r \geqq 0,\ x_s \geqq 0}{\mathrm{Max}} \left\{ \sum_r x_r p_r + \sum_s x_s p_s + \sum_s \lambda_s \left(y_s - x_s - \sum_r a_{rs} x_r\right) \right\}$$

are

(11) $p_r - \sum_s \lambda_s a_{rs} \begin{Bmatrix} \leqq \\ = \end{Bmatrix} 0 \qquad \text{if } \bar{x}_r \begin{Bmatrix} = \\ > \end{Bmatrix} 0$

(12) $p_s - \lambda_s \begin{Bmatrix} \leqq \\ = \end{Bmatrix} 0 \qquad \text{if } \bar{\bar{x}}_s \begin{Bmatrix} = \\ > \end{Bmatrix} 0$

The parameters λ_s satisfy a constraint

(13) $\lambda_s \begin{Bmatrix} \geqq \\ = \end{Bmatrix} 0 \qquad \text{if } x_s + \sum_r a_{rs} x_r \begin{Bmatrix} = \\ < \end{Bmatrix} y_s$

From (12) it follows that $\lambda_s = 0$ is impossible and hence from (13) that the "=" sign is attained in all constraints (10). The λ_s are the internal prices of materials s, and (11) and (12) state the usual profitability conditions.

5. We return now to the earlier problem of finding the equilibrium conditions of a household with respect to all commodities whether consumed or not. The piecewise linear utility function shall be defined as the intersection of halfspaces

$$y \leqq \sum_r a_r (i_1, \ldots, i_n)\, x_r + b(i_1, \ldots, i_n)$$

Here the $i=(i_1, \ldots, i_n)$ denote lattice points with integers as coordinates. We postulate that these half-spaces be so chosen as to make each point defined by $x_r = i_r$ $\qquad r = 1,\ldots n$

$$y = \sum_r a_r(i)\, i_r + b(i)$$

a vertex of the polyhedron. This is possible in a convex body. With each vertex i then is associated one facet whose equation is

(14) $y = \sum_r a_r(i)\, x_r + b(i)$

For each point of the utility function one or several equations (14) hold according as the point is in the interior or at the boundary of a facet. Equilibrium is achieved when y is maximized with respect to the x_r subject to the conditions

(15) $y \leqq \sum_r a_r(i)\, x_r + b(i)$

for all i,

(16) $x_r \geqq 0$ and

(17) $\sum_r x_r\, p_r \leqq M.$

In terms of the LMR we have the problem of finding

(18) $\underset{y, x_r \geqq 0}{\text{Max}} \left\{ y + \sum_i \lambda(i) \left(\sum_r a_r(i)\, x_r + b(i) - y\right) + \mu \left(M - \sum_r p_r\, x_r\right) \right\}$

with some set of non-negative $\lambda(i)$, μ such that

$$\lambda(i) = 0 \qquad \text{if } \sum_r a_r(i)\, x_r + b(i) > y$$

$$\mu = 0 \qquad \text{if } \sum_r x_r\, p_r < M.$$

The solutions $\bar{y}$, $\bar{x}_r$ satisfy necessary and, in connection with the constraints (15), (16), (17) sufficient conditions, obtained by differentiating with respect to y and x_r of (18):

(19) $1 = \sum_i \lambda(i) \qquad \lambda_i \geqq 0$

$$(20)\qquad \sum_i a_r(i)\ \lambda(i) \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} \mu p_r \qquad \text{if} \quad \bar{x}_r \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0$$

Case I If the "=" sign in constraint (15) is taken on for one facet i only, then (20) takes the form

$$(21)\qquad a_r(i)\ \lambda(i) \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} \mu p_r \qquad \text{for} \quad \bar{x}_r \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0$$

Hence the marginal rates of substitution of commodities consumed in positive quantities, $\frac{a_s}{a_r}$, must equal the price ratio $\frac{p_s}{p_r}$.

Case II Otherwise $\lambda(i) > 0$ for several i. By convexity these facets i must be adjacent. One has

$$(22)\qquad \sum_i a_r(i)\ \lambda(i) \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} \mu p_r \qquad \text{for} \quad \bar{x}_r \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0.$$

Here the left side represents a weighted average, with weights $\lambda(i) \geqq 0$ and $\sum_i \lambda(i) = 1$ (equation (19)) of the marginal utilities for the facets i on whose common boundary the equilibrium point is located. The fact to be emphasized is that marginal inequalities hold with respect to all commodities consumed in zero amounts. We note in passing that a utility function of this type can give rise to piecewise linear Engel curves similar to those discussed by [Houthakker]. These Engel curves can be derived from a finite number of equilibrium points, each of which can be computed by some of the familiar methods of linear activity analysis. A drawback of this approach is that one can no longer predict the effect of datum variations in as simple a fashion as by means of the Slutsky equations.

If the changes are small enough to leave the quantity vector on the same facet, the effects of income changes are indeterminate and of price changes are trivial (movement to the boundary). But beyond a given facet there is no simple prediction.[2/]

6. In the same way as demonstrated for the elementary theory of consumption, it would be possible to derive the characteristic marginal equations and inequalities that govern the equilibrium of a firm confronted with a family of piecewise linear and convex production functions. It will be more interesting instead to analyse the short run allocation problem [Dorfman] in more concrete terms, admitting imperfections of the labor and loanable fund markets, and assuming a fixed stock of capital equipment. We single out for separate treatment the problem of optimum inventory policy.

Assume that the firm can choose between n different processes, so defined that they may run concurrently and independently of each other. Let the price differential between finished product and raw material inputs for each process be given. Assume that for each process a fixed input ratio holds with respect to the various types of labor and capital and with respects to funds, independently of the process level. The problem is that of allocating resources so as to maximize profits. The following tables summarize data and notations.

resource		availability	amounts used	money costs per unit	index range
labor	given force	ℓ_s	x_s	t_s	$S=1,\dots S_\ell$
	newly hired	unlimited	y_s	w_s	
Capital equipment		c_s		0	$S=1,\dots S_c$
funds		m_s	z_s	i_s	$S=1,\dots S_m$
Materials		unlimited			

process level	per unit input of labor	of capital	of funds	profit on materials
u_r	L_{rs}	C_{rs}	M_r	P_r

Problem: Find conditions under which the maximum profit

$$\text{Max}\ (\sum_r u_r p_r - \sum_s x_s t_s - \sum_s y_s w_s - \sum_s z_s i_s)$$

is obtained by the firm subject to the limits existing with respect to the following factors

labor employed: $\sum_r u_r L_{rs} \leqq x_s + y_s$

capital capacity: $\sum_r u_r C_{rs} \leqq c_s$

funds borrowed: $\sum_r u_r M_r \leqq \sum_s z_s$

old labor force: $x_s \leqq \ell_s$

borrowing limits: $z_s \leqq m_s$

and subject to the non-negativity constraints

$$u_r = 0\ , \quad x_s \geqq 0\ , \quad y_s \geqq 0\ , \quad z_s \geqq 0$$

Solution of this problem with the LMR reveals the existence of efficiency prices for each type of labor (λ_s), capital equipment (μ_s), and for funds (ν) not discriminating between various sources . A line of production is carried on at a positive or zero level respectively, if the (marginal or average) cost calculated in terms of these efficiency prices is equal or greater than/or equal to marginal returns from the product.

(22) $$\bar{u}_r \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \qquad \text{if } p_r - \sum_s L_{rs} \lambda_s - \sum_s C_{rs} \mu_s - M_r \nu \left\{ \begin{matrix} = \\ \leqq \end{matrix} \right\} 0$$

Efficiency prices and external prices may be different: the efficiency price of labor is equal to the marginal wage rate; and the efficiency price of funds is equal to the marginal rate of interest plus possibly a premium in case the total limit of borrowing is reached. Finally there is an imputation of efficiency rent to the standing equipment. Again rent is zero where capacity cannot be fully utilized. To summarize, the following are necessary and sufficient conditions for maximization of profits:

1. The profitability principle: activities resulting in negative profit rates are to be dropped.

2. The marginal cost principle: efficiency prices of factors obtained in the market are equal to the market price of the last (most expensive) units purchased.

3. The rent principle: resources available to the firm are evaluated at efficiency rents that absorb any positive profit. Rents are zero if there remains unused capacity.

The reader will find no difficulty in extending the analysis to more general situations, e.g., limitations on material availabilities or piece-wise linear, but convex input relations. In the latter case, not all profit is absorbed by efficiency rents so that an intra-marginal profit emerges. The model then approaches the familiar theory of the firm.[3] A distinctive feature is again the presence of marginal inequalities with respect to all quantities which enter at zero levels, and also the fact that these conditions can be proved strictly sufficient for an efficient allocation.

In the same manner it is possible to analyse the equilibria of other complexes, for instance of production at various locations and between different countries engaged in international trade. The results of such analyses are not very surprising. A less obvious situation seems to arise only with regard to the time dimension of production. We shall present here a very simplified problem, that of optimum inventory accumulation under perfect certainty.[4]

7. We assume that

the cost of production per unit of output is given as a function of discrete time = $c(t)$

$t = 0, 1, \ldots$

the production capacity of the plant is fixed = b

the price of the firm's product is known for all future = $p(t)$

no upper limit exists on inventories but the lower limit is zero;

the cost of holding a unit of inventory stock over the unit of time (including interest charges) = i

the initial inventory is given = a

Our unknowns are:

sales $q(t)$

the level of output $x(t)$

and inventories $s(t)$

Let T be a finite horizon. The problem is to find

$$(23) \quad \max_{q,x,s} \sum_{t=0}^{T} [\, p(t)\, q(t) - c(t)\, x(t) - s(t)\, i\,]$$

subject to

$$q(t) \gtreqqless 0 \qquad s(t) \geqq 0$$

$$x(t) \geqq 0 \qquad x(t) \leqq b$$

$$s(t+1) = s(t) - q(t) + x(t)$$

$$s(0) =$$

The L.M.R. reduces this to finding the

$$(24) \quad \underset{x \geqq 0\ q \geqq 0\ s \geqq 0}{\mathrm{Max}} \sum_{0}^{T} \Big\{ p(t)\, q(t) - c(t)\, x(t) - is(t)$$
$$- \lambda(t)[s(t+1) - s(t) + q(t) - x(t)] - \mu(t)\, [x(t) - b] \Big\}$$

with suitable non-negative multipliers $\lambda(t)$, $\mu(t)$ which vanish when the "=" is not attained in the corresponding constraints.

Differentiation yields

$$(25)\quad p(t) - \lambda(t) \begin{Bmatrix} = \\ \leq \end{Bmatrix} 0 \qquad \text{if } q(t) \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

$$(26)\quad - c(t) + \lambda(t) - \mu(t) \begin{Bmatrix} = \\ \leq \end{Bmatrix} 0 \qquad \text{if } x(t) \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

$$(27)\quad - i + \lambda(s) - \lambda(s-1) \begin{Bmatrix} = \\ \leq \end{Bmatrix} 0 \qquad \text{if } s(t) \begin{Bmatrix} > \\ = \end{Bmatrix} 0 .$$

$\lambda(t)$ is an internal efficiency price of the commodity at time t, which never falls below the market price (25). It increases over time by the amount of storage cost, so long as positive inventories are being held, but never by more than this. (27) The (marginal) costs of production are equal to the internal price if the former is raised by the scarcity rent $\mu(t)$ on productive capacity. (26) It follows incidentally that if production is efficient at all, it is also efficient at the maximum output level; and that it is efficient exclusively at the maximum level if costs are positively below the internal efficiency price (the future sales price, discounted by the costs of storage). The time-relating condition (27) is similar in structure to the spatial efficiency condition (7) of section 3.

An important difference appears however in the boundary or initial conditions. In a closed space economy the "equation of continuity" serves as a boundary condition forcing total originations to equal total terminations of commodity flows, and no extra condition is required to make the system determinate. In the present case, we have an initial condition $s(0) = a$ and obtain from (23) directly the terminal condition $s(T) = 0$, since otherwise the right hand sum would not be maximal. However there is no reason to expect that the solutions converge if the horizon T goes to infinity.5/ Thus if the axis of time is considered open at its future end, an indeterminate situation arises. This remains true also if profits are evaluated at a discount and total profits have a finite value. This seems to constitute a remarkable contrast between allocation in space and in time.

The purpose of this paper has been to show the pertinence of activity analysis to problems usually treated in a different way in the context of the general theory of price. To bridge the differences in method, recourse has been had to a reformulation of principal results of activity analysis in terms of a "Lagrangean Multiplier Rule" and thereby to calculus methods that seem more familiar to students of economic theory.

Appendix

Necessary conditions for the maximum of a (piece-wise continuously differentiable) function subject to (piece-wise continuously differentiable) inequalities as constraints. Let $f(x)$ be the (scalar) maximand, x the (vector) variable and $g(x) \geqq 0$ the (vector) inequality as constraint.

Lemma:

x is a maximum point only if there exists some non-negative parameter (vector) $\lambda = (\lambda_k)$ such that

$$\text{grad } [f + \lambda' g] = 0$$

and $\qquad \lambda_k = 0 \qquad \text{if } g_k(x) > 0.$

Proof:

Clearly it is necessary for a constrainted maximum, that it cannot be improved by small changes satisfying the constraints:

$df \leqq 0$ for all dx subject to $dg_k(x) = 0$ for all k with $g_k(x) = 0$. By a well-known lemma of Minkowski [Gale-Kuhn-Tucker] a necessary and sufficient condition for this linear inequality subject to linear constraints to be satisfied is the existence of a non-negative vector $\mu = (\mu_{k'})$ such that

$$\frac{-\partial f}{\partial x_i} = \sum_{k'} \mu_{k'} \frac{\partial g_{k'}}{\partial x_i}$$

where k' runs over all k for which $g_k(x) = 0$. Putting $\lambda_k = \begin{cases} \mu_{k'} & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases}$, yields at once the assertion.

Corollary:

A necessary condition for x to be a maximum point of $f(x)$ subject to the constraints

$$g(x) = 0$$

$$h(x) \geqq 0$$

$$x \geqq 0$$

is the existence of parameter vectors λ, μ with $\mu_k \begin{Bmatrix} \geqq \\ = \end{Bmatrix} 0$ if $h_k(x) \begin{Bmatrix} = \\ > \end{Bmatrix} 0$ such that

$$\frac{\partial}{\partial x_i} \left[f(x) + \lambda' g(x) + \mu' h(x) \right] \begin{Bmatrix} = \\ \leqq \end{Bmatrix} 0 \text{ if } x_i \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

The proof is straightforward with the lemma applied to constraints

$$g(x) \geqq 0$$

$$-g(x) \geqq 0$$

$$h(x) \geqq 0$$

$$x \geqq 0 .$$

The (weak) LMR stated in the corollary contains the mathematical formulation of the principle of marginal <u>in</u>equalities and equalities that supplants the usual marginal relations if the economic variables are non-negative.

Footnotes

1. It might seem to follow from this proof that $\bar{\mu}$ or $\bar{x}$ are arbitrary except for the stipulation that some components be positive. However the zero conditions on the $\bar{\mu}(x)$ and $\bar{x}(\mu)$ simultaneously, force a set of well-defined value(s) (the saddle-point(s)) to be attained.

2. Samuelson's Le Chatelier Principle makes possible some qualitative statements, however, if the duality principle is employed. [Samuelson].

3. Cf. for instance [Hart] for the behavior of the firm under capital rationing.

4. This is a further study of an inventory problem analysed in somewhat different form by [Hohn - Modigliani] [Morin].

5. Any problem that yields almost periodical functions $s(t)$ as solutions can serve as an example of non-convergence.

References

Charnes, A., Cooper, W. W., and Mellon, B., "Blending Aviation Gasolines - A Study in Programming Interdependent Activities in an Integrated Oil Company", Econometrica, Volume 20, No. 2 (1952) pp. 135-159.

Courant, R., and Hilbert, D., Methoden der Mathematischen Physik, Volume I, Chapter IV, Berlin, 1924.

Dantzig, G. B., "A Proof of the Equivalence of the Programming Problem and the Game Problem", Monograph 13, Chapter XX, pp. 330-335.

Dorfman, R. A., "Application of Linear Programming to the Theory of the Firm", Berkeley and Los Angeles, 1951.

Gale, D., Kuhn, H. W. and Tucker, A. W., "Linear Programming and the Theory of Games", Monograph 13, Chapter XIX, pp. 318-329.

Hart, A. G., "Anticipation, Uncertainty and Dynamic Planning", The Journal of Business of the University of Chicago, Volume XIII, No. 4, (1940) p. 98.

Hicks, J. R., "Theorie Mathématique de La Valeur en Régime de Libre Concurrence", Paris 1937.

Hitchcock, F. L., "The Distribution of a Product from Several Sources to Numerous Localities", Journal of Mathematics and Physics, Massachusetts Institute of Technology, Volume 20 (1941), pp. 224 - 230.

Houthakker, H. S., "La Forme des Courbes d'Engel", Cahiers du Séminaire d'Econometrie, Volume 2.

References cont'd

Koopmans, T. C., "Analysis of Production as an Efficient Combination of Activities", Monograph 13, Chapter III, pp. 33-97.

Koopmans, T. C., and Reiter, S., "A Model of Transportation", Chapter XIV, Monograph 13, pp. 222-259.

Kuhn, H. W. and Tucker, A. W., "Non-linear Programming", Second Berkeley Symposium on Mathematical Statistics and Probability, 1951.

Modigliani, F. and Hohn, F., "Solution of Certain Problems of Production Planning over Time Illustrating the Effect of the Inventory Constraint", Appendix to Cowles Commission Discussion Paper, Economics 2038.

Morin, F., "Note on an Inventory Problem Discussed by Modigliani and Hohn", Cowles Commission Discussion Paper, Economics 2042.

Samuelson, P. A., "The Le Chatelier Principle in Linear Programming", unpublished RAND paper, August 4, 1949.

Slater, M., "Lagrangean Multipliers Revisited", Cowles Commission Discussion Paper, Mathematics 403.

von Neumann, J., and Morgenstern, O., Theory of Games and Economic Behavior, Second edition, Princeton, 1947.