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Efficient Transportation in Networks[1],[2]

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1. Introduction.

1.1. The Setting of the Problem. Determination of the social cost of transportation is central to any theory of efficient location. This problem has some interesting aspects in the presence of congestion, a prominent case of a discrepancy between private and social costs. This paper attempts to analyse the costs of congestion by way of the following problem which is connected with earlier activity analysis of transportation.

(1) Given a network of transportation lines (highways, railroads, canals, etc.) whose capacities are given as functions of speeds, (2) given also the capacities of line intersections, independent of speeds, (3) given finally a transportation program specifying net amounts of commodities to be shipped from or received at given points; what is the allocation of traffic flows to roads and the system of traffic speeds that minimizes total money costs of transportation?

A problem of this kind is not without interest in itself. Since in the real world available road capacities and transportation programs are fixed in the short run, we here have on a theoretical plane the problem with which traffic authorities are concerned in their day to day activities of coping with the demands of traffic. The economics of this subject falls back to a simple formula. Yet it might be said that its significance is far from being realized. It seems therefore worth while to present a somewhat detailed analysis of what would otherwise appear to be a simple matter.

1.2. Economic Plausibility Considerations. The phenomenon of congestion may be approached in two steps. Assume first that speeds are fixed on all roads. In that case congestion is present whenever the demand of traffic for the use of a given road exceeds its (single-valued) capacity. Putting it this way shows at once the economic reasons for congestion. Roads tending to be congested are scarce resources rather than free goods for the public. Since there is however no market price to restrict the demand for this good, a mechanism of waiting queues must in the absence of direct measures of control reduce the demand to available capacity.

In the second place let speeds be variable and the capacities of a road be a function of (average) speed, having a maximum for some intermediate value of (average) speed. The notion of congestion arises with respect to the speed preferences of traffic. Congestion is present if the volume of traffic reduces the average speed thereby preventing the attainment of the preferred speed distribution. In the absence of direct allocation or of prices that will equilibrate demand and capacity, the process of speed retardation expands the supply, and the building up of waiting queues dispels demand until a correspondence is reached.

The break-down of normal flow conditions in traffic, which is what congestion amounts to, thus goes back in both cases to a failure of outside mechanisms to work toward an equilibrium. Of these mechanisms the two most obvious ones are those using allocation by direct control and those employing market prices. According to the theory of welfare economics, the latter procedure is, generally speaking, most efficient. Let us consider briefly how the installment of a price mechanism would affect the traffic equilibrium.

In the case of fixed speeds and therefore fixed capacities, a system of road tolls can presumably cut down the demand for each road to capacity and thereby eliminate both queues and congestion, provided of course that the (static) transportation program is altogether possible within the limitations of the given network. Under speed variability the results of a balanced road toll system are less conspicuous. Generally speaking, congestion will not altogether disappear, since some degree of congestion on a road in high demand may be economical as compared with the cost of transportation over alternatives. But queues will be absent and traffic conditions more balanced. Considering the resulting speeds given and fixed each vehicle may choose its preferred route, and capacity at this speed will exactly permit the resulting traffic.

We may look at the tolls from still a different angle. In the case of fixed speed, a vehicle on a congested road causes a certain opportunity cost to traffic. This is the highest of the additional costs that must be borne by vehicles excluded from the use of this road. In equilibrium the toll is just equal to this social cost. Under conditions of variable speed the social cost entailed by a vehicle on a congested road equals the cost to all other vehicles on the road of the delay caused by this vehicle. In equilibrium speeds and tolls are so balanced as to make tolls equal to this social cost.

These considerations cannot tell us, however, whether the marginal cost principle is sufficient to ensure an optimum allocation of resources and how to determine the equilibrium system of prices and flows. The substitution relations between flows of traffic and the interdependence of speeds, capacities and costs do not appear so simple as to permit definite conclusions on the strength of economic intuition.

<u>1.3. Plan of this Paper</u>. This paper attempts to buttress our intuitive arguments with an activity analysis of a network model of transportation. The objectives are: 1) to study the efficiency problem with reference to several physical inputs desirable in themselves, but of unlimited availability rather than with respect to money cost. 2) to establish that the marginal cost conditions of equilibrium are sufficient for efficiency in a broad class of cases. 3) to identify the present problem with a problem of saddle values for which computational procedures are well developed.

As a side objective it is hoped to obtain some information as to possibilities and limitations of the little explored field of nonlinear activity analysis.

The following parts of this section introduce the notations, discuss the assumptions and state theorems of activity analysis which are used later on. In the second section we take up our problem in a simplified version, considering speeds as given and fixed. This converts the problem into one of linear activity analysis which can be tackled with standard methods. The third section drops this restriction and proceeds from the conclusions obtained for the linear case. Its objective is to show the existence of efficiency prices in terms of which the obvious conditions on marginal cost become necessary and sufficient for a problem solution. The efficiency prices are geared to a saddle value problem which reduces this computation to standard methods. The (fourth) concluding section summarizes the outcome in terms of a simple allocation game.

1.4. Table of Notations.

Indices	i	of location
	ij	of road
	m	of commodity
	n	of transportation resources
Variables	x_{ij}^m	commodity flow
	u_{ij}	vehicle speed
Inputs	k_{ij}^n	transportation resources
	c_{ij}	road capacity
	c_i	road intersection capacity
Outputs	q_i^m	programmed net receipts
Efficiency prices	p_i^m	price of commodity
	p^n	price of resource
	p_{ij}	road toll
	p_i	intersection toll

For convenience the following abbreviations are also used.

x_{ij} for $\sum_m x_{ij}^m$

x for the vector of x_{ij}^m

u for the vector of u_{ij}

p for the vector of p_i^m, p_i, p_{ij} .

The key concepts in our problem are those of <u>network</u>, <u>flow</u>, <u>transportation program</u>, <u>capacity</u>, and <u>transportation input</u>. Presently we shall define these and state our assumptions concerning them.

1.5. Definitions and Assumptions.

1.5.1. By network we mean a connected system of traffic channels, of their junctions and of origination and destination points of traffic (Fig. 1). For the sake of convenience we shall speak of roads, road-intersections and terminals respectively, intersections and terminals being denoted by indices i, roads by the (ordered) double indices ij referring to the adjacent location.

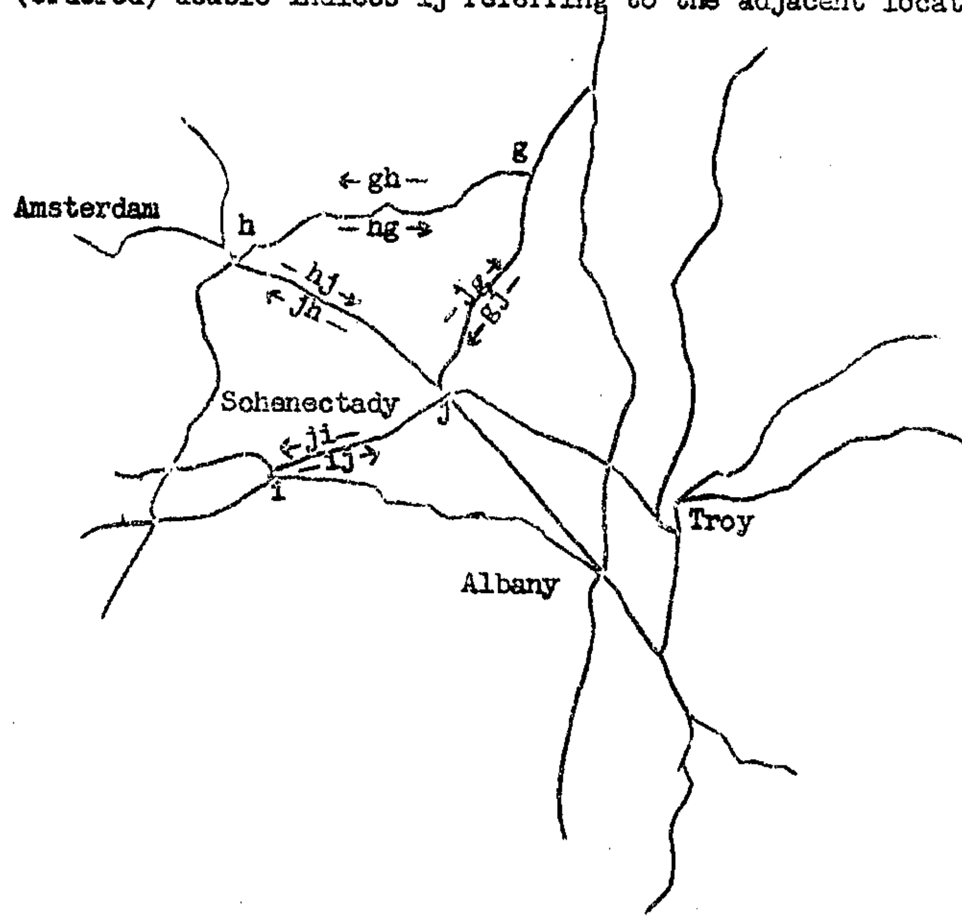

Figure 1

Section from the highway map of New York State

(Four points are denoted by gh and ij respectively, and their connecting roads are marked by ij, gh, etc.)

We adopt the convention that any sums of terms with double indices ij shall extend only over roads contained in the network.

1.5.2. A basic notion is that of commodity flow: the amount of commodity m that leaves point i per unit of time on vehicles entering road ij, x_{ij}^m. This concept, in fact, makes possible the formulation of our essentially discrete problem (optimum choice among a discrete number of route combinations) in terms of the minimization of a differentiable function of the continuous flows, subject, of course, to the obvious constraints of flow continuity in the network (cf. (1.3) infra). By convention

(1.1) $\quad x_{ij}^m \geqq 0$

We define traffic flow as the number of vehicles entering the road per unit of time, x_{ij}. By a proper choice of the units, traffic flow can be made equal to aggregate commodity flow, barring indivisibilities

$$x_{ij} = \sum_m x_{ij}^m \quad .$$

1.5.3. The notion of vehicle speed u_{ij} is obvious. We assume uniform speed for all vehicles on a given road so as to exclude the problems involved in the passing of vehicles. While the latter constitute an important aspect of traffic as such, it is believed that they do not change sensibly the formal relationships between speed and capacity, a concave capacity function with respect to "average speed" replacing the differently shaped but also concave capacity function defined for uniform speed (cf. infra).[3/]

In accordance with the convention (1.1)

(1.2) $\quad u_{ij} \geqq 0$

1.5.4. The transportation program q_i^m specifies the net amount of commodity m to be received by terminal i per unit of time. The term net amount denotes the excess of shipments over receipts of that commodity and thus does not contain transhipments. If a net amount of m is shipped from point i,

q_i^m is negative. At all road intersections which are not terminals $q_i^m = 0$ for all m. The simplest program is of course that with one origination and one termination point for a commodity (e.g., the commodity "passenger").

The program is assumed to be static or otherwise to be subject only to changes slow enough as to not sensibly disturb the stationary flow patterns.

Now the excess of flows to location i

$$\sum_j x_{ji}^m$$

over the flows from location i

$$\sum_j x_{ij}^m$$

of each commodity must equal the net flow terminating at i, that is the transportation program q_i^m

$$\sum_j (x_{ji}^m - x_{ij}^m) = q_i^m \quad . \tag{1.3}$$

The combined transportation programs of commodities proper determines the residual program of empty vehicle movements. For by definition of a stationary flow pattern

$$\sum_m q_i^m \sum_j (x_{ij} - x_{ji}) = 0 \quad .$$

Hence if 0 is the index of empty equipment

$$q_i^0 = - \sum_m{}' q_i^m$$

where the ' denotes that 0 is to be omitted in the summation. We may assume the transportation program to be either fixed or a given function of the commodity prices at the location. It will be apparent in the course of the analysis that the two cases do not necessitate a separate treatment.

The transportation network constitutes a closed economic system in the following sense. All flows into and out of the system may be expressed in the transportation programs of boundary locations. With this convention, programs of each commodity must balance

$$\sum_i q_i^m = 0$$

and the same must hold for empty vehicles

$$\sum_i q_i^o = -\sum_m \sum_i q_i^m = 0 .$$

1.5.5. For every given speed u_{ij}, there exists a limit to the traffic flow x_{ij} to be called the road capacity function $c_{ij}(u_{ij})$.

(1.4) $$x_{ij} \leq c_{ij}(u_{ij}) .$$

Independently of the road capacities there may be effective limits on the traffic through intersections. This calls for a notion of intersection capacity c_i. As a first approximation, intersection capacity may be conceived of as a limit on aggregate flow (through the intersection) in which speed, being fixed at a safety level, does not enter.

(1.5) $$\sum_j (x_{ij} + x_{ji}) \leq c_i$$

A detailed analysis of intersection capacity is possible only with reference to the individual layout and signalling apparatus of road intersections, a task well beyond the scope of this paper.

From traffic engineering data $c_{ij}(u_{ij})$ is known to be a concave function with a finite maximum having approximately the shape indicated in the following diagram (Figure 2). We assume it to have a continuous derivative $\frac{dc_{ij}}{du_{ij}}$.

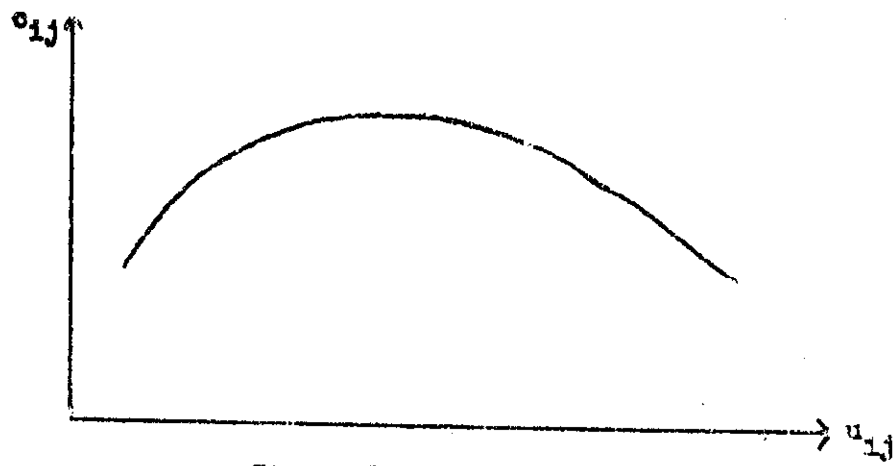

Figure 2

[Cf. O. K. Normann and the Committee on Highway Capacity, Department of Traffic and Operations, Highway Research Board. "Highway Capacity: Practical Applications of Research," Public Roads, 25:202-3.]

The constraints imposed on the traffic by the capacity limits of the network must be compatible with the transportation program, that is to say, there must exist at least one combination of flows and speeds which sustains the program within the limits set. In other words: There exist x_{ij}, u_{ij} such that (1.1)...(1.5) are all satisfied.

1.5.6. By transportation inputs we mean the mobile resources (as distinct from road capacity) employed in transportation. As an example take labor, rolling equipment (vehicles), fuel, and money costs. The assumption is made that these resources are available in unlimited quantities but that they are desirable in themselves.[4] Hence our problem will be to minimize (whatever that may mean) the amounts of these inputs.

Since we do not take account here of indivisibilities the assumption is reasonable that, for a given transportation technique, all inputs into transportation on a given road are strictly proportional to flow. Since we are considering only one type of transportation (at a time), say trucking, we may rule out the possibility of several techniques so that we have only one transportation activity on each road. Let k^n_{ij} represent the quantity of resource n used up by a unit of flow on road ij. The input of n into the whole of transportation activities is then

$$k^n = \sum_{ij} k^n_{ij} x_{ij} \quad .$$

Now the inputs k^n_{ij} will be a function of the vehicle speed u_{ij}, a higher speed of transportation resulting in an increase in the use of some resources (fuel, money) and a decrease in the use of others (equipment, labor). But in either case it is fairly safe to assume inputs to be a convex function of speed.[5] In the case of money and fuel this convexity may be justified as "increasing marginal cost"; with respect to equipment and labor, convexity follows from the fact that these inputs are (approximately) proportional to transportation time, which is the reciprocal of speed, a clearly convex function. We assume the existence of continuous derivatives $\frac{dk^n_{ij}}{du_{ij}}$.

With respect to fuel costs traffic engineering data confirm the convex nature of this input function over the relevant range [3]. [John Beakey, "The Effect of Surface Type, Alignment and Traffic Congestion on Vehicular Fuel Consumption." Oregon State Highway Commission, Highway Department, <u>Technical Bulletin</u>, No. 17 (1944), pp. 85 ff.]

1.5.7. We are now in a position to state what is meant by an efficient combination of transportation activities. Efficiency is here defined in the sense of Pareto-optimality, the familiar concept that underlies the general theory of activity analysis. Let

$$k = (k^1, \ldots, k^n, \ldots)$$
$$\bar{k} = (\bar{k}^1, \ldots, \bar{k}^n, \ldots)$$

be input vectors for a given program (q^m_i). Then $\bar{k}$ dominates k

$$\bar{k} = k \qquad \text{if and only if for all n}$$

$$\bar{k}^n \geqq k^n \qquad \text{but not } \bar{k}^n = k^n .$$

An input vector $\bar{k}$ is called minimal subject to a given program if there exists no k (for the given program) such that $\bar{k} \geqq k$.

A combination of transportation activities is said to be <u>efficient</u> if its input vector is minimal subject to the given program (q_i^m).

In general, the input vectors $k_{ij} = (k_{ij}^1, \ldots, k_{ij}^n)$ are not proportional, some roads being more expensive in terms of fuel than labor (say). But if all roads have comparable cost conditions the inputs maintain a fixed proportion to each other, and the analysis can be reduced to the consideration of just one resource, money (say). But even if this proportionality does not prevail and substitution among the inputs is possible via a substitution among routes of transportation, the analysis will show that each efficiency point is characterised by a set of efficiency price ratios between the various resources. These may be interpreted as (inverse) ratios of convertibility of one resource to another so that we are back again in the case of one input. It is because of this (fictitious) convertibility that there exists a well defined notion of transportation cost.

We may also mention here another concept of transportation efficiency. It is possible to compare with a given program (q_i^m) programs of transportation $x_o q_i^m$ which are proportional to the given one by a factor to be termed the program level x_o.

A combination of transportation activities may be called <u>efficient</u> if it maximizes the program level within the constraints of the network and within given limits for the mobile transportation resources [capacities].

It is plausible and will be shown later on that the two definitions of efficiency are equivalent if the limitations on the mobile transportation resources become effective at a program level $x_0 = 1$. We shall limit ourselves therefore to the consideration of the first (input) version of the efficiency problem. This may now be stated, mathematically, as follows. Let $k(x,u)$ denote the row vector

$$\left(\sum_{ij} k^1_{ij}(u_{ij})\, x_{ij}, \;\dots\; \sum_{ij} k^n_{ij}(u_{ij})\, x_{ij} \;\dots\right).$$

Problem 1.

Find $\bar{x} = (\bar{x}^m_{ij})$, $\bar{u} = (\bar{u}_{ij})$ such that $k(\bar{x}, \bar{u}) = \min\limits_{x,\, u} k(x, u)$ subject to the constraints

(1.1) $x^m_{ij} \geqq 0$

(1.2) $u_{ij} \geqq 0$

(1.3) $\sum\limits_j (x^m_{ij} - x^m_{ji}) + q^m_i = 0$

(1.4) $x_{ij} \leqq c_{ij}(u_{ij})$

(1.5) $\sum\limits_j (x_{ij} + x_{ji}) \leqq c_i$.

1.6. Tools. Before we proceed to the details of the problem solution, we shall state the two principal theorems on which the analysis is based. The first one which may be called the main theorem of linear activity analysis, is stated in [4, Theorem 5.4.1, (p. 82)] to be referred to here as Lemma 1: "A necessary and sufficient condition that an attainable point y be efficient according to Definition 5.2 is that there exists a vector p, normal to the possible cone (A) in y, which has positive components for all final commodities, nonnegative components for all primary commodities whose availability limit is reached in y, and zero components for all primary commodities whose availability limit is not reached in y, (5.8) $\qquad p_{fin} > 0, \qquad p_{pri=} \geqq 0, \qquad p_{pri>} = 0.$ "

The second theorem is a summary of well known extremum conditions in the calculus.

__Lemma 2:: Let $F(x,y)$ be differentiable, jointly convex in x, and jointly concave in y; $x = (x_i)$, $y = (y_j)$. Then F has a saddle value over the positive orthant of x, y precisely where__

$$\frac{\partial F}{\partial x_i} \left\{ \begin{array}{c} = \\ \geqq \end{array} \right\} 0 \qquad \text{for} \qquad x_i \left\{ \begin{array}{c} > \\ = \end{array} \right\} 0$$

$$\frac{\partial F}{\partial y_j} \left\{ \begin{array}{c} = \\ \leqq \end{array} \right\} 0 \qquad \text{for} \qquad y_j \left\{ \begin{array}{c} > \\ = \end{array} \right\} 0$$

[Cf. 5; Lemma 3 (p. 485)].

2. Analysis of Flows.

__2.1. A Linear Problem.__ Assume that speeds are given and fixed, e.g., at their optimal values. Then input and capacity functions are given linear functions and we obtain a linear problem.

__Problem 2.__

Find $\quad \bar{x} = (\bar{x}^m_{ij})$ such that $k(\bar{x}, u) = \min\limits_{x} k(x, u)$ subject to

$$x^m_{ij} \geqq 0$$

$$\Sigma\, x^m_{ij} - x^m_{ji} + q^m_i = 0$$

$$\sum_m x^m_{ij} \leqq c_{ij}$$

$$\sum_{j,m} (x^m_{ij} + x^m_{ji}) \leqq c_i \,.$$

In treating extremum problems of linear vector functions subject to linear inequalities as constraints we are on familiar grounds. However, Problem 2 is not in the form of a standard activity analysis problem. Presently we show that it can be reduced to the following one.

Problem 3

activities

shipments x_{ij}^m

primary commodities (inputs)

desired in themselves:

mobile transportation resources $k^n = -\sum_{ij} k_{ij}^n x_{ij}$

subject to availability limitations:

road capacity resources $-x_{ij} = -\sum x_{ij}^m \geqq -c_{ij}$

intersection capacity resources $-x_i = -\sum_{j,m} (x_{ij}^m + x_{ji}^m) \geqq -c_i$

net shipments from origination points $-x_i^m = \sum_j (x_{ji}^m - x_{ij}^m) \geqq q_i^m$

$(i,m) \in \left\{ i,m : q_i^m \leqq 0 \right\}$

final commodities (outputs)

net receipts at destination points $y_i^m = \sum_j (x_{ji}^m - x_{ij}^m)$

$(i,m) \in \left\{ i,m : q_i^m > 0 \right\}$

Denote for given m the set of i $\left\{ i : q_i^m \leqq 0 \right\}$ by $S(m)$. From the definition of x_i^m, y_i^m the following contraint on the amount of final commodities results

$$0 = \sum_i (x_i^m - y_i^m) = \sum_{i \in S(m)} x_i^m - \sum_{i \notin S(m)} y_i^m = -\sum_{i \in S(m)} q_i^m - \sum_{i \notin S(m)} y_i^m \quad \text{or}$$

$$\sum_{i \notin S(m)} y_i^m = -\sum_{i \in S(m)} q_i^m = \sum_{i \notin S(m)} q_i^m .$$

Therefore the point $y_i^m = q_i^m$, $(i,m) \in \left\{ i,m : q_i^m > 0 \right\}$ in the space of final commodities is efficient provided that the inputs of desired commodities are minimized. This shows that the solutions of problem 2 can be obtained as a subset of the efficient point set of (activity analysis problem 3.

This subset is characterized by the condition

$$\sum_j (x^m_{ij} - x^m_{ji}) + q^m_i = 0 \qquad \text{all } i,m$$

The verification that the postulates of activity analysis [4, pp. 48-55] are satisfied by problem 3 is straightforward and is left to the reader. For instance, postulates C and D are nothing but our previous assumption on the possibility of the transportation program within the limits set by the network.

We are now able to apply to Problem 3 the main theorem of activity analysis [4, 5.4.1, p. 82, in conjunction with definitions 5.1, 5.2 of p. 79] characterizing the efficient point set.

Theorem 1: Necessary and sufficient for $\bar{x}^m_{ij}$ to be a solution of Problem 2 is the existence of nonnegative numbers p^m_i, p_i, p_{ij} such that in addition to the constraints (2.1) - (2.4)

$$\text{(2.1)} \qquad x^m_{ij} \geqq 0$$

$$\text{(2.2)} \qquad \sum_m x^m_{ij} \leqq c_{ij}$$

$$\text{(2.3)} \qquad \sum_{j,m} (x^m_{ij} + x^m_{ji}) \leqq c_i$$

$$\text{(2.4)} \qquad \sum_j (x^m_{ij} - x^m_{ji}) + q^m_i = 0$$

the following conditions are satisfied

$$\text{(2.13)} \qquad p^m_j - p^m_i \left\{ \begin{matrix} = \\ \leqq \end{matrix} \right\} \sum_n p^n k^n_{ij} + p_{ij} + p_i + p_j \qquad \text{if } \bar{x}^m_{ij} \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0$$

where

$$\text{(2.13a)} \qquad p^m_j \left\{ \begin{matrix} > \\ \geqq \end{matrix} \right\} 0 \quad \text{if } (i,m) \left\{ \begin{matrix} \in \\ \notin \end{matrix} \right\} \left\{ i,m : q^m_i > 0 \right\}$$

$$(2.14) \qquad p_{ij} \begin{Bmatrix} = \\ \geq \\ = \end{Bmatrix} 0 \qquad \text{if} \qquad \sum_m x_{ij}^m \begin{Bmatrix} < \\ = \end{Bmatrix} c_{ij}$$

$$(2.15) \qquad p_i \begin{Bmatrix} = \\ \geq \\ = \end{Bmatrix} 0 \qquad \text{if} \qquad \sum_{j,m} (x_{ij}^m + x_{ji}^m) \begin{Bmatrix} < \\ = \end{Bmatrix} c_i$$

$$(2.16) \qquad p^n > 0 .$$

2.3. Efficiency Prices. The equations (2.13)...(2.16) may be called the efficiency conditions. They supplement the constraints by definition, of the network, and of the program for a complete determination of the solution of Problem 2.

If we regard the "Lagrangean parameters" p as prices in the following way —

p^n	unit cost of resource n
p_i	toll at intersection i
p_{ij}	toll on road ij
p_i^m	price of commodity m at terminal i

then the efficiency conditions take on the following economic meaning reminiscent of the role of real prices and tolls.

1. Geographical price differences are less than or equal to the cost of transportation consisting of costs for all resources used plus tolls.

2. If transportation of a commodity actually takes place between two given points the price difference between the receiving and the shipping point is equal to transportation cost. In particular costs of transportation are the same on all alternative roads used.

3. Prices of desirable resources are never zero, but tolls are zero whenever demand falls short of capacity.

4. Transportation costs being positive under all circumstances, circular shipments are excluded since the geographical price difference would be zero. This does not entail the possibility of closed paths in the flow pattern. But the net flow in traversing a closed flow path is zero (a so called neutral circuit [1, p. 248].

5. The resource prices and tolls do not discriminate among commodities. In particular the commodities excluded from the use of a desirable road are those for which alternative routes are least costly.

6. If resource prices and tolls are given the problem becomes a Koopmans-Hitchcock transportation problem independently in each commodity [4, pp. 222-259] with $\sum_n p^n k^n_{ij} + p_{ij} + p_i + p_j$ the cost of transportation.

Presently we shall connect the efficiency tolls with the marginal costs of transportation in the network. As a preliminary we consider a corollary of Theorem 1, in which the conditions for a solution are summarized in a form that is suggestive of the computational processes required.

2.4. A Related Saddle Value Problem.

Lemma 5: The function

$$\begin{aligned}(2.17)\qquad \phi(x, p; u) = \sum_{ij,m,n} p^n k^n_{ij} x^m_{ij} &+ \sum_{ij} p_{ij} \left(\sum_m x^m_{ij} - c_{ij}\right) \\ &+ \sum_i p_i \left(\sum_{j,m} x^m_{ij} + x^m_{ji} - c_i\right) \\ &+ \sum_{i,m} p^m_i \left(\sum_j x^m_{ij} - x^m_{ji} + q^m_i\right)\end{aligned}$$

with $p^n > 0$, has a saddle point $\bar{x}$, $\bar{p}$ for every solution $\bar{x}$ of Problem 2.

$$\phi(\bar{x}, p; u) \leqq \phi(\bar{x}, \bar{p}; u) \leqq \phi(x, \bar{p}; u) .$$

Here the parameter u in $\emptyset(x, p; u)$ indicates that the right side of (2.17) still depends on u. The proof is straightforward using the extremum conditions for linear functions (Lemma 2, end of Section 1).

<u>Corollary</u>: $\sum_{ij,m} k_{ij} \bar{x}^m_{ij} = \emptyset(\bar{x}, \bar{p})$.

Proof: Trivial.

<u>2.5. Marginal Social Cost Identified</u>. We proceed to show that for any pair of points ij and any commodity m, $p^m_j - p^m_i$ is the marginal cost of transportation of m from i to j.

In Lemma 3 substitute $q^m_i = \overset{\circ}{q}{}^m_i + z \Delta q^m_i$. The rate of increase with respect to z of

$$\max_x \, [k_o \, x_o - \sum_{ij,n} p^n \, k^n_{ij} \, x_{ij}]$$

subject to (2.1)...(2.6) is given by the differential quotient with respect to z of (2.17) with $\overset{\circ}{q}{}^m_i + z \Delta q^m_i$ substituted for q^m_i. This differential quotient is

$$\frac{d\emptyset}{dz} = \sum \left\{ \frac{\partial \emptyset}{\partial \bar{x}^m_{ij}} \cdot \frac{d\bar{x}^m_{ij}}{dz} + \frac{\partial \emptyset}{\partial p^m_i} \cdot \frac{dp^m_i}{dz} + \frac{\partial \emptyset}{\partial p_i} \frac{dp_i}{dz} + \frac{\partial \emptyset}{\partial p_{ij}} \frac{dp_{ij}}{dz} + \frac{\partial \emptyset}{\partial z} \right\}$$

As we shall argue in greater detail at a later point all the composite terms vanish since (for instance) either $\frac{\partial \emptyset}{\partial \bar{x}^m_{ij}} = 0$, or $\bar{x}^m_{ij} = 0$ in the neighborhood of $\bar{x}^m_{ij}$. For $\frac{\partial \emptyset}{\partial z}$ at $z = 0$ one obtains

$$\sum_{i,m} \Delta q^m_i \, p^m_i \quad .$$

Now choose

$$\Delta q^m_i = \left\{ \begin{matrix} -1 \\ 1 \end{matrix} \right\} \text{ for } \left\{ \begin{matrix} i \\ j \end{matrix} \right\}$$

and let $z = 0$. The differential quotient with respect to z of (2.17) then becomes the marginal cost of transportation from i to j and is found to be equal to $p_j^m - p_i^m$. This proves our assertion. In particular if there exists a road ij in the network, the marginal cost becomes

$$k_{ij} + p_{ij} + p_i + p_j \, .$$

It follows now that the p_i, p_{ij} are bounded,[7] for the marginal cost cannot exceed the sum $\sum_{ij,n} p^n, k^n_{ij}$, say.

Since the marginal cost of transportation increases by jumps as the various roads are filled to capacity and new circuits become necessary, some of the p_i^m, p_i, p_{ij} also increase by jumps. Then the efficiency prices are step functions of the program level x_o.

2.6. <u>Equivalence of the Efficiency Concepts</u>. The asserted relation between the two efficiency problems, the input minimization problem and the program maximization problem, is easily demonstrated. The latter can be formulated as

<u>Problem 4.</u>

Find $\quad \max x_o$ subject to

$$- \sum_{ij,m} k^n_{ij} \, x^m_{ij} \geqq - k^n$$

$$- \sum_{m} x^m_{ij} \geqq - c_{ij}$$

$$- \sum_{j,m} (x^m_{ij} + x^m_{ji}) \geqq - c_i$$

$$- \sum_{j} (x^m_{ij} - x^m_{ji}) - q^m_i \, x_o = 0$$

$$x^o \geqq 0$$

$$x^m_{ij} \geqq 0 \quad .$$

This problem can be attacked in the same way as Problem 3, the first three constraints to be interpreted as restrictions on primary commodity availabilities and the fourth one as the condition that net output of intermediate commodities is zero. Application of the same theorem of activity analysis results in

<u>Lemma 6: The flows $\bar{x}^m_{ij}$ are an efficient combination in the sense of Problem 4 if and only if there exist parameters λ^n, λ_i, λ_{ij}, λ^m_i such that</u>

$$\lambda^m_j - \lambda^m_i \left\{\begin{matrix} = \\ \leq \end{matrix}\right\} \sum_n \lambda^n k^n_{ij} + \lambda_{ij} + \lambda_i + \lambda_j \quad \text{if} \quad \bar{x}^m_{ij} \left\{\begin{matrix} > \\ = \end{matrix}\right\} 0$$

$$\lambda^n = 0 \qquad \text{if} \qquad \sum_{ij,m} k^n_{ij}\, x^m_{ij} < k^n$$

$$\lambda_i = 0 \qquad \text{if} \qquad \sum_m x^m_{ij} < c_{ij}$$

$$\lambda_{ij} = 0 \qquad \text{if} \qquad \sum_{j,m} (x^m_{ij} + x^m_{ji}) < c_i \quad .$$

Thus if the limits k^n are set in such a way as to make all $\lambda^n > 0$, the solution of Problem 4 becomes identical with that of Problem 3.

<u>3. Analysis of Flows and Speeds.</u>

<u>3.1. The Original Problem.</u> In this section we turn back to our original Problem 1. Lemma 5 and Corollary imply that Problem 2 can be written in the form

Find $\quad \min_{x \geq 0} \max_{p \geq 0} \phi(x, p; u) \quad .$

Since every solution of Problem 2 satisfies automatically the constraints of Problem 1 and because of the Corollary of Lemma 5 one obtains the following unconstrained version of Problem 1.

<u>Problem 5.</u>

Find $\quad \min_{u \geq 0} \min_{x \geq 0} \max_{p \geq 0} \phi(x, p; u) \ .$

In terms of the solution of Problem 2 this becomes

$$(3.1) \qquad = \min_{u \geq 0} \phi(\bar{x}(u), \bar{p}(u); u)$$

First we shall study necessary conditions for a solution of Problem 5.

3.2. Necessary Conditions of Efficiency. Since Problem 5 arises from Problem 2 by making some parameter variable the conditions for a solution of Problem 2 clearly remain necessary. However the $\bar{x}^m_{ij}$, $\bar{p}^m_i$, $\bar{p}_i$, $\bar{p}_{ij}$ appearing in these conditions are now functions of the new variables u_{ij}, as are $c_{ij} = c_{ij}(u_{ij})$ and $k^n_{ij} = k^n_{ij}(u_{ij})$.

We shall proceed under the assumption (plausible enough, but tedious to prove) that $\phi(\bar{x}(u), \bar{p}(u); u)$ is a continuous function of u and that the $\bar{x}^m_{ij}$, $\bar{p}_i$, $\bar{p}_{ij}$, $\bar{p}^m_i$ have piecewise continuous derivatives with respect to the components of u except at a finite number of points at which however the right and left limits (in the u_{ij}) of these derivatives exist. Let these right hand and left hand limits of the derivatives be denoted by $\frac{\partial^+}{\partial u_{ij}}$, $\frac{\partial^-}{\partial u_{ij}}$. A necessary condition for a minimum of $\phi(u)$ is then that for $u = \bar{u}$

$$(3.2) \qquad \frac{d^-\phi}{du_{ij}} \leq 0 \qquad \text{and} \qquad \frac{d^+\phi}{du_{ij}} \geq 0 .$$

(3.2) now gives rise to

$$\frac{d^-\phi}{du_{ij}} = \frac{\partial^-\phi}{u_{ij}} + \sum_{gh,m} \frac{\partial \phi}{\partial x^m_{gh}} \frac{d^-\bar{x}^m_{gh}}{du_{ij}} + \sum_{g} \frac{\partial \phi}{\partial p_g} \frac{d^-\bar{p}_g}{du_{ij}} + \sum_{gh} \frac{\partial \phi}{\partial p_{gh}} \frac{d^-\bar{p}_{gh}}{du_{ij}} + \sum_{g,m} \frac{\partial \phi}{\partial p^m_g} \frac{d^-\bar{p}^m_g}{du_{ij}}$$

and the corresponding equation for $\frac{d^+\phi}{du_{ij}}$. Here we have written $\frac{d^-\bar{x}^m_{gh}}{du_{ij}}$ for the left hand limit of derivatives with respect to u_{ij}, etc.

Presently we show that all composite terms in (3.2) will vanish. Since the argument is the same for all terms, consider the $\frac{\partial \phi}{\partial p_{gi}} \frac{d^-\bar{p}_{gi}}{du_{ij}}$ (say).

For given gh, $\bar{p}_{gh}$ is either identically zero in a left hand neighborhood (exclusive) of $\bar{u}_{ij}$, or positive there. In the first case $\frac{d^- \bar{p}_{gh}}{du_{ij}} = 0$ for $u = \bar{u}$ and in the second case $\frac{\partial \phi}{\partial p_{gh}} = \sum_m \bar{x}^m_{gh} - c_{gh} = 0$ by the condition (2.14) of Theorem 1. In this way one concludes that

$$\frac{d^- \phi}{du_{ij}} = \frac{\partial^- \phi}{\partial u_{ij}} \qquad \frac{d^+ \phi}{du_{ij}} = \frac{\partial^+ \phi}{\partial u_{ij}} .$$

From the definition of ϕ (cf. Lemma 5) and of $\frac{d^-}{du_{ij}}$ if follows now that

$$(3.3) \qquad \begin{aligned} \frac{\partial^- \phi}{\partial u_{ij}} &= \frac{dk_{ij}}{du_{ij}} \, {}_-\bar{x}_{ij} - {}_-\bar{p}_{ij} \frac{dc_{ij}}{du_{ij}} \\ \frac{\partial^+ \phi}{\partial u_{ij}} &= \frac{dk_{ij}}{du_{ij}} \, {}_+\bar{x}_{ij} - {}_+\bar{p}_{ij} \frac{dc_{ij}}{du_{ij}} \end{aligned}$$

where we have used notations $\quad k_{ij} = \sum_n p^n k^n_{ij}$

$$\lim_{u_{ij} \leq \bar{u}_{ij}} \bar{p}_{ij}(u_{ij}) = {}_-\bar{p}_{ij}(u_{ij})$$

$$\lim_{u_{ij} \geq \bar{u}_{ij}} \bar{p}_{ij}(u_{ij}) = {}_+\bar{p}_{ij}(u_{ij})$$

and similarly for ${}_-\bar{x}^m_{ij}$ and ${}_+\bar{x}^m_{ij}$. Instead of $\sum_m x^m_{ij}$ we may however write c_{ij} since $\sum \bar{x}^m_{ij} < c_{ij}$ entails $\bar{p}_{ij} = 0$ (equation 2.14) and the inequalities remain unchanged. We have thus proved

Lemma 7. Suppose that $\bar{x}^m_{ij}$, $\bar{u}_{ij}$ are solutions of Problem 1. Then the conditions (2.13)...(2.16) must be satisfied and, in addition, the inequalities

$$(3.4) \qquad \begin{aligned} &\frac{dk_{ij}}{du_{ij}} \, c_{ij} \quad - {}_-\bar{p}_{ij} \frac{dc_{ij}}{du_{ij}} \leqq 0 \\ &\frac{dk_{ij}}{du_{ij}} \, c_{ij} \quad - {}_+\bar{p}_{ij} \frac{dc_{ij}}{du_{ij}} \geqq 0. \end{aligned}$$

It follows as an incidental fact, that if

$$\frac{dk_{ij}}{du_{ij}} > 0 \qquad \text{for } u = \bar{u}$$

then

$$_{-}\bar{p}_{ij} > {}_{+}\bar{p}_{ij}$$

and conversely for the case when $\frac{dk_{ij}}{du_{ij}} < 0$. This may be explained as follows. If (in the neighborhood of $\bar{u}_{ij}$) money cost of transportation increases with speed and speed is fixed below the optimum $\bar{u}_{ij}$, a relatively higher toll is required to discourage traffic sufficiently from using this road which has become too cheap.

<u>3.3. Sufficiency of the Necessary Conditions for Efficiency.</u> Condition (3.4) can only be satisfied if

$$(3.5) \qquad \text{sign } \frac{dk_{ij}}{du_{ij}} = \text{sign } \frac{dc_{ij}}{du_{ij}} \qquad \text{for } u = \bar{u} \text{ and all } ij.$$

Since c_{ij} is assumed concave and k_{ij} convex, (3.5) can be satisfied only for a piece of either the increasing or the decreasing branch of the capacity curve c_{ij} (cf. Figure 3).

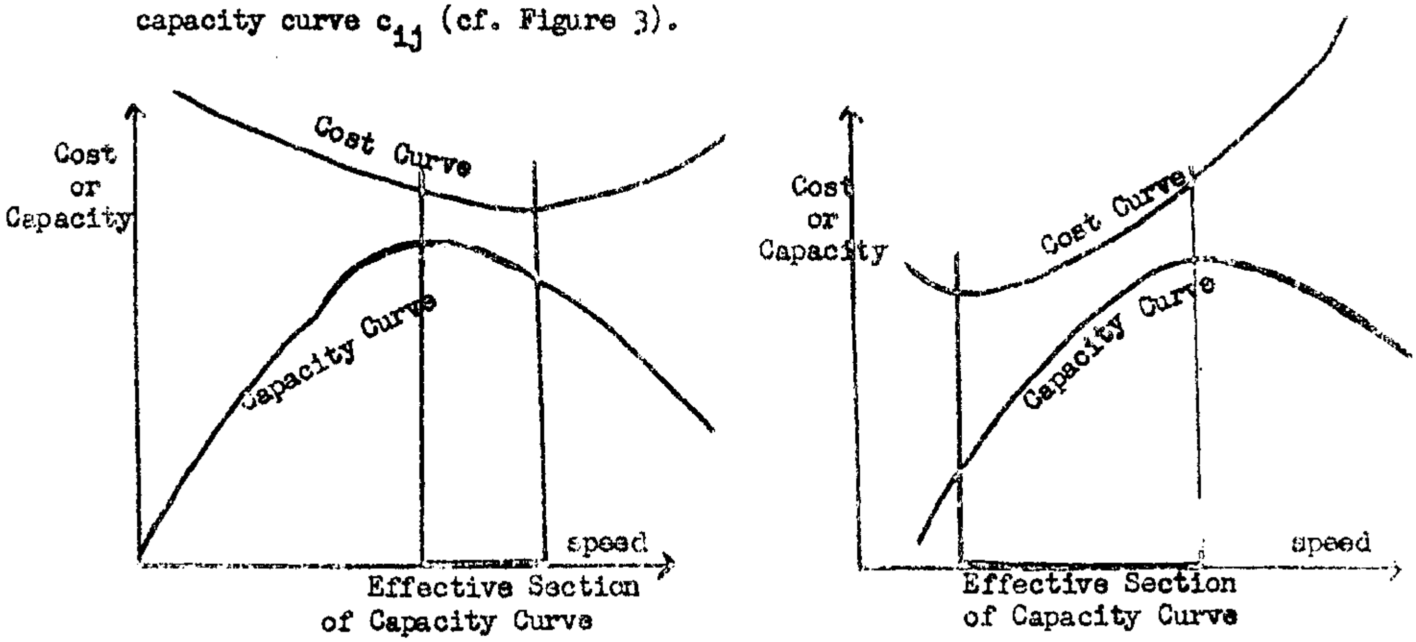

Figure 3

<u>Location of Cost and Capacity Curves</u>

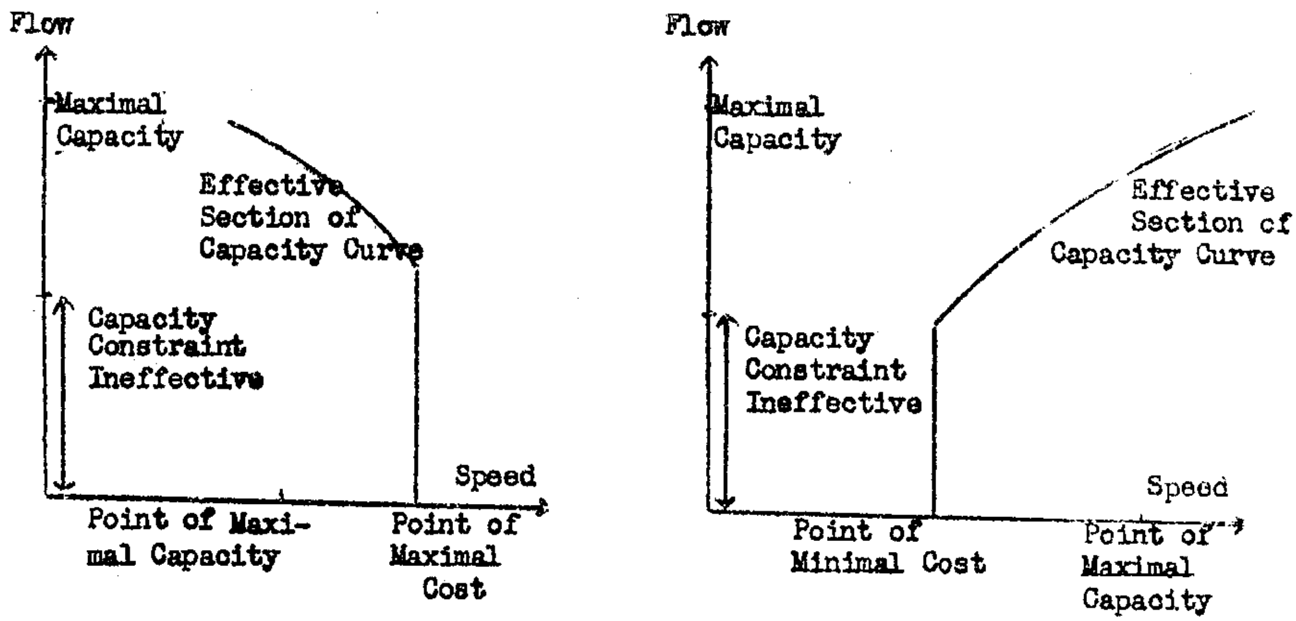

Figure 4

Derived Flow Curves

Whenever the capacity constraint

$$(2.2) \qquad x_{ij} \leq c_{ij}(u_{ij})$$

is not reached (3.4) requires that

$$\frac{dk_{ij}}{du_{ij}} = 0 \quad .$$

This is indicated in Figure 4 by the vertical (opportunity) line through the point of minimal cost. The figures also show that it depends solely on the relative location of minimal cost point and maximal capacity point, whether the desired flow curve is increasing or decreasing. Since the flow curves, thus defined, are monotonic, the inverse functions exist.

$$(3.6) \qquad u_{ij} = u_{ij}(x_{ij}) \; .$$

Since (3.6) is a consequence of the necessary condition (3.4) it is itself a necessary condition on the solution $\bar{x}, \bar{u}$ of Problem 1. Thus the solution of Problem 4 must be of the form $u_{ij} = u_{ij}(x_{ij})$ and therefore the independent variable u may be eliminated in Problem 4. This leads to the equivalent formulation.

Find $\min\limits_{x \geq 0} \max\limits_{p \geq 0} \phi(x, p; u(x))$.

In explicit form, leaving out the term

$$\sum_{ij} p_{ij}(x_{ij} - c_{ij}) \quad \text{from} \quad \phi(x, p; u)$$

which by our substitution for u vanishes identically, we have

Problem 6.

Find $\min\limits_{x \geq 0} \max\limits_{p \geq 0} \Psi(x, p)$ where

$$\Psi(x, p) = \sum_{ij,m} k_{ij}(u_{ij}(x_{ij}))\, x^m_{ij} \tag{3.7}$$

$$+ \sum_i p_i \left[\sum_{j,m} (x^m_{ij} + x^m_{ji} - c_i \right] + \sum_{i,m} p^m_i \left(\sum_j x^m_{ij} - x^m_{ji} + q^m_i\right) .$$

Suppose for a moment (as we shall prove later) that

$$k_{ij}(u_{ij}(x_{ij}))\, x^m_{ij}$$

is a convex function of x^m_{ij}. then Ψ is a convex function of x and a (linear and hence) concave function of p.

In this case the vanishing of the first derivatives with respect to x and p (corrected for zeros of x and p) is necessary and sufficient for a minimax point of (x, p). That means then

$$
(3.8)\quad \left\{
\begin{array}{llll}
\dfrac{\partial \psi}{\partial x_{ij}^{m}} \left\{\begin{array}{c} = \\ \geq \end{array}\right\} 0 & \text{if} & \bar{x}_{ij}^{m} \left\{\begin{array}{c} > \\ = \end{array}\right\} 0 \\[2ex]
\dfrac{\partial \psi}{\partial p_{i}} \left\{\begin{array}{c} = \\ \leq \end{array}\right\} 0 & \text{if} & \bar{p}_{i} \left\{\begin{array}{c} > \\ = \end{array}\right\} 0 \\[2ex]
\dfrac{\partial \psi}{\partial p_{ij}} \left\{\begin{array}{c} = \\ \leq \end{array}\right\} 0 & \text{if} & \bar{p}_{ij} \left\{\begin{array}{c} > \\ = \end{array}\right\} 0 \\[2ex]
\dfrac{\partial \psi}{\partial p_{i}^{m}} \left\{\begin{array}{c} = \\ \leq \end{array}\right\} 0 & \text{if} & \bar{p}_{i}^{m} \left\{\begin{array}{c} > \\ = \end{array}\right\} 0
\end{array}
\right.
$$

are necessary and sufficient conditions for a solution $\bar{x}$, $\bar{u}$ of Problem 6 where $\quad \bar{u} \equiv (\bar{u}_{ij}) \equiv (u_{ij}(\bar{x}_{ij}))$.

Consider the first of the equations (3.8), which reads explicitly,

$$
k_{ij} + \frac{dk_{ij}}{du_{ij}} \; \frac{du_{ij}}{dx_{ij}^{m}} + p_i \left\{\begin{array}{c} = \\ \geq \end{array}\right\} p_j^m - p_i^m
$$

if $\quad \bar{x}_{ij}^{m} \left\{\begin{array}{c} > \\ = \end{array}\right\} 0$.

For the second term we have

$$
\begin{array}{ccc}
\dfrac{dk_{ij}}{du_{ij}} \Big/ \dfrac{dc_{ij}}{du_{ij}} & \text{if} & x_{ij} = c_{ij}(u_{ij}) \\[2ex]
0 & \text{if} & x_{ij} < c_{ij}(u_{ij})
\end{array}
$$

by (3.4). Since the two derivatives in the first alternative have the same sign (by (3.11)), the term $\dfrac{dk_{ij}}{du_{ij}} \; \dfrac{du_{ij}}{dx_{ij}^{m}}$ is always nonnegative. We may put it equal to an "efficiency price" p_{ij}^{*} with

$$
p_{ij}^{*} \geq 0 \;, \qquad p_{ij}^{*} = 0 \qquad \text{if} \qquad x_{ij} < c_{ij}(u_{ij}) \;.
$$

Except for the convexity of $k_{ij}(u_{ij}(x_{ij}))\, x_{ij}$ we have proved

<u>Theorem 2: The system $\bar{x} = (\bar{x}^m_{ij}), \bar{u} = (\bar{u}_{ij})$ is a solution of Problem</u> 1 <u>precisely if it satisfies the following (efficiency) conditions (in addition to the constraints of the problem)</u>

$$(3.9) \qquad p^m_j - p^m_i \begin{Bmatrix} = \\ \leq \end{Bmatrix} k_{ij} + p^*_{ij} + p_i + p_j \quad \text{if} \quad \bar{x}^m_{ij} \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

$$(3.10) \qquad \frac{\partial k_{ij}}{\partial u_{ij}} c_{ij} - p^*_{ij} \frac{\partial c_{ij}}{\partial u_{ij}} = 0$$ [8]

<u>where</u>

$$p^*_{ij} \begin{Bmatrix} \geq \\ = \end{Bmatrix} 0 \quad \text{if} \quad \sum_m x^m_{ij} \begin{Bmatrix} = \\ \leq \end{Bmatrix} c_{ij}$$

$$p_i \begin{Bmatrix} \geq \\ = \end{Bmatrix} 0 \quad \text{if} \quad \sum_{j,m} x^m_{ji} + x^m_{ij} \begin{Bmatrix} = \\ \leq \end{Bmatrix} c_i .$$

<u>3.4. Assertion about Convex Functions.</u> The verification of the convexity of $k_{ij}(u_{ij}(x_{ij}))\, x^m_{ij}$ is routine by means of the following statements which are easily checked.

1. The inverse of a nondecreasing concave function is nondecreasing and convex.
2. The inverse of a nonincreasing concave function is nonincreasing and concave.
3. A nondecreasing convex function of a nondecreasing convex function is nondecreasing and convex.
4. A nonincreasing convex function of a nonincreasing concave function is nondecreasing and convex.
5. The product of two nonnegative, nondecreasing, convex functions is nondecreasing and convex.

Statement	with functions	proves
1, 2	$c_{ij}(u_{ij})$	$u_{ij}(x_{ij})$ is either nondecreasing and convex or nonincreasing and concave
3, 4	$k_{ij}(u_{ij})$, $u_{ij}(x_{ij})$	$k_{ij}(u_{ij}(x_{ij}))$ is nondecreasing and convex
5	$k_{ij}(u_{ij}(x_{ij}))$, x_{ij}^m	$k_{ij}(u_{ij}(x_{ij}))x_{ij}^m$ is nondecreasing and convex

4. Conclusion.

4.1. The Marginal Cost Principle. In summary let us reconsider the main equation

$$(2.13) \qquad p_j^m - p_i^m \left\{\begin{matrix} = \\ \leq \end{matrix}\right\} \sum_n k_{ij}^n\, p^n + p_{ij} + p_i + p_j$$

with the supplementary equation for variable speeds

$$(3.10) \qquad p_{ij} = \frac{d}{du_{ij}} \left[\sum_n k_{ij}^n(u_{ij})\ p^n\right] \Big/ \frac{dc_{ij}(u_{ij})}{du_{ij}} .$$

1. We know from p. 2[illegible] that $p_j^m - p_i^m$ denotes the marginal cost of the program from adding one unit of m to the receipts of j and to the shipments from i, respectively.

This marginal social cost exceeds the marginal private cost $\sum_n k_{ij}^n\, p^n$ by a nonnegative amount $p_i + p_{ij}$. Necessary and sufficient for private cost to equal social cost is therefore that charges equivalent to $p_i + p_{ij}$ be levied on the users of road ij.

2. Suppose that alternative routes are used by flows of commodity m from i to j. Then $p_j^m - p_i^m$ being the same for all routes it is clear that $p_i + p_{ij}$ must compensate for larger values of $\sum_{gh \in R} \sum_n p^n k_{gh}^n$ on the alternative

routes R.

Now p_i is common to all routes passing i. It is therefore obviously a shadow price compensating for the benefit of using this intersection instead of being detoured. This same interpretation applies to p_{ij}, but a more direct explanation is available in equation (3.10). The right side of it clearly denotes the cost to all traffic on road ij arising from the change in speed caused by the marginal vehicle. The equality of both kinds of marginal cost, that to traffic on the same road and that to traffic excluded from this road constitutes an important aspect of the optimal allocation.

Since the right side of (3.9) does not contain m, the marginal social cost of traffic is the same for all commodity flows on that road. It is also the same for all alternative routes taken by flows that could be directed over ij under an efficient traffic allocation. This is the content of the marginal cost principle in traffic allocation: equality of marginal social costs among the alternative routes taken by flows of given commodity (neutral circuits) and among the commodities flowing on a given route.

The main purpose of this paper has been to establish that the marginal cost principle is sufficient here to reserve efficiency in the allocation of traffic. This is not trivial since the aggregate social cost of traffic is not a jointly convex function of flows and speeds. Therefore detailed analysis was required in lieu of the general principle.

4.2. Other Conclusions. In the course of this study occasional light was also thrown on other aspects of efficient transportation in networks. Thus the problems of computation were shown to be of the familiar type of finding the saddle point(s) of a function convex in one given set of variables and concave in the complement set of the variables. The question of uniqueness of the problem solution was touched upon in the marginal cost principle. Equality of marginal costs on alternative routes and among several commodities on one of these routes entails the possibility of substituting flows of these commodities among these routes without a change in social cost.

4.3. An Allocation Game. It may be instructive to review the role of efficiency prices in our model in terms of an (intellectual rather than practicable) allocation game copied in all essential features from Koopmans' Analysis of Production [4; pp. 93, 94].

The participants in this allocation game will be called traffic coordinators (one in charge of each road and one in charge of each road intersection), custodians (in charge of roads, intersections and other resources), shippers (in charge of transportation activities), and program managers (one in charge at each terminal). The rules of conduct as prescribed by (2.1)...(2.6), (2.13)...(2.16), (3.10) are the following.

For shippers: Trade and ship commodities between different locations whenever such activity yields a nonnegative profit. Profits are to be calculated as the geographical price difference minus cost of transportation resources including tolls (per commodity unit) for the use of roads and intersections (2.2.1). Do not engage in transportation activities with negative profits (2.2.2).

For the program managers: At origination points of a commodity sell the quantities specified by the transportation program at the highest prices for which shippers can be induced to take them over. At termination points of a commodity, buy the quantities required by the transportation program at the lowest prices for which they can be obtained from shippers.

For the custodian of resources: Charge prices in accordance with the needs of the rest of the economy. Be prepared to lower your prices to zero if available stocks exceed demand.

For the custodians of roads and intersections: Charge tolls just high enough to adjust the demand for passage to the road (intersection) capacity. Charge no tolls if the demand does not reach the capacity limit.

For the traffic coordinators: Make levies of a fixed proportion to profits on shippers and road custodians, paying back a similar proportion in the case of losses. Regulate speed so as to maximize the income from these levies which is calculated on the basis of expected profits in the following way.

> To calculate expected profits of shippers assume future shipments to be equal to present ones and assume future cost to be based on present tolls and future speeds. To calculate expected toll incomes take present tolls and derive future traffic flows by inserting future speeds into the capacity functions.

The conclusion of this paper is that under these rules an efficient combination of activities once established will be preserved. In other words, once the efficiency prices and speeds are computed traffic can be coordinated without direct control by communicating information about these tolls and speeds to all participants. Instead of directly computing the speeds and prices, the game may be set up in such a way as to lead automatically to the attainment of the solution. Since proportional adjustment will not do [7, pp. 17-22, 74-78], it is necessary to prescribe specifically the magnitudes of changes in prices and speeds with which the road (intersection) custodians and traffic coordinators shall react upon observed discrepancies.

Arrow and Hurwicz [8] have suggested a set of concrete rules of behaviour for output (flow) and price-controllers such that the optimum is attained from all initial positions through the automatic moves of all participants. If we wish, we may identify the players in this allocation game with entrepreneurs and the efficiency prices with market prices which would result under the reign of perfect competition. All roads would be toll roads owned by private firms (although some roads might yield zero tolls). But practical difficulties would arise, not only in the collection of tolls efficient enough to prevent new congestion, but also in the enforcement of competition among road custodians and traffic coordinators.

4.4. Outlook. The present study may serve as a master pattern for the analysis of more general network problems involving, e.g., the time component of variable programs. Some extensions are straightforward, such as the admission of several types of equipment with different cost characteristics (leading perhaps to a separation of slow from fast traffic on various roads), or the replacement of one way roads by two way roads with the possibility of continuous substitution of flows (the tolls p_{ij} and p_{ji} must be equal). The replacement of uniform speeds by speed distributions will be the subject of a separate paper.

References

[1] T. C. Koopmans and S. Reiter, "A Model of Transportation," Chapter XIV, Activity Analysis of Production and Allocation, Cowles Commission Monograph No. 13, John Wiley, 1951.

[2] Cf. Part 2, p. 10.

[3] Cf. Part 2, p. 12.

[4] T. C. Koopmans, "Analysis of Production as an Efficient Combination of Activities," Chapter III, Activity Analysis of Production and Allocation, Cowles Commission Monograph No. 13, John Wiley, 1951.

[5] H. W. Kuhn and A. W. Tucker; "Nonlinear Programming," Second Berkeley Symposium on Mathematical Statistics and Probability, 1951.

[6] P. A. Samuelson, "The Le Chatelier Principle in Linear Programming," RAND, RM 210, 1949.

[7] P. A. Samuelson, "Market Mechanism and Maximization," RAND, D-69, 1949.

[8] K. Arrow and L. Hurwicz, "A Gradient Method for the Lagrangean Problem," Cowles Commission Discussion Paper, Economics 296.

1. Research undertaken under contract between the Cowles Commission for Research in Economics and the RAND Corporation.

2. The author has benefitted from comments by I. N. Herstein, H. Markowitz, and M. Slater. His main indebtedness is to T. C. Koopmans, who has suggested this problem, stimulated its treatment in various discussions, and presented an earlier version of it at the Logistic Conference, January, 1952, Washington, D. C.

3. The discussion of capacity formulas for nonuniform speed distribution is deferred to a separate paper. Cf. also B. McGuire, "Highway Capacity and Traffic Congestion: A Preliminary Study," Cowles Commission Discussion Paper, Economics No. 2048.

4. If these resources are subject to limitations these restrictions will, in general, conflict with the road capacity constraints in such a way as to render the marginal cost conditions insufficient for efficiency. The scope of the present paper does not permit our going further into this, but the study of Section 4 will convince the reader that the persistence of relative convexity of inputs with respect to flows is highly sensitive to the nature of the restrictions imposed.

5. A function is convex if (and only if) the chord spanned by any two points on its graph lies above the graph. Analytically: Let $0 \leq \lambda \leq 1$; then $f(x)$ is convex if (and only if)

$$f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2).$$

6. It may be seen directly from these conditions or, alternatively, from the general theorem of activity analysis mentioned [4; Theorem 5.6 (p. 86)] that any $\bar{x}$ satisfying (2.7)...(2.12) is also a solution of the problem.

Find $\max\limits_{x} (k_o x_o - \sum\limits_{ij,n} p^n k_{ij}^n x_{ij})$

subject to the constraints (2.1)...(2.6). Now

$$\max_{(x_{ij}^m)} - \sum_{ij,n} p^n k_{ij}^n x_{ij}$$

subject to (2.1)...(2.6),

is a continuous function of x_o whose derivative to the right is piecewise continuous in $0 \leq x_o \leq 1$. This derivative is also a piecewise continuous function of the p^m in the domain $0 \leq p^n \sum\limits_n p^n = 1$. It is therefore only necessary to choose

$$k_o > \max_{\substack{(p^n) \\ p^n \geq 0 \\ \sum_n p^n = 1}} \max_{\substack{x_o \\ 0 \leq x_o \leq 1}} \frac{d^+}{dx_o} \left[\begin{array}{l} \min\limits_{(x_{ij}^m)} \sum\limits_{ij,n} p^n k_{ij}^n x_{ij} \\ \text{subject to } (2.1)\ldots(2.6) \end{array} \right]$$

A rigorous analysis of the invoked continuity relations lie outside the scope of this study.

7. This in conjunction with (2.7) can be used to show that in (2.7) for k_o sufficiently large π_o must become positive and hence $\bar{x}_o = 1$. This demonstration fails however if the program for $x_o = 1$ cannot be sustained by the network. For then the marginal cost may increase indefinitely.

8. Comparing (3.10) with (3.4) shows that

$$ {}_{+}\bar{p}_{ij} \leqq p^*_{ij} \leqq {}_{-}\bar{p}_{ij} \qquad \text{if} \qquad \frac{dc_{ij}}{du_{ij}} > 0 $$

and (3.8)

$$ {}_{-}\bar{p}_{ij} \leqq p^*_{ij} \leqq {}_{+}\bar{p}_{ij} \qquad \text{if} \qquad \frac{dc_{ij}}{du_{ij}} < 0 $$

p^*_{ij} may therefore be different from either limit of $\bar{p}_{ij}$.