COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 2046

LINEAR ACTIVITY ANALYSIS AND INTERNATIONAL TRADE

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June 16, 1952

1. This paper is an attempt to use the linear activity analysis approach in a study of the benefits of economic integration.

In the most general terms, the problem can be stated as follows: the constraints to which a complex industrial concern or economic system must comply can be conceived as applying either to the system as a whole, or to each of its parts independently; clearly, even if the nature of the constraints is the same in both cases, the latter alternative is more restrictive in the sense that, without removing the over-all limitations set in the former case, the latter one may, in addition, prevent each sector from having its activities adjusted for the benefit of all. With regard to external limitations, partitioning an economic system is therefore a cause of loss, because it can only restrict the range of possible achievements available to the system.

That is the sort of loss we are concerned with: assuming that some constraints are to be imposed on a system, we seek to evaluate the drawbacks of an arbitrary discrimination in their enforcement within the system.

The method developed under the name of activity analysis provides quite naturally a general formulation which applies to a large class of problems sharing that same peculiarity. Two examples are presented here: the first deals with a technical problem of ocean transportation; the second is an economic appraisal of international trade barriers. The formal model, which will appear to fit both, will be fully studied in the second context only (Parts II and III); the short presentation of the first (Part I) is only intended to facilitate a comparison of the two very different instances, which may yield an insight into their common features, the implications of which are precisely the object of our study.

Unlike most problems of activity analysis, and nearly all problems of marginal analysis, the present one can, in a rather broad range of cases, be worked out, and be given a solution in general form, without need of specifying the numerical data and resorting to computational devices. This advantage is paid for by what some will think to be an over-simplification of reality; but it should also convince them that this approach to the problem perhaps deserves further study, and, at least, is not a blind alley.

I. FORMULATION OF A TRANSPORTATION PROBLEM

2. A simple case.

The clearest picture of the problem we are to deal with can be given by this simple case:

Two goods (subscripts 1 and 2) are transported in two ships (superscripts 1 and 2); for our purpose each good is described by two physical characteristics, namely the weight (a_1, a_2 respectively) and the volume (b_1, b_2) of the unit quantity (of course, the unit can be chosen so that its weight, or volume, is 1). Each ship, which can carry both goods jointly, is described by its capacity in terms of these characteristics, namely its weight capacity (w^1, w^2 respectively), and cubic capacity (v^1, v^2). What possibilities of transportation does the fleet present?

As for the first ship only, each physical characteristic brings about a limitation in the amounts x_1^1, x_2^1, that the ship can transport jointly; the following inequalities must hold:

$$\text{(1)} \quad \begin{cases} a_1 x_1^1 + a_2 x_2^1 \leq w^1 \\ b_1 x_1^1 + b_2 x_2^1 \leq v^1 \\ x_1^1 , x_2^1 \geq 0. \end{cases}$$

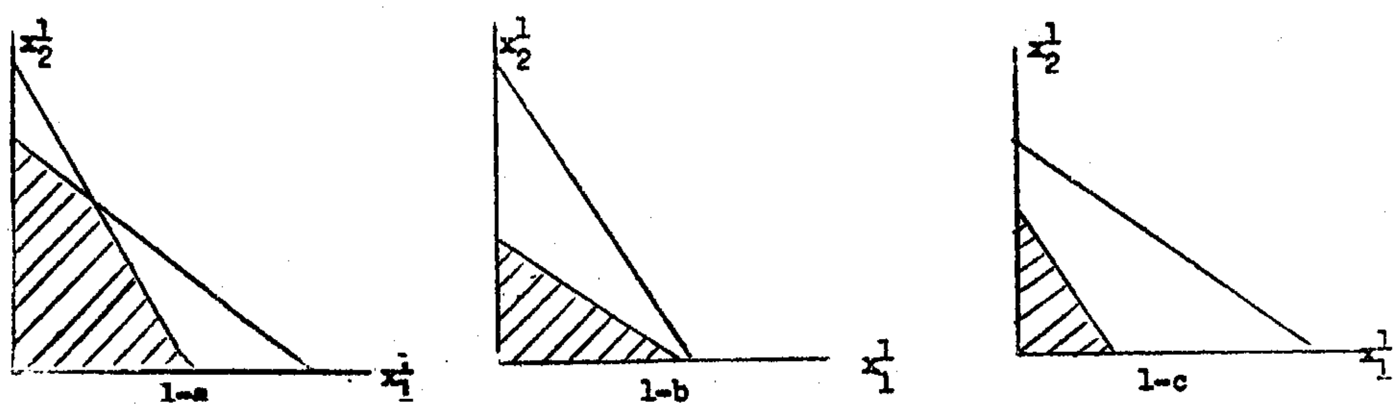
Graphically, the set of all possible points (x_1^1 , x_2^1) is the hatched area in fig. 1, where the two straight lines (which can have the 3 relative positions 1-a, 1-b, 1-c) are given by the equations

$$a_1 x_1^1 + a_2 x_2^1 = w^1$$

$$b_1 x_1^1 + b_2 x_2^1 = v^1 ,$$

the limits of the first two inequalities (1).

Fig. 1



A similar graph can be drawn for the second ship; it will depict the possible point set in the (x_1^2 , x_2^2)-space, defined by:

It is conceivable that neither of the two goods, when loaded up to the full capacity of the first ship, can fill entirely the space available on that ship, - while, at the same time, the weight capacity of the other ship cannot be reached either. Any improvement on that situation, where both space and tonnage are wasted, would require that "weight capacity" or "space availability" be transferable from one ship to the other. This extreme case clearly shows a possibility of loss originating from the fact that physical limitations apply to each ship separately, and not to the fleet as a whole. A detailed study will discover other possibilities of loss in less obvious cases. In other words, although inequalities (1')-(2') and definition (3) together entail

(4) $$A\,y \leq c^1 + c^2 , \quad y \geq 0 ,$$

yet the converse is not always true: it may be possible to find a vector y which satisfies (4) but which cannot be partitioned into two vectors x^1 , x^2 , satisfying (1') and (2').

Applied to this very simple example, our problem would be to compare the set defined in the y-space by (1')-(2')-(3), to the set (4). Let us remark that fig. 1 suggests that, even here, a direct treatment would involve considering 2x3=6 cases successively (2 ships, 3 possible cases for each one).

3. General case.

A generalization is conceivable in three ways; we can: (a) not restrict ourselves to 2 different ships, but allow their number K to be arbitrarily large; (b) not restrict ourselves to 2 goods, but study the case of N goods; (c) not restrict ourselves to 2 limiting physical characteristics of these goods (weight and volume), but take into consideration any other sort of limitation that may arise.

The problem is of the same nature as that previously stated, except that:

(a) generalization (a) allows for any number of partial sets like (1') or (2') to be "added up"; y becomes a sum of K vectors $x^1 + \ldots + x^k + \ldots + x^K$;

(b) generalization (b) makes the vectors x^k and y have N components, that is, the left hand members of inequalities (1) for instance have N terms;

(c) generalization (c) makes the vector inequalities like (1') stand for (say) P scalar inequalities instead of 2.

4. Formulation of the problem.

Finally:

Let A be the matrix

$$A = \begin{bmatrix} a_1^1 & \cdots\cdots & a_n^1 & \cdots\cdots & a_N^1 \\ \vdots & & & & \\ a_1^p & & a_n^p & & a_N^p \\ \vdots & & & & \\ a_1^P & & a_n^P & & a_N^P \end{bmatrix}$$

The K vectors x^k $(k = 1,\ldots,K)$ are subject to the K sets of conditions:

(5-k)
$$\begin{cases} A\,x^k \leqq c^k & \text{(P scalar inequalities for each k)} \\ x^k \geqq 0 & \text{(N scalar inequalities for each k)} \end{cases}$$

What do these conditions imply with regard to the sum

(6)
$$y = \sum x^k \; ?$$

Compare the so defined attainable point set in the y-space to the set

(7)
$$\begin{cases} A\,y \leqq \sum c^k \\ y \geqq 0. \end{cases}$$

Furthermore, we do not content ourselves with finding a convenient criterion as to whether or not a point y is attainable (i. e., whether a given transportation program can be carried out); we need in addition a method of determining at least one set of possible x's corresponding to it (i. e., at least one possible way of effectively dividing that total load among our ships).

II. INTERNATIONAL TRADE BARRIERS

5. The model.

Exactly the same formulation applies to the following problem:

Let us consider K different countries (superscript, k) whose industries are, or could be, engaged in the production of N different goods (subscript, n). The production of each good requires the use of P natural resources (second superscript, p). The amounts of resources available in each country are c^k.

The bill of goods produced in each country is

$$\text{(country } k) \qquad x^k = \begin{bmatrix} x_1^k \\ \vdots \\ x_n^k \\ \vdots \\ x_N^k \end{bmatrix}$$

The technology is supposed to be the same for all countries, and is represented by a nonnegative matrix (P by N)

$$A = \begin{bmatrix} a_1^1 & \cdots\cdots & a_n^1 & \cdots\cdots & a_N^1 \\ \vdots & & & & \\ a_1^p & & a_n^p & & a_N^p \\ \vdots & & & & \\ a_1^P & & a_n^P & & a_N^P \end{bmatrix},$$

such that, in the absence of international trade, the attainable point set for a particular country k is given by

(identical with (5-k))
$$\begin{cases} A\,x^k \leqq c^k \\ x^k \geqq 0 \end{cases} \qquad \text{where } c^k = \begin{bmatrix} c^{k1} \\ \vdots \\ c^{kp} \\ \vdots \\ c^{kP} \end{bmatrix}$$

The total output $y = \sum x^k$ is thus limited to a range of possibilities that we are to investigate.

If these countries were willing and/or able to pool their resources together, the total output would be limited only by:

(identical with (7))
$$A\,y \leqq \sum c^k, \quad y \geqq 0.$$

What possibilities are they losing? What others remain to them? What specializations will trade barriers induce? Those questions are a new translation of the problem fully formulated in section 4.

The remainder of this paper gives a complete solution for the case of any number of countries, any number of goods, and 2 resources only.

The results are summed up in sections 14 and 15.

III. ANALYTICAL SOLUTION: N GOODS, K COUNTRIES, 2 RESOURCES

6. Notation.

We shall normalise the matrix A, which has 2 rows only:

$$A = \begin{bmatrix} 1 \ldots\ldots 1 \ldots\ldots 1 \\ \alpha_1 \ldots\ldots \alpha_n \ldots\ldots \alpha_N \end{bmatrix},$$

where the α's are supposed all different, and numbered so that

$$\alpha_1 < \alpha_2 < \ldots\ldots < \alpha_n < \ldots\ldots \alpha_N.$$

In the same way, we write the resource vectors:

$$c^k = \begin{bmatrix} \gamma^k \\ \gamma^k \rho^k \end{bmatrix} = \gamma^k \begin{bmatrix} 1 \\ \rho^k \end{bmatrix} .$$

A point y will be called attainable if there can be found a system of points x^k satisfying $y = \sum x^k$ and such that all x^k are attainable, that is $A x^k \leqq c^k$, $x^k \geqq 0$.

The attainable point set in the y-space will be denoted by (AtY). It is easy to show that (AtY) is convex.

7. Study of the attainable point set in the x^k-space.

Let us concentrate upon a particular country, k. The quantities of goods that it can produce are given by the inequalities (where the superscript k is fixed):

(8)
$$\begin{cases} x^k \geqq 0 \\ \dfrac{\sigma_1^k}{\alpha_1} = \sum_n x_n^k \leqq \gamma^k \\ \sigma_N^k = \sum_n \alpha_n x_n^k \leqq \gamma^k \rho^k \end{cases}$$

The maximum of the linear form $\sigma_N^k = \sum_n \alpha_n x_n^k$ 1/, in which all x_n^k's are nonnegative and are subject to $\sum_n x_n^k \leqq \gamma^k$, is $\alpha_N \gamma^k$ (α_N is the largest α); so the second inequality (8) implies

$$\sum_n \alpha_n x_n^k \leqq \alpha_N \gamma^k .$$

And in the same way, the third condition (4) implies

$$\sum x_n^k \leqq \frac{\gamma^k \rho^k}{\alpha_1} .$$

1/ The reason for denoting these sums by σ_1, σ_N, will appear later on.

Thus, all conditions (4) together are equivalent to the following ones, which provide a new definition of the attainable point set in the x^k-space:[2/]

$$x^k \geq 0$$

(9) $$\sigma_1^k = \alpha_1 \sum_n x_n^k \leq \gamma^k \min(\alpha_1, \rho^k)$$

$$\sigma_N^k = \sum_n \alpha_n x_n^k \leq \gamma^k \min(\alpha_N, \rho^k).$$

8. <u>Other consequences of (4).</u>

Consider the following sum, in which n is an arbitrary number smaller than N:

$$\sigma_n^k = \alpha_1 x_1^k + \alpha_2 x_2^k + \dots\dots + \alpha_{n-1} x_{n-1}^k + \alpha_n (x_n + \dots\dots + x_N).$$

We have

$$\sigma_N^k - \sigma_n^k = (\alpha_{n+1} - \alpha_n) x_{n+1}^k + \dots\dots + (\alpha_N - \alpha_n) x_N^k \geq 0,$$

and:

$$\frac{\alpha_n}{\alpha_1} \sigma_1^k - \sigma_n^k = (\alpha_n - \alpha_1) x_1^k + \dots\dots + (\alpha_n - \alpha_{n-1}) x_{n-1}^k \geq 0.$$

So: $$\sigma_n^k \leq \min\left(\sigma_N^k, \frac{\alpha_n}{\alpha_1} \sigma_1^k\right),$$

or: $$\sigma_n^k \leq \gamma^k \min\left(\alpha_N, \rho^k, \alpha_n \frac{\alpha_n}{\alpha_1} \rho^k\right).$$

or, finally:

(10) $$\sigma_n^k \leq \gamma^k \min(\alpha_n, \rho^k).$$

2/ The symbol $\min(\alpha_1, \rho^k)$ means: "α_1 or ρ^k, whichever is the smaller."

Let us define a square matrix M by $M_{ij} = \min(\alpha_i, \alpha_j)$,

or, more explicitly:

$$M = \begin{vmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \cdots\cdots \alpha_1 & \cdots\cdots \alpha_1 \\ \alpha_1 & \alpha_2 & \alpha_2 & \cdots\cdots \alpha_2 & \cdots\cdots \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots\cdots \alpha_n & \cdots\cdots \alpha_n \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots\cdots \alpha_n & \cdots\cdots \alpha_N \end{vmatrix},$$

and a vector μ^k by

$$\mu^k = \begin{vmatrix} \min(\alpha_1, \rho^k) \\ \min(\alpha_n, \rho^k) \\ \min(\alpha_N, \rho^k) \end{vmatrix}$$

Finally, μ will designate the matrix $\mu = [\mu^1 \ldots \mu^k \ldots \mu^K]$.

With these definitions, the equations (9)-(10) can be summed up in the following form:

$$(11) \qquad \begin{cases} x^k \geqq 0 \\ Mx^k \leqq \gamma^k \mu^k. \end{cases}$$

These are equivalent to (8), but will prove more convenient. Note that among the latter conditions, at least N-2, which have been derived from the other 2, are otiose.

9. <u>"Extreme point" in the x^k-space.</u>

The set $M x^k = \gamma^k \mu^k$, $x^k \geqq 0$, is not empty. It contains at least:

-if $\rho^k < \alpha_1$, the point $\begin{cases} x_1^k = \dfrac{\gamma^k \rho^k}{\alpha_1} \\ \text{all other } x\text{'s} = 0; \end{cases}$

-if $\alpha_{n-1} \leqq \rho^k < \alpha_n$, the point $\begin{cases} x_{n-1}^k = \gamma^k \dfrac{\alpha_n - \rho^k}{\alpha_n - \alpha_{n-1}} \\ x_n^k = \gamma^k \dfrac{\rho^k - \alpha_{n-1}}{\alpha_n - \alpha_{n-1}} \\ \text{all other } x\text{'s} = 0; \end{cases}$

-if $\rho^k \geq \alpha_N$, the point $\begin{cases} x_N^k = \gamma^k \\ \text{all other } x\text{'s} = 0. \end{cases}$

But the determinant $|M|$ is not vanishing, as one can see immediately:

$$|M| = \alpha_1 (\alpha_2 - \alpha_1)(\alpha_3 - \alpha_2) \ldots\ldots (\alpha_N - \alpha_{N-1}).$$

Therefore, the solution we have found for the equations $M x^k = \gamma^k \mu^k$ is unique; the corresponding ppint will be called the "extreme" point of the set x^k, and will be denoted by $\bar{x}^k$:

$$\bar{x}^k = \gamma^k M^{-1} \mu^k.$$

10. Attainable points in the y-space.

In the y-space, let us consider the set (where M is the matrix defined above):

$$(12) \qquad \begin{aligned} y &\geq 0 \\ My &\leq \sum \gamma^k \mu^k = \mu\gamma. \end{aligned}$$

In spite of a similar appearance, the structure of such a set differs from that of set (11) in the fact that none of the inequalities (12) are necessarily redundant, as will be shown later on.

We shall argue that the set (12) is the attainable point set (AtY), defined in section 6.

11. Extreme point of the set (AtY).

The point $\bar{y} = \sum \bar{x}^k$ belongs to (AtY), since all the $\bar{x}^k$ are attainable. Obviously, it belongs also to the set (12); moreover, we have

$$M\bar{y} = M \sum \gamma^k M^{-1} \mu^k = \sum \gamma^k \mu^k = \mu\gamma,$$

and, since M is of rank N, $\bar{y}$ is the only point which satisfies this equation. It will be called the extreme point of (AtY).

12. <u>All points of (AtY) satisfy (12).</u>

This follows from the fact that the equations (12) are merely a sum of the corresponding equations (11), which are verified by all attainable points in the x^k-spaces.

13. <u>All points verifying (12) are attainable.</u>

This proposition is true when there is only one good: every positive number y verifying $\alpha_1 y \leqq \sum \gamma^k \min(\alpha_1, \rho^k)$ may be considered as the sum of K positive numbers x^k verifying $\alpha_1 x^k \leqq \gamma^k \min(\alpha_1, \rho^k)$.

Let us assume that the proposition is true in an (N-1)-good space. Then, in the N-good space, let y^* be any point satisfying (12).

If $y^* = \bar{y}$, our proposition is proved.

If not, let us consider the point, variable on the straight line $\bar{y}\, y^*$:

$$y = y^* + \lambda(y^* - \bar{y}).$$

At least one of the coordinates of $y^* - \bar{y}$ is negative, because (1) all cannot vanish, since $y^* \neq \bar{y}$, and (2) all cannot be nonnegative, with one at least strictly positive, since their sum $\sum_n y^*_n - \sum_n \bar{y}_n$ is nonpositive:

$$\sum_n y^*_n \leqq \frac{\sum_k \gamma^k \min(\alpha_1, \rho^k)}{\alpha_1} = \sum_n \bar{y}_n .$$

It follows that at least one of the coordinates of y is a decreasing function of λ. Let λ° be the smallest value of λ for which any one of them vanishes, and let y° be the corresponding point; y° satisfies (12), because

$$y^\circ \geqq 0$$

$$M y^\circ = M[y^* + \lambda^\circ (y^* - \bar{y})] = (1 + \lambda^\circ) M y^* - \lambda^\circ M\bar{y} = (1 + \lambda^\circ) M y^* - \lambda^\circ \mu \gamma$$

$$\leqq (1 + \lambda^\circ)\mu\gamma - \lambda^\circ \mu \gamma = \mu\gamma.$$

But since y° has only N-1 non-vanishing coordinates, which in their (N-1)-dimensional space satisfy a set of inequalities similar to (12), it follows from the initial hypothesis that it is attainable (in the (N-1)-commodity space); therefore it is also attainable in the N-commodity space.

Finally, $y^* = \dfrac{y^\circ + \lambda^\circ \bar{y}}{1 + \lambda^\circ}$, which is on the segment joining the two attainable points y° and $\bar{y}$, is attainable.

14. <u>To sum up:</u>

The attainable points in the y-space are the nonegative points satisfying the N scalar inequalities expressed by the vector inequality

$$M\, y \leqq \mu\, \gamma,$$

where M is a square NxN matrix, and μ a KxN matrix, both of which can be written directly from the data:

$$M_{ij} = \min\ (\alpha_i, \alpha_j) \text{ and } \mu_{ij} = \min\ (\alpha_j, \rho^i).$$

The procedure given in section 13 enables us to find at least one set of x corresponding to any given attainable y.

The attainable point set is a polyhedron, restricted to the positive region, and bounded by the N hyperplanes

(13) $$M\, y = \mu\, \gamma.$$

All of these hyperplanes pass through a common vertex $\bar{y}$, the only solution of $M\, \bar{y} = \mu\, \gamma$ (the "extreme point"). Fig. 2 gives an illustration for the case of 3 goods.

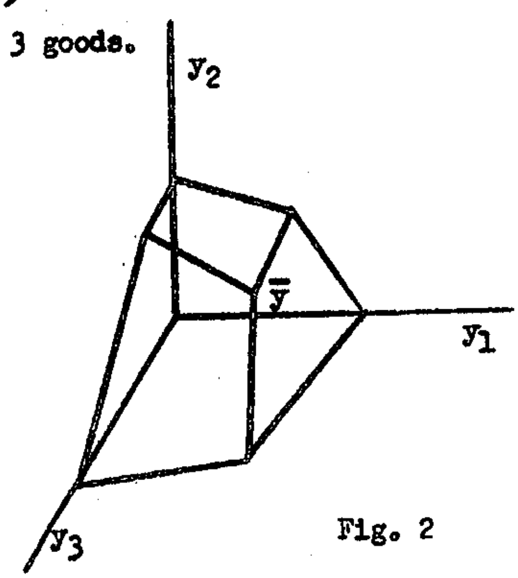


Fig. 2

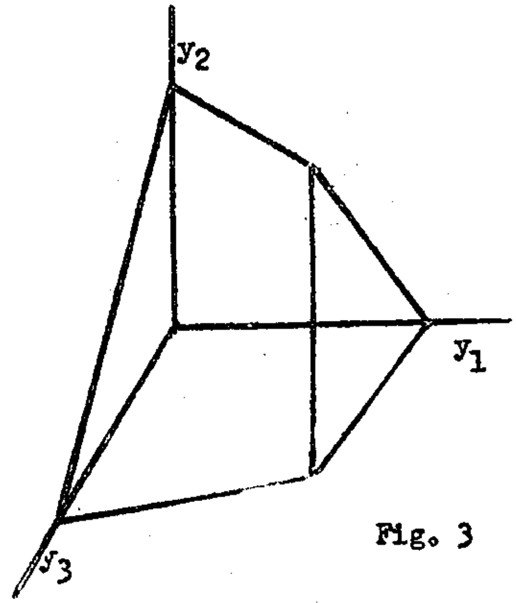


Fig. 3

15. <u>Comparison with an integrated economy.</u>

If resources were transferable the total production would be limited by $A\,y \leqq \sum c^k$, $y \geqq 0$, where A is the given 2xN technology matrix; that is, the attainable point set would be a polyhedron bounded by the two hyperplanes (fig. 3):

$$A\,y = \sum c^k \tag{14}$$

Fig. 4 provides a schematic comparison of the two cases (with 3 goods). Two of the hyperplanes (13) are parallel to the two hyperplanes (14), as it can be seen from the coefficients of the first and the last rows of M: on fig. 4, planes P and Q are parallel to P' and Q' respectively.

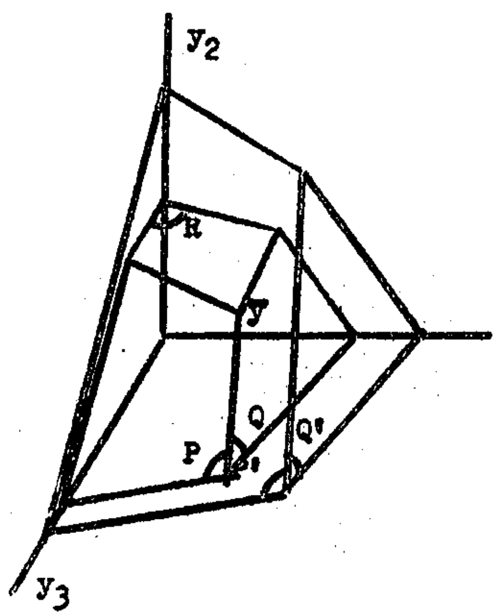


Fig. 4

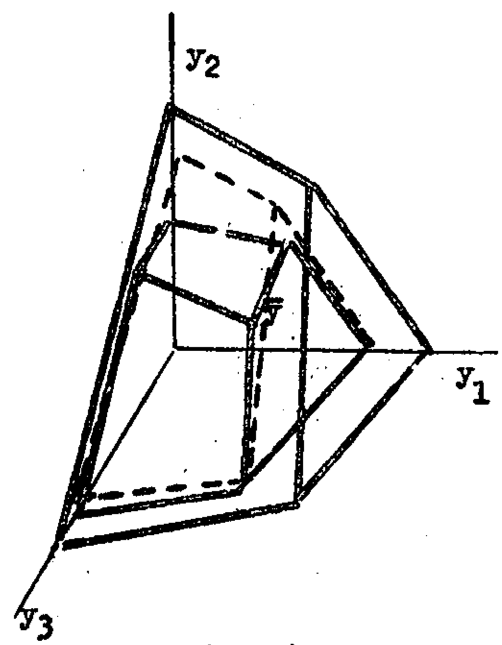


Fig. 5

Fig. 4 helps one visualize how non-transferability of resources restricts production possibilities. The two sets differ in two ways, corresponding to two different kinds of loss, which we shall term (1) waste of resources, and (2) loss of opportunities.

(1) Waste of resources.

The waste of resources appears in the fact that the above mentioned hyperplanes, though parallel, need not coincide; the distances between the hyperplanes of each of the two couples measure the waste in each of the two resources. More precisely: the waste amounts to the quantity of resources which, if free transfers were allowed, could be given up without precluding any previously possible total output. To evaluate it, we consider that smallest polyhedron of the "free-transfer" type (type of fig. 3) which contains (AtY) entirely; this polyhedron is shown in dotted lines in fig. 5; its equations are

$$A\, y \leq \begin{vmatrix} r^1 \\ r^2 \end{vmatrix}, \quad y \geq 0$$

, where $\begin{vmatrix} r^1 \\ r^2 \end{vmatrix}$ expresses the total amounts of resources to which it corresponds. To compute these amounts, we can write

$$r^1 = \sum_n \bar{y}_n = \frac{1}{\alpha_1} \sum_k \gamma^k \min\,(\alpha_1, \rho^k)$$

$$r^2 = \sum_n \alpha_n y_n = \sum_k \gamma^k \min\,(\alpha_N, \rho^k)$$

since we know that the two bounding planes, $\sum_n y_n = r^1$ and $\sum_n \alpha_n y_n = r^2$, pass through $\bar{y}$.

The waste is then:

(1st resource) $\sum \delta^k - r^1 = \frac{1}{\alpha_1} \sum \delta^k\, [\alpha_1 - \min\,(\alpha_1, \rho^k)] = \frac{1}{\alpha_1} \sum \delta^k \max\,(\alpha_1 - \rho^k, 0)$

(2nd resource) $\sum \delta^k \rho^k - r^2 = \sum \delta^k\, [\rho^k - \min\,(\alpha_N, \rho^k)] = \sum \delta^k \max\,(\rho^k - \alpha_N, 0)$.

(2) Loss of opportunities.

In addition to the waste of resources just described, there is a loss of opportunities, represented, on fig. 5 for instance, by the space lying between the dotted polyhedron and the initial attainable point set PQR. This loss originates from the N-2 intermediate among the N restrictions $M\,y \leq \mu\,\delta$. These N-2 intermediate inequalities which put further limitation on the aggregate output are irrelevant in an integrated economy, which is subject only to the two extreme inequalities ($A\,y \leq \sum c^k$).

Of course a model with 2 goods (e.g., the shipping problem discussed in section 2) cannot reveal such a loss; in fact, the larger the number of goods, in excess of 2, the greater is the loss of opportunities.

16. Concluding remarks.

As was pointed out at the outset, the model discussed here, though fitting the transportation problem quite realistically, falls short of any such realism in the context of international economics. And in addition to all the weaknesses of the linear approach, the solution developed here has the shortcoming of not taking into account more than two resources.

It is hoped however that it may help to show how economic segregation entails various losses, and cast light on some features which would reappear in similar, but more elaborate studies which are not altogether foreign to reality.