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The Problem of the Musical Chairs and an Equivalent 2-Person Game[1/]

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1. Statement of the Problem.

The problem of the musical chairs is as follows. Given are n plants and n locations, a set of real numbers $a_{\chi m}$ representing the flows from the χth plant to the m^{th} plant, and a set of non-negative real numbers k_{ij} representing the cost of transportation from location i to location j. What is the assignment of plants to locations which minimizes the total cost of transportation as among plants?

This model has serious limitations. First, the total flows originating at plant χ are exhausted by the $a_{\chi m}$ only when the notion of plant is also taken to encompass retail outlets and, ultimately, households. Secondly, substitution among flows from plants of the same industry is disregarded. However, the problems involved in indivisibilities are serious enough to make a rather simple model desirable at the outset.

This problem is related to the optimum assignment problem [l.c., footnote] which may be formulated in terms of plants and locations as follows.

1/ This paper was stimulated by an unpublished paper of J. von Neumann, "The Problem of Optimal Assignment and a Certain 2-Person Game," (October 26, 1951), of which extensive use is made in the proofs to follow. Errors are, of course, my own.

Given n plants and n locations and a set of numbers c_{ij} representing the cost of production for plant i at location j, what is the assignment of plants to locations which minimizes total production cost? One will note that in the former problem the cost to each plant of transportation is a function of the locations of all other plants.

von Neumann has shown [ℓ.c.] that the optimum assignment problem is equivalent to the following game. Let there be n x n double indexed cells, say, fields in a matrix. Player I hides in one cell. Player II attempts to "find" I by guessing either of the indices of the cell in which player I has hidden. The payoff to player II is c_{ij} if I is found in cell i, j and zero otherwise.

The purpose of this paper is to show the equivalence of the musical chairs' problem with the following game. Two n x n chess boards are given. On each of these player I chooses a row, player II a column. The choices are compared and player I pays to player II the amount $k_{ij} \cdot a_{\ell m}$ where i, ℓ are the rows chosen, j, m the columns chosen. This game may also be formulated abstractly in terms of strategies. For player I strategy g is defined as follows. Let $g = i.n + \ell$ where i, ℓ are positive integers not exceeding n. Then strategy g is to choose row i on the first and row ℓ on the second chess board. Similarly for player II let strategy $h = j.n + m$ be to choose columns j and m respectively. The game equivalent to the musical chairs' problem is the two person zero-sum game whose payoff matrix is given by

$$b_{gh} = k_{ij} \cdot a_{\ell m}$$

where $g = i.n + \ell$ $\qquad$ $h = j.n + m.$

It will have been apparent that both the optimal assignment problem and the musical chairs' problem are of the nature of permutation problems. The gain from the equivalent game formulations lies essentially in the reduction of this difficult type of a problem to a continuous one of manageable properties.

2. Mathematical discussion.

Let $\mathring{A} = (\mathring{a}_{\ell m})$ $\qquad \mathring{a}_{\ell\ell} = 0$

$\qquad a_{\ell m} \geqq 0$

$K = (k_{ij})$, $\qquad k_{ii} = 0$

$\qquad k_{ij} > 0 \qquad i \neq j$

be given n x n matrices, P an n x n permutation matrix[2/]

Problem 1

Find $\bar{P}$ such that $\text{trace}\,(\mathring{A}\,\bar{P}'\,K\,\bar{P}) = \min\limits_{P} \text{trace}\,(\mathring{A}\,P'\,K\,P)$

Let $A = \mathring{A} + D$

where D a diagonal matrix. Since the diagonal of $P'\,K\,P$ contains only zero elements $\text{trace}\,(\mathring{A}\,P'\,K\,P) = \text{trace}\,(A\,P'\,K\,P)$. The subject of this discussion paper is

Lemma 1

For every given matrix $\mathring{A}$ there exists a matrix $\mathring{A} + D = A = (a_{\ell m})$ (with D a diagonal matrix) such that problem 1 is equivalent to

Problem 2

Find numbers $\bar{x}_{i\ell}$, $\bar{y}_{jm}$ $\quad i, j, \ell, m = 1, \ldots n$ such that

$$\sum_{i,j,\ell,m=1}^{n} k_{ij}\, a_{\ell m}\, \bar{x}_{i\ell}\, \bar{y}_{jm} = \min_{x_{i\ell}} \max_{y_{jm}} \sum_{i,j,\ell,m=1}^{n} k_{ij}\, a_{\ell m}\, x_{i\ell}\, y_{jm}$$

subject to the conditions

(1) $\qquad x_{i\ell} \geqq 0$

$\qquad y_{jm} \geqq 0$

2/ Since there is no danger of confusion, the letters P, Q shall be used in in three different ways
as a index of a permutation (u_P)
as the n x n matrix of a permutation $(A'\,P'\,K\,P)$
as the operator permuting an index into (i^P) another index.

$$(2)\qquad \sum_{\ell} x_{i\ell} = 1 \qquad \sum_{m} y_{jm} = 1$$

$$(3)\qquad \sum_{i} x_{i\ell} = 1 \qquad \sum_{j} y_{jm} = 1$$

<u>Proof:</u>

Following von Neumann [ℓ.c. p. 4] we consider the two vector spaces R = set of all vectors $z = (z_{ij})$ in n^2 dimensions such that

$$(4)\qquad z_{ij} \geqq 0$$

$$(5)\qquad \sum_{j} z_{ij} = 1 \qquad \sum_{i} z_{ij} = 1$$

S = set of all vectors $z = (z_{ij})$ in n^2 dimensions such that $z_{ij} = \delta_{ij}^{Q}$ for some permutation Q of the integers $1, \ldots n$.

<u>Lemma 2</u>

[G. Birkhoff, 1946] R = convex hull of S. [For a proof cf. von Neumann: ℓ.c. p. 5 – 8].

Therefore every vector $x = (x_{gh})$ in R can be represented as a linear combination of vectors $z_Q = (z_{gh}) = (\delta_{gh}^{Q})$ of S with coefficients u_Q

(6) $$0 \leqq u_Q \leqq 1$$

$$x = \sum_Q u_Q \, z_Q \, .$$ Thus

(7) $$x_{gh} = \sum_Q u_Q \, \delta_{gh}^{Q}$$

and

(8) $$1 = \sum_g x_{gh} = \sum_Q u_Q \sum_g \delta_{gh}^{Q} = \sum_Q u_Q$$

Similarly

(9) $$y_{gh} = \sum_Q v_Q \, \delta_{gh}^{Q} \quad \text{with} \quad 0 \leqq v_Q \leqq 1$$

(10) $$\sum_Q v_Q = 1 \, .$$ Hence

(11) $$\sum_{i,j,\ell,m} k_{ij} \, a_{\ell m} \, x_{i\ell} \, y_{jm} = \sum_{\substack{i,j,\ell m \\ P,Q}} k_{ij} \, a_{\ell m} \, u_P \, \delta_{i\ell}^{P} \, v_Q \, \delta_{jm}^{Q}$$

$$= \sum_{\chi,m=1}^{n} k_\chi{}^P{}_m{}^Q a_{\chi m} u_P v_Q$$

with (8) $\sum_P u_P = 1$ (10) $\sum_Q v_Q = 1$

We show next that for fixed $x = (x_{i\chi})$

(12) $$\max_{y_{jm}} \sum_{i,j,\chi,m=1}^{n} k_{ij} a_{\chi m} x_{i\chi} y_{jm} =$$

subject to (1). (2). (3)

$$\max_{v_Q} \sum_{\substack{i,\chi,m=1\\Q}}^{n} k_{im}{}^Q a_{\chi m} x_{i\chi} v_Q$$

subject to (6), (8).

For brevity write for the left side M. Let M be taken on for $y_{jm} = y^o_{jm}$.

According to (7) y^o_{jm} may be represented in the form

$y^o_{jm} = \Sigma v^o_Q \; \delta_{jm}{}^Q$ with some v^o_Q satisfying (6) and (8). Hence

$$M = \sum_{\substack{i,j,\chi,m=1\\Q}}^{n} a_{\chi m} k_{ij} x_{i\chi} v^o_Q \, \delta_{jm}{}^Q = \sum_{i,\chi,m=1}^{n} a_{\chi m} k_{im}{}^Q x_{i\chi} v^o_Q$$

The right side is clearly

(13) $$\leq \max_{v_Q} \sum_{\substack{i,\chi,m=1\\Q}}^{n} a_{\chi m} k_{im}{}^Q x_{i\chi} v_Q \qquad \text{for } v_Q$$

subject to (6) (8). Let this maximum be taken on for $v_Q = v^*_Q$. Assume that the "<" sign holds in (13). Now $\sum_Q v_Q \; \delta_{jm}{}^Q$ defines a vector$(y_{jm})$by (7) which would yield a value of $\Sigma a_{\chi m} k_{ij} x_{i\chi} y_{jm} > M$. This is a contradiction, and therefore the equality sign holds in (13) and so in (12).

In the same way it is seen that

(14) $$\min_{x_{i\chi}} \max_{y_{jm}} \sum_{i,j,\chi,m=1}^{n} k_{ij} a_{\chi m} x_{i\chi} y_{jm}$$

subject to (1),(2),(3)

$$= \min_{u_P} \max_{u_Q} \sum_{\substack{\chi, m=1 \\ P,Q}}^{n} k_\chi \, {}^{P}{}_{m}{}^{Q} \, a_{\chi m} \, u_P \, v_Q$$

subject to (6), (8).

For the sum $\sum_{\chi, m=1}^{n} k_\chi \, {}^{P}{}_{m} \, {}^{Q} \, a_{\chi m}$

one may also write trace (A' P' K Q), so that the right side of (14) becomes

$$\min_{u_P} \max_{v_Q} \sum_{P,Q} \text{trace}\,(A'\ P'\ K\ Q)\ u_P\ v_Q$$

subject to (6), (8).

We proceed to prove that for a suitable D this expression is equal to $\min_P$ trace (A' P' K P). The first step is to show that for fixed u_P

(15) $$\max_{\sum_Q v_Q = 1} \sum_Q \text{trace}\,(A'\ P'\ K\ Q)\ v_Q = \text{trace}\,(A'\ P'\ K\ P)$$

For this it is sufficient that trace $(\mathring{A}'\ P'\ K\ Q) < 0$ if $P \neq Q$.

Presently we show that this can be achieved by choosing

$$A = \mathring{A} + cI$$

where I is the unit matrix and c a sufficiently small constant, say,

$$c < - \frac{\max_{P,Q} \ \text{trace}\,(\mathring{A}'\ P' K\ Q)}{\min_{i \neq j} \ k_{ij}}$$

Suppose that $P \neq Q$, hence that $k_\chi \, P_\chi \, Q \neq 0$ for some χ trace (A'P'KQ) may then be written

$$\sum_{\chi m} k_\chi \, {}^{P}{}_{m} \, {}^{Q} \, \mathring{a}_{\chi m} + \sum_{\chi} k_\chi \, {}^{P}{}_{\chi} \, {}^{Q} \cdot c$$

$$\leq \max_{P,Q} (A^{o'}\ P'\ K\ Q) + \sum_{\chi} k_\chi \, {}^{P}{}_{\chi} \, {}^{Q} \cdot c$$

$$< 0 \leq \text{trace}\,(\overset{o}{A}{}' \, P' \, K \, P)$$

$$= \text{trace}\,(A' \, P' \, K \, P)$$

This establishes that

$$\max_{\sum_Q v_Q = 1} \sum_Q \text{trace}\,(A' \, P' \, K \, Q) \, v_Q = \text{trace}\,(A' \, P' \, K \, P)$$

It follows now at once that

$$(16) \qquad \min_{u_P} \max_{v_Q} \sum_{P,Q} \text{trace}\,(A' \, P' \, K \, Q) \; u_P \, v_Q$$

subject to (6), (8)

$$= \min_P \text{trace}\,(A' \, P' \, K \, P) = \text{trace}\,(A' \, \bar{P}' \, K \, \bar{P}) \, .$$

(16) in conjunction with (14) establishes the assertion of lemma 1.

In equation (14) the solution is

$$(17) \qquad u_P = \begin{cases} 1 \text{ for } P = \bar{P} \\ 0 \text{ for } P \neq \bar{P} \end{cases}$$

$$(18) \qquad v_Q = \begin{cases} 1 \text{ for } Q = \bar{P} \\ 0 \text{ for } Q \neq \bar{P} \end{cases}$$

Returning to equations (7) and (9) of p. 5 we note that the solution of problem 2 becomes

$$(19) \qquad \bar{x}_{i\chi} = \sum_Q \bar{u}_Q \, \delta_{i\chi} \, Q = \delta_{i\chi} \, \bar{P}$$

$$(20) \qquad \bar{y}_{jm} = \sum_Q \bar{v}_Q \, \delta_{jm} \, Q = \delta_{jm} \, \bar{P} \, .$$

3. To find equivalent games is only a special case of the more general task of finding analogous continuous problems for the permutation problems considered. For instance in the same way as before it can be shown that

3.1 the optimal assignment problem (o. a. p.) is equivalent to

Problem 4

$$\text{Minimise}_{x_{ij}} \sum_{i,j=1}^{n} c_{ij} \, x_{ij}$$

subject to

(3.1) $\sum_{j} x_{ij} \geqq 1$

(3.2) $\sum_{i} x_{ij} \geqq 1$

(3.3) $x_{ij} \geqq 0$

3.2 the musical chairs problem (m.c.p.) is equivalent to

<u>Problem 5</u>

$$\underset{x_{ij}}{\text{minimize}} \sum_{i,j,\ell,m=1}^{n} k_{ij}\, a_{m\ell}\, x_{i\ell}\, x_{jm}$$

subject to

(3.1) $\sum_{j} x_{ij} \geqq 1$

(3.2) $\sum_{i} x_{ij} \geqq 1$

(3.3) $x_{ij} \geqq 0$

The usefulness of problem 4 lies in the fact that it permits the formulation of necessary and sufficient conditions for the solution of the o.a.p. in terms of efficiency prices.

<u>Definition</u> The location associated with a plant by the solution of the o.a.p. or the m.c.p. rexpectively is called the socially optimal location for that plant (with respect to that problem).

<u>Definition</u> A set of non-negative numbers r_i associated with the locations i is called a set of efficiency rents if for each plant the sum of transportation costs and rent takes its minimum when all plants are at their socially optimal locations. Let P be the permutation which assigns to each plant its socially optimal location. The definition of the efficiency rents r_i for the o.a.p. is

(3.4) $c_j Q_j + r_j Q \geqq c_j P_j + r_j P$ for all i, Q.

And for the m.c.p.:

(3.5) $$\sum_m k_{\chi Q} \, {}_m Q \cdot a_{\chi m} + r_{\chi Q} \geq \sum_m k_{\chi P} \, {}_m P \cdot a_{\chi m} + r_{\chi P}$$

for all χ, Q.

Suppose that a permutation P satisfies (3.4) Then clearly

$$\sum_i [c_j \, Q_j + r_j \, Q] \geq \sum_i [c_j \, P_j + r_j \, P] \text{ or}$$

$$\sum_i c_j \, Q_j \geq \sum_i c_j \, P_j.$$

Any solution P of (3.4) is therefore a solution of the o.a.p. Similarly it follows from

$$\sum_{\chi, m} (k_\chi \, Q_m \, Q \cdot a_{\chi m} + r_\chi \, Q) \geq \sum_{\chi, m} (k_\chi \, P_m \, P \cdot a_{\chi m} + r_\chi P)$$

at once that any solution P of (3.5) solves the m.c.p.

Hence the existence of efficiency prices such that (3.4) or (3.5) hold is a sufficient condition for P to be a solution of the o. a. p. or the m. c. p., respectively. That it is a necessary condition for the o.a.p. can be seen from the equivalent problem 4 as follows.

The minimand of problem 4 is linear and hence convex in x_{ij}. The constraints (3.1), (3.2), (3.3) are linear and hence concave in x_{ij}. By the theorem of Kuhn and Tucker [Second Berkeley Symposium p. 486] on the existence of Lagrangean parameters one has

(3.6) $$\min \sum_{i,j=1}^{n} c_{ij} \, x_{ij} = \max_{\lambda_i, \mu_j} \; \min_{x_{ij}} \left\{ \sum c_{ij} \, x_{ij} \right.$$

subject to (3.1), (3.4), (3.3)

$$= \max_{\lambda_i, \mu_j} \; \min_{x_{ij}} \left\{ \sum c_{ij} \, x_{ij} - \sum_i \lambda_i \left(\sum_j x_{ij} - 1 \right) - \sum_j \mu_j \left(\sum_i x_{ij} - 1 \right) \right\}$$

subject to

(3.7) $x_{ij} \geqq 0$

(3.8) $\lambda_i \geqq 0$

(3.9) $\mu_j \geqq 0$

Necessary for a saddle point $\bar{x}_{ij}$, $\bar{\lambda}_i$, $\bar{\mu}_j$ of the linear function on the right side of (3.6) is that the following conditions hold

(3.10) $$c_{ij} - \bar{\lambda}_i - \bar{\mu}_j \begin{Bmatrix} = \\ \geqq \end{Bmatrix} 0 \text{ if } \bar{x}_{ij} \begin{Bmatrix} > \\ = \end{Bmatrix} 0.$$

Making use of the fact that $\bar{x}_{ij}$ is of the form $\bar{x}_{ij} = \delta_{ij} P$

on obtains

$$c_{ij} - \bar{\lambda}_i - \bar{\mu}_j \begin{Bmatrix} = \\ \geqq \end{Bmatrix} 0 \text{ if } i \begin{Bmatrix} = \\ \neq \end{Bmatrix} j^P$$

or

$$c_j Q_j - \bar{\lambda}_{jQ} \geqq c_j P_j - \bar{\lambda}_j P$$

Defining $r_i = a - \lambda_i \geqq 0$ with sufficiently large a, one has as a necessary condition for a solution of the o.a.p. that

(3.4) $$c_j Q_j + r_j Q \geqq c_j P_j + r_j P$$

as asserted.

This proof does not carry through in the case of the m.c.p. In fact, the existence of efficiency prices is not a necessary condition for the solution of the m.c.p. This can be seen from the following example.

$$A' = \begin{pmatrix} 0 & 2.1 & 1.5 \\ 2 & 0 & 2 \\ 3 & 1 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Q	(123) & (213)	(132) & (231)	(321) & (312)
$\sum_m a_{1m} k_1 Q_m Q$	5.7	5.1	3.6
$\sum_m a_{2m} k_2 Q_m Q$	6	4	8
$\sum_m a_{3m} k_3 Q_m Q$	4	7	4
$\sum_m a_{\chi m} k_\chi Q_m Q$	15.7	16.1	15.6

The last row shows that the solution is given by the permutations of the third column. Now with $Q_1 = (123)$ $Q_3 = (321)$ and $\chi = 2$ one has $j^{Q_1} = j^{Q_2} = 2$ and $6 + r_2 < 8 + r_2$ in contradiction to (3.5).3/

3/ It is of course possible to introduce some notion of efficiency prices in terms of which the m.c.p. can be formulated. Let K be a constant greater than or equal to the maximum of $\sum_m a_{\chi m} k_{\chi P_m P}$, over all χ, where P is the permutation that solves the m.c.p. Define

$$K - \sum_m a_{\chi m} k_{\chi P_m P} = r_{\chi P} .$$

Then the total costs (transportation costs plus rent) of each firms is less than or equal to K only if every plant is at its socially optimal location. These prices do not achieve a complete decentralization of decision making. Rather they serve to induce locational exchange through certain coalitions, which however in some cases will have to encompass the total of n plants.