The Problem of the Musical Chairs and an Equivalent 2-Person Game.1/

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1. Statement of the Problem.

The problem of the musical chairs is as follows. Given are n plants and n locations, a set of real numbers $a_{im}$ representing the flows from the $i$th plant to the $m$th plant, and a set of non-negative real numbers $k_{ij}$ representing the cost of transportation from location $i$ to location $j$. What is the assignment of plants to locations which minimizes the total cost of transportation among plants?

This model has serious limitations. First, the total flows originating at plant $i$ are exhausted by the $a_{im}$ only when the notion of plant is also taken to encompass retail outlets and, ultimately, households. Secondly, substitution among flows from plants of the same industry is disregarded. However, the problems involved in indivisibilities are serious enough to make a rather simple model desirable at the outset.

This problem is related to the optimum assignment problem [1.e., footnote] which may be formulated in terms of plants and locations as follows.

1/ This paper was stimulated by an unpublished paper of J. von Neumann, "The Problem of Optimal Assignment and a Certain 2-Person Game," (October 26, 1951), of which extensive use is made in the proofs to follow. Errors are, of course, my own.
Given \( n \) plants and \( n \) locations and a set of numbers \( c_{ij} \) representing the cost of production for plant \( i \) at location \( j \), what is the assignment of plants to locations which minimizes total production cost? One will note that in the former problem the cost to each plant of transportation is a function of the locations of all other plants.

von Neumann has shown \([I.c.]\) that the optimum assignment problem is equivalent to the following game. Let there be \( n \times n \) double indexed cells, say, fields in a matrix. Player I hides in one cell. Player II attempts to "find" I by guessing either of the indices of the cell in which player I has hidden. The payoff to player II is \( c_{ij} \) if I is found in cell \( i, j \) and zero otherwise.

The purpose of this paper is to show the equivalence of the musical chairs' problem with the following game. Two \( n \times n \) chess boards are given. On each of these player I chooses a row, player II a column. The choices are compared and player I pays to player II the amount \( k_{ij} \cdot a_{jm} \) where \( i, j, k \) are the rows chosen, \( j, m \) the columns chosen. This game may also be formulated abstractly in terms of strategies. For player I strategy \( g \) is defined as follows. Let \( g = i.n + j \) where \( i, j \) are positive integers not exceeding \( n \). Then strategy \( g \) is to choose row \( i \) on the first and row \( j \) on the second chess board. Similarly for player II let strategy \( h = j.n + m \) be to choose columns \( j \) and \( m \) respectively. The game equivalent to the musical chairs' problem is the two person zero-sum game whose payoff matrix is given by

\[
b_{gh} = k_{ij} \cdot a_{jm}
\]

where \( g = i.n + j \), \( h = j.n + m \).

It will have been apparent that both the optimal assignment problem and the musical chairs' problem are of the nature of permutation problems. The gain from the equivalent game formulations lies essentially in the re-
duction of this difficult type of a problem to a continuous one of manageable properties.

2. Mathematical discussion.

Let \( \mathbf{\hat{A}} = (\hat{a}_{m}) \)  
\( \hat{a}_{m} = 0 \)  
\( a_{m} \neq 0 \)

\( K = (k_{ij}) \)

\( k_{ii} = 0 \)

\( k_{ij} > 0 \quad i \neq j \)

be given \( n \times n \) matrices, \( P \) an \( n \times n \) permutation matrix.  

**Problem 1**

Find \( P \) such that \( \text{trace} (\mathbf{\hat{A}} P K P) = \min \text{trace} (\mathbf{\hat{A}} P P K P) \)

Let \( \mathbf{A} = \mathbf{\hat{A}} + \mathbf{D} \)

where \( \mathbf{D} \) a diagonal matrix. Since the diagonal of \( P \mathbf{P} K P \) contains only zero elements

\( \text{trace} (\mathbf{\hat{A}} P P K P) = \text{trace} (\mathbf{\hat{A}} P P K P) \). The subject of this discussion paper is

**Lemma 1**

For every given matrix \( \mathbf{\hat{A}} \) there exists a matrix \( \mathbf{\hat{A}} + \mathbf{D} = \mathbf{A} = (a_{m}) \)

(with \( \mathbf{D} \) a diagonal matrix) such that problem 1 is equivalent to

**Problem 2**

Find numbers \( \bar{x}_{i} \), \( \bar{y}_{j m} \) \( i, j, \lambda, m = 1, \ldots, n \) such that

\[
\sum_{i,j, \lambda, m=1}^{n} k_{ij} \cdot a_{m} \cdot \bar{x}_{i} \cdot \bar{y}_{j m} = \min \max \sum_{i,j, \lambda, m=1}^{n} k_{ij} \cdot a_{m} \cdot \bar{x}_{i} \cdot \bar{y}_{j m}
\]

subject to the conditions

\[
\bar{x}_{i} \geq 0 \quad (1)
\]

\[
\bar{y}_{j m} \geq 0
\]

\( ^{2/} \) Since there is no danger of confusion, the letters \( P, \lambda \) shall be used in three different ways

as a index of a permutation \( (\lambda_{P}) \)
as the \( n \times n \) matrix of a permutation \( (A P P K P) \)
as the operator permuting an index into \( (i^{P}) \) another index.
\[ \sum_{i} x_{ij} = 1 \]
\[ \sum_{m} y_{jm} = 1 \]
\[ \sum_{i} x_{ij} = 1 \]
\[ \sum_{j} y_{jm} = 1 \]

Proof:

Following von Neumann [2, p. 4] we consider the two vector spaces

\[ R = \text{set of all vectors } s = (s_{ij}) \text{ in } n^2 \text{ dimensions such that} \]

\[ s_{ij} \geq 0 \]
\[ \sum_{j} s_{ij} = 1 \]
\[ \sum_{i} s_{ij} = 1 \]
S = set of all vectors \( z = (z_{ij}) \) in \( n^2 \) dimensions such that \( z_{ij} = \delta_{ij}^Q \) for some permutation \( Q \) of the integers 1, ..., \( n \).

Lemma 2

[O. Birkhoff, 1946] \( R \) = convex hull of \( S \). [For a proof cf. von Neumann: loc. cit. p. 5-8].

Therefore every vector \( x = (x_{gh}) \) in \( R \) can be represented as a linear combination of vectors \( z_Q^* = (z_{gh}) = (\delta_{gh}^Q) \) of \( S \) with coefficients \( u_Q \).

\[
\begin{align*}
(6) \quad 0 &\leq u_Q \leq 1 \\
(7) \quad x_{gh} = \sum_Q u_Q \delta_{gh}^Q \\
\text{and} \\
(8) \quad 1 = \sum_g x_{gh} = \sum_Q u_Q \delta_{gh}^Q \\
\end{align*}
\]

Similarly

\[
\begin{align*}
(9) \quad y_{gh} = \sum_Q v_Q \delta_{gh}^Q \quad \text{with} \quad 0 \leq v_Q \leq 1 \\
(10) \quad \sum_Q v_Q = 1 \\
\text{Hence} \\
(11) \quad \sum_{i,j,m}^\Sigma k_{ij} a_{\lambda m} x_{ij} y_{jm} = \sum_{i,j,m}^\Sigma k_{ij} a_{\lambda m} u_p \delta_{iP}^Q v_Q \delta_{jm}^Q
\end{align*}
\]
\[
= \sum_{i,j,m=1}^{n} k_{ij}^m a_{ij}^m u_p v_q
\]

with (8) \(\sum_{p} u_p = 1\) (10) \(\sum_{Q} v_q = 1\)

We show next that for fixed \(x = (x_{ij})\)

(12) \[
\max_{y_{jm}} \sum_{i,j,m=1}^{n} k_{ij}^m a_{ij}^m x_{ij} y_{jm}
\]

subject to (1), (2), (3)

\[
\max_{v_q} \sum_{i,j,m=1}^{n} k_{im}^Q a_{ij}^m x_{ij} v_q
\]

subject to (6), (8).

For brevity write for the left side \(M\). Let \(M\) be taken on for \(y_{jm} = y_{jm}^o\).

According to (7) \(y_{jm}^o\) may be represented in the form

\[
y_{jm}^o = \sum_{Q} \nu_{jm}^o \quad \text{with some } \nu_{jm}^o \text{ satisfying (6) and (8). Hence}
\]

\[
M = \sum_{i,j,m=1}^{n} a_{ij}^m k_{ij} x_{ij} \nu_{jm}^o \quad \text{with } \nu_{jm}^o = \sum_{i,j,m=1}^{n} a_{ij}^m k_{im}^Q x_{ij} \nu_{jm}^o
\]

The right side is clearly

(13) \[
\leq \max_{v_q} \sum_{i,j,m=1}^{n} a_{ij}^m k_{im}^Q x_{ij} v_q
\]

subject to (6) (8). Let this maximum be taken on for \(v_q = v_q^*\). Assume that the \(\leq\) sign holds in (13). Now \(\sum_{Q} \nu_{jm}^o \delta_{jm}^Q\) defines a vector \(y_{jm}\) by (7) which would yield a value of \(\sum_{Q} \nu_{jm}^o \delta_{jm}^Q > M\). This is a contradiction, and therefore the equality sign holds in (13) and so in (12).

In the same way it is seen that

(14) \[
\min_{x_{ij}^m y_{jm}} \max_{i,j,m=1}^{n} k_{ij}^m a_{ij}^m x_{ij}^m y_{jm}
\]

subject to (1), (2), (3)
\[ -\min \max \sum_{\lambda, m=1}^{n} k_{\lambda} \beta_{m} a_{\lambda m} u_{\lambda} v_{\lambda} \]

subject to (6), (8).

For the sum \[ \sum_{\lambda, m=1}^{n} k_{\lambda} \beta_{m} a_{\lambda m} \]

one may also write trace \( \langle A' P' K Q \rangle \), so that the right side of (14) becomes

\[ \min \max \sum_{\lambda} \text{trace} \langle A' P' K Q \rangle u_{\lambda} v_{\lambda} \]

subject to (6), (8).

We proceed to prove that for a suitable \( D \) this expression is equal to \( \min \text{trace} \langle A' P' K P \rangle \). The first step is to show that for fixed \( u_{\lambda} \)

\[ \max \sum_{Q} \text{trace} \langle A' P' K Q \rangle v_{Q} = \text{trace} \langle A' P' K P \rangle \]

For this it is sufficient that trace \( \langle A' P' K Q \rangle < 0 \) if \( P \neq Q \).

Presently we show that this can be achieved by choosing

\[ A = \hat{A} + \alpha I \]

where \( I \) is the unit matrix and \( \alpha \) a sufficiently small constant, say,

\[ \alpha < -\frac{\max \text{trace} \langle \hat{A}' P' K Q \rangle}{\min \sum_{i \neq j} k_{ij}} \]

Suppose that \( P \neq Q \), hence that \( k_{\lambda} \beta_{\lambda} Q \neq 0 \) for some \( \lambda \). trace \( \langle A' P' K Q \rangle \) may then be written

\[ \sum_{\lambda} k_{\lambda} \beta_{\lambda} P_{m} Q_{a_{\lambda m}^0} + \sum_{\lambda} k_{\lambda} \beta_{\lambda} Q_{a_{\lambda m}^0} \leq \max_{P, Q} (A^0 P^0 K Q) + \sum_{\lambda} k_{\lambda} \beta_{\lambda} Q_{a_{\lambda m}^0} \]
\[ < 0 \leq \text{trace} \left( A^o \ F^o \ K \ P \right) = \text{trace} \left( A^i \ F^i \ K \ P \right) \]

This establishes that
\[
\max_{\Sigma v_Q = 1} \Sigma \text{trace} \left( A^o \ P^o \ K \ Q \right) v_Q = \text{trace} \left( A^i \ F^i \ K \ P \right)
\]

It follows now at once that

\[
\min_{u_P} \max_{v_Q} \Sigma p_Q \text{trace} \left( A^o \ F^o \ K \ Q \right) u_P v_Q
\]

subject to (6), (8)

\[
= \min_{u_P} \text{trace} \left( A^i \ F^i \ K \ P \right) = \text{trace} \left( A^i \ F^i \ K \ P \right).
\]

(16) in conjunction with (14) establishes the assertion of lemma 1.

In equation (14) the solution is

\[
u_P = \begin{cases} 
1 & \text{for } P = P \\
0 & \text{for } P \neq P
\end{cases}
\]

(17)

\[
v_Q = \begin{cases} 
1 & \text{for } Q = P \\
0 & \text{for } Q \neq P
\end{cases}
\]

(18)

Returning to equations (7) and (9) of p. 5 we note that the solution of problem 2 becomes

\[
\bar{F}_{\bar{X}} = \Sigma Q \bar{v}_Q \delta_{\bar{X} \bar{Q}} = \delta_{\bar{X} \bar{P}}
\]

(19)

\[
\bar{F}_{\bar{X}} = \Sigma Q \bar{v}_Q \delta_{\bar{X} \bar{Q}} = \delta_{\bar{X} \bar{P}}
\]

(20)

3. To find equivalent games is only a special case of the more general task of finding analogous continuous problems for the permutation problems considered. For instance in the same way as before it can be shown that

3.1 the optimal assignment problem (o.a.p.) is equivalent to

Problem 4

Minimise \[ \sum_{i,j=1}^{n} c_{ij} x_{ij} \]
subject to

\[(3.1) \quad \sum_{j} x_{ij} \geq 1\]

\[(3.2) \quad \sum_{i} x_{ij} \geq 1\]

\[(3.3) \quad x_{ij} \geq 0\]

3.2 the musical chairs problem (m.o.p.) is equivalent to

Problem

\[\begin{align*}
\text{minimize} & \quad \sum_{i,j,m=1}^{n} k_{ij} s_{mj} x_{ij} x_{jm} \\
\text{subject to} & \quad (3.1) \quad \sum_{j} x_{ij} \geq 1 \\
& \quad (3.2) \quad \sum_{i} x_{ij} \geq 1 \\
& \quad (3.3) \quad x_{ij} \geq 0
\end{align*}\]

The usefulness of problem 4 lies in the fact that it permits the formulation of necessary and sufficient conditions for the solution of the o.a.p. in terms of efficiency prices.

Definition The location associated with a plant by the solution of the o.a.p. or the m.o.p. respectively is called the socially optimal location for that plant (with respect to that problem).

Definition A set of non-negative numbers \(r_i\) associated with the locations \(i\) is called a set of efficiency rents if for each plant the sum of transportation costs and rent takes its minimum when all plants are at their socially optimal locations. Let \(P\) be the permutation which assigns to each plant its socially optimal location. The definition of the efficiency rents \(r_i\) for the o.a.p. is

\[(3.4) \quad c_{ij} q_j + r_j Q \geq c_{ij} P_j + r_j P \quad \text{for all} \quad i, Q.\]

And for the m.o.p.:
(3.5) \[ \sum_{m} k_{m} x_{m} q_{m} + r_{q} x_{q} q_{q} \geq \sum_{p} k_{p} p_{p} x_{p} + r_{p} x_{p} \]

for all \( q_{m}, q_{q} \).

Suppose that a permutation \( P \) satisfies (3.4). Then clearly

\[ \sum_{i} c_{i} q_{j} + r_{j} q_{j} \geq \sum_{j} c_{j} p_{j} + r_{j} p_{j} \]

or

\[ \sum_{i} c_{i} q_{j} \geq \sum_{j} c_{j} p_{j}. \]

Any solution \( P \) of (3.4) is therefore a solution of the c.a.p. Similarly it follows from

\[ \sum_{m} (k_{m} x_{m} q_{m} + r_{q} x_{q} q_{q}) \geq \sum_{m} (k_{m} p_{m} + r_{p} x_{p}) \]

at once that any solution \( P \) of (3.5) solves the m.c.p.

Hence the existence of efficiency prices such that (3.4) or (3.5) holds is a sufficient condition for \( P \) to be a solution of the c.a.p. or the m.c.p., respectively. That it is a necessary condition for the c.a.p. can be seen from the equivalent problem 4 as follows.

The minimand of problem 4 is linear and hence convex in \( x_{ij} \).

The constraints (3.1), (3.2), (3.3) are linear and hence concave in \( x_{ij} \). By the theorem of Kuhn and Tucker [Second Berkeley Symposium p. 486] on the existence of Lagrangean parameters one has

(3.6) \[ \min_{i,j} \sum_{i,j} c_{ij} x_{ij} = \max_{\lambda_{i}, \mu_{j}} \min_{x_{ij}} \left\{ \sum_{i,j} c_{ij} x_{ij} \right\} \]

subject to (3.1), (3.4), (3.3)

\[ = \max_{\lambda_{i}, \mu_{j}} \min_{x_{ij}} \left\{ \sum_{i,j} c_{ij} x_{ij} - \sum_{i} \lambda_{i} (\sum_{j} x_{ij} - 1) - \sum_{j} \mu_{j} (\sum_{i} x_{ij} - 1) \right\} \]

subject to
(3.7) \( x_{ij} \geq 0 \)

(3.8) \( \lambda_i \geq 0 \)

(3.9) \( \mu_j \geq 0 \)

Necessary for a saddle point \( \overline{x}_{ij}, \overline{\lambda}_i, \overline{\mu}_j \) of the linear function on the right side of (3.6) is that the following conditions hold

(3.10) \( c_{ij} - \overline{\lambda}_i - \overline{\mu}_j \left\{ \begin{array}{c} > 0 \text{ if } \overline{x}_{ij} \left\{ \begin{array}{c} = 0 \text{ if } \overline{x}_{ij} > 0. \end{array} \right. \end{array} \right. \)

Making use of the fact that \( \overline{x}_{ij} \) is of the form \( \overline{x}_{ij} = \delta_{ij} \rho \)

on obtains

\[ c_{ij} - \overline{\lambda}_i - \overline{\mu}_j \left\{ \begin{array}{c} > 0 \text{ if } \overline{x}_{ij} > 0. \end{array} \right. \]

or

\[ c_{ij} - \overline{\lambda}_i - \overline{\mu}_j \geq c_{ij} + \overline{\lambda}_i \]

Defining \( r_i = a - \lambda_i \geq 0 \) with sufficiently large \( a \), one has as a necessary condition for a solution of the o.s.p. that

(3.14) \( c_{ij} + r_i \geq c_{ij} + r_i \)

as asserted.

This proof does not carry through in the case of the m.c.p. In fact, the existence of efficiency prices is not a necessary condition for the solution of the m.c.p. This can be seen from the following example.

\[
A' = \begin{pmatrix} 0 & 2 & 1 & 1.5 \\ 2 & 0 & 2 \\ 3 & 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}
\]
\[\begin{array}{|c|c|c|c|}
\hline
& (123) \& (213) & (132) \& (231) & (321) \& (312) \\
\hline
\sum_{m} a_{1m} k_{1} q_{m} q & 5.7 & 5.1 & 3.6 \\
\hline
\sum_{m} a_{2m} k_{2} q_{m} q & 6 & 4 & 8 \\
\hline
\sum_{m} a_{3m} k_{3} q_{m} q & 4 & 7 & 4 \\
\hline
\sum_{m} a_{\gamma m} k_{\gamma} q_{m} q & 15.7 & 16.1 & 15.6 \\
\hline
\end{array}\]

The last row shows that the solution is given by the permutations of the third column. Now with \(Q_{1} = (123)\), \(Q_{2} = (321)\) and \(\gamma = 2\) one has
\[q_{1} = q_{2} = 2\] and \(6 + r_{2} < 8 + r_{2}\) in contradiction to \((3.5)\).

It is of course possible to introduce some notion of efficiency prices in terms of which the m.c.p. can be formulated. Let \(K\) be a constant greater than or equal to the maximum of \(\sum_{m} a_{\gamma} k_{\gamma} f_{\gamma}^{P}\), over all \(\gamma\), where \(P\) is the permutation that solves the m.c.p. Define
\[K = \sum_{m} a_{\gamma} k_{\gamma} f_{\gamma}^{P} r_{\gamma}^{P}\]

Then the total costs (transportation costs plus rent) of each firm is less than or equal to \(K\) only if every plant is at its socially optimal location. These prices do not achieve a complete decentralization of decision making. Rather they serve to induce locational exchange through certain coalitions, which however in some cases will have to encompass the total of \(n\) plants.