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The Aggregate Allocation of Land and Labor

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Earlier discussions of the aggregation problem ([1] and papers there quoted) were not yet affected by the emphasis on non-negativity which is characteristic of the modern approach to production and allocation [2]. As an example of the effect of this complication we shall discuss the average allocation of a single factor of production between a number of purposes under highly simplified assumptions. One possible interpretation of this model, if we ignore several additional features of reality, is the allocation of land between various crops under certainty or single-valued expectations as to prices and yields. (cf. C.C.D.P. Economics 2036, p. 4). Another interpretation, with similar qualifications, is the choice of occupations by individuals, recently discussed by Tinbergen [4] and A. D. Roy [3]. Tinbergen's use of calculus methods, however, implies a serious neglect of the incontestable fact that most people have only one job at a time. Roy's important contribution is not subject to this defect, but some readers may shirk the task of translating his otherwise admirable presentation back into mathematics.

For convenience of exposition we shall first express our analysis in terms of land. Consider a region A with a given total area X, each point of which is specified by a parameter α (e.g. its geographical coordinates). Each point can produce a yield of $y_i(\alpha)$ units of the i-th crop ($i = 1, 2, \ldots, n$), the price per unit of that crop is p_i. Transportation costs are not introduced explicitly. The yield vector $y(\alpha)$ is supposed to be continuously variable on A with a continuous frequency distribution $F(y_1, \ldots, y_n)$ and a continuous density function $f(y_1, \ldots, y_n)$.

The micro-economic problem is to determine what proportion $x_i(\alpha)$ of each point of land shall be devoted to the i-th crop so as to maximize the revenue

$$(1) \qquad r(\alpha) = \sum_{i=1}^{n} p_i \, x_i(\alpha) \, y_i(\alpha)$$

subject to

$$(2) \qquad x_i(\alpha) \geq 0; \quad \sum_{i=1}^{n} x_i(\alpha) = 1.$$

Clearly $x_i(\alpha)$ can only be 0, 1 or indeterminate. The latter case will occur if two or more of the $p_i \, y_i(\alpha)$ are equal, but we shall assume that the points where this holds form a set of measure zero. This is clearly not a restriction of much consequence; it will be fulfilled if we have a number of "wheat belts", "corn belts" etc., each of them sharply delimited though varying in extent as prices change.

If a price changes we always have

$$(3) \qquad \Delta x_i \; \Delta p_i \geq 0 \qquad (i=1, 2, \ldots, n)$$

where Δx_i is either 0 or 1 (apart from a set of measure zero).

The average proportion X_i of land allocated to crop i is given by

$$(4) \qquad X_i = \int_0^{\infty} \cdots \int_0^{\infty} x_i \, f(y_1, \ldots, y_n) \, dy_1 \ldots dy_n$$

Since $x_i = 1$ if and only if $p_i \, y_i > p_j \, y_j$ ($j=1, \ldots, i-1, i+1, \ldots, n$) this can be written

(5)
$$X_i = \int_0^\infty dy_i \int_0^{\frac{p_i y_i}{p_1}} dy_1 \cdots \int_0^{\frac{p_i y_i}{p_n}} dy_n \, f(y_1,\ldots,y_n)$$

We call

(6)
$$\frac{\partial}{\partial y_i} F(y_1,\ldots,y_n) = F_i(y_1,\ldots,y_n)$$

(7)
$$\frac{\partial^2}{\partial y_i \partial y_j} F(y_1,\ldots,y_n) = F_{ij}(y_1,\ldots,y_n)$$

and then

(8)
$$X_i = \int_0^\infty F_i\left(\frac{p_i y_i}{p_i},\ldots,\frac{p_i y_i}{p_n}\right) dy_i$$

As infinite yields do not occur we may regard the domain of integration as effectively bounded and therefore do not have to consider convergence properties. The density function was assumed to be continuous; hence the integrand possesses continuous derivatives with respect to its arguments and hence with respect to p_i. Consequently (for $i \neq j$)

(9)
$$\frac{\partial X_i}{\partial p_j} = \frac{\partial}{\partial p_j} \int_0^\infty F_i\left(\frac{p_i y_i}{p_1},\ldots,\frac{p_i y_i}{p_n}\right) dy_i$$

$$= \int_0^\infty \frac{\partial}{\partial p_j} F_i(\ldots)\, dy_i$$

Put $p_i y_i = t_i$; $\frac{t_i}{p_j} = z_{ij}$, then

(10)
$$\frac{\partial X_i}{\partial p_j} = \frac{1}{p_i} \int_0^\infty \frac{\partial}{\partial p_j} F_i\left(\frac{t_i}{p_1},\ldots,\frac{t_i}{p_n}\right) dt_i$$

$$= \frac{1}{p_i} \int_0^\infty \frac{\partial F_i(\ldots)}{\partial z_{ij}} \frac{\partial z_{ij}}{\partial p_j} dt_i$$

$$= -\frac{1}{p_i\, p_j^2} \int_0^\infty t_i F_{ij} \left(\frac{t_i}{p_1}, \ldots, \frac{t_i}{p_n}\right) dt_i$$

Similarly

(11) $$\frac{\partial X_j}{\partial p_i} = -\frac{1}{p_j p_i^2} \int_0^\infty t_j F_{ji} \left(\frac{t_j}{p_1}, \ldots, \frac{t_j}{p_n}\right) dt_j$$

and since $F_{ij} = F_{ji}$

(12) $$p_j \frac{\partial X_i}{\partial p_j} = p_i \frac{\partial X_j}{\partial p_i}$$

In the same way we can show that if Y_i is the average yield of the i-th crop on all land in A, then

(13) $$\frac{\partial Y_i}{\partial p_j} = \frac{\partial Y_j}{\partial p_i}$$

If in the region as a whole something is produced of every crop (3) can be written, as a result of the assumed continuity of the density function

(14) $$\frac{\partial X_i}{\partial p_i} > 0$$

The integrability conditions (12) and (13) and the convexity condition (14) together imply the existence of a constrained maximum in the X_i similar to that in the x_i (cf. (1) and (2)). These conditions arise out of the maximization of

(15) $$R = \sum_{i=1}^{n} p_i\, Y_i\, (X_i)$$

subject to

(16) $$\sum_{i=1}^{n} X_i = 1$$

The first-order equations for a maximum are, in addition to (16)

(17) $$p_i \frac{\partial Y_i}{\partial X_i} - \mu = 0$$

where μ is a Lagrange multiplier, to be interpreted as the marginal value product of land as a whole. We see that the marginal value product of each category of land is the same, but this need not hold for non-marginal points, i.e. points whose use is not affected by small price changes.

The second-order conditions for a constrained maximum of R require that the leading principal minors of the matrix

$$(18) \qquad H = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & p_1 \dfrac{\partial^2 Y_1}{\partial X_1^2} & 0 & \cdots & 0 \\ 1 & 0 & p_2 \dfrac{\partial^2 Y_2}{\partial X_2^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 1 & 0 & 0 & \cdots & p_n \dfrac{\partial^2 Y_n}{\partial X_n^2} \end{bmatrix}$$

be alternatively positive and negative. By differentiating (16) and (17) it can easily be shown that

$$(19) \qquad \frac{\partial X_i}{\partial p_i} = - \frac{\partial Y_i}{\partial X_i} \frac{H_{ii}}{H} > 0$$

in accordance with (14). If we take the second leading principal minor of H we find that

$$(20) \qquad p_i \frac{\partial^2 Y_i}{\partial X_i^2} + p_j \frac{\partial^2 Y_j}{\partial X_j^2} < 0 \quad (i, j = 1, 2, \ldots n;\ i \neq j)$$

so that there can be at most one crop which has ~~diminishing~~ increasing returns; in a situation like the present one diminishing returns are the rule. Since

$$(21) \qquad \frac{\partial X_i}{\partial p_j} = - \frac{\partial Y_j}{\partial X_j} \frac{H_{ij}}{H}$$

the truth of (12) can be verified, and as

$$(22) \qquad \frac{\partial Y_i}{\partial p_j} = \frac{\partial Y_i}{\partial X_i} \frac{\partial X_i}{\partial p_j}$$

in conjunction with (17) the same holds for (13).

The remarkable aspect of this transformation from a micro-economic into a macro-economic maximum problem is the simple form of the aggregate production function implied in (15). No matter what shape the joint frequency distribution of the yields may have, the average yield of each class of land depends only on the area of that class and not directly on how much is grown of other crops.

If we interpret the analysis in terms of labor X_i is the proportion of the population working in the i-th occupation, Y_i the average output in that occupation (e.g., the number of bricks laid) and p_i the piece rate for one unit of output. In that case special interest is attached to the distribution of incomes, where the Pareto curve still provides a challenge to economic theory. If we denote the proportion of incomes below M by N(M), then, using (8) and the transformation specified after (9) we have

$$(23) \qquad N(M) = \sum_{i=1}^{n} \frac{1}{p_i} \int_0^{\frac{M}{p_i}} F_i \left(\frac{t}{p_1}, \ldots, \frac{t}{p_n} \right) dt$$

Assuming that prices do not change we can put them all equal to one, so that (23) becomes

$$(24) \qquad N(M) = \sum_{i=1}^{n} \int_0^{M} F_i \, (t, \ldots, t) \, dt$$

The simplicity of this formula is somewhat deceptive, and the derivation of the Pareto distribution by this method, if possible at all, would no doubt require additional assumptions.

It would be interesting to apply similar aggregation procedures to more complicated micro-economic problem in the hope of getting rid of the analytical difficulties connected with non-negativity. An important application might for instance be made in the field of consumption theory (cf. C.C.D.P. Economics 2030). We have no results along these lines to offer yet.

References

[1] Klein, L. R. "Remarks on the Theory of Aggregation," Econometrica 14 (1946), 303.

[2] Koopmans, T. C. (Ed.) Activity Analysis of Production and Allocation New York, 1951.

[3] Roy, A. D. "Some Thoughts on the Distribution of Earnings", Oxford Economic Papers, N. S. 3 (1951), 135.

[4] Tinbergen, J. "Some Remarks on the Distribution of Labour Incomes", International Economic Papers 1 (1951), 195 (translation of a paper of 1947).