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Note on an Inventory Problem

Discussed by Modigliani and Hohn

François Morin

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In this note a solution to the inventory problem treated by F. Modigliani and F. Hohn in the Appendix to CCDP Economics 2038 is developed.

Let us restate the problem and introduce the notations:

Find the most profitable schedule of production of a commodity over a given period of time (0, T) so as to meet the following requirements:

(a) the initial inventory, h_o, is given;

(b) the sales schedule is given; let S(t) for instance be the cumulative sales from 0 to t; ($\frac{dS}{dt}$ need not be assumed continuous);

(c) the inventory can never be negative;

(d) the terminal inventory is 0;

(e) the cost of production is given: let f(x) be the cost, per unit time, of producing x units of product per unit time ($\frac{f(x)}{x}$ is the average cost; $f'(x)$ is the marginal cost[1];

(f) the marginal cost is increasing ($f'' > 0$);

[1] In the following ' denotes a first derivative
" denotes a second derivative.

(g) the cost of storage is α per unit of product and per unit time.

The unknown is the production schedule $X(t)$, which we define as:

$$X(t) = h_o + \text{cumulative output up to time } t.$$

This definition insures that X is continuous; but apart from that, no restriction other than those resulting from (a) - (g) is imposed on X; in particular, we do not assume the rate of production X' to be continuous.

In this respect, the difference between the problem solved here and that solved by F. Modigliani and F. Hohn rests on two points:

The first difference is that we drop the assumption that the sales rate remains constant during every period of one year; the sales function can be any function whatsoever.

The second is, that the same restriction is also dropped with regard to the production function: we permit ourselves to seek the solution among the most general class of functions. The solution so reached will of course be a more profitable one, but its realization would involve what may appear as a practical inconvenience (rate of output changing every day). And finally, whether that difference will be considered as an improvement or an unsatisfactory assumption will depend on the point of view one adopts.

<u>Formal statement of the problem.</u>

$X(t)$ is subject to the following restrictions:

(1) $X(0) = h_o$

(2) $X(T) = S(T)$

(3) $X(t) \geqq S(t)$ for any $0 \leqq t \leqq T$.

(4) $X(t)$ is continuous and non-decreasing.

The costs of storage and production per unit time, at time t, are respectively:

$$\alpha(X - S) \text{ and } f(X')$$

the total cost is:

(5) $$C = \int_0^T [\alpha(X - S) + f(X')]\, dt$$

The problem is to find the function X which minimizes (5) subject to (1) - (4).

Solution.

The solution is derived straightforwardly from 3 classical formulas of Calculus of Variations, which deal with the minimization of an integral of the form

$$\int_{t_0}^{t_1} \psi[t, X(t), X'(t)]\, dt$$

Those formulas are:

(a) the solution X must satisfy the following differential equation (equation of the extremals):

(6) $$\frac{\partial \psi}{\partial X} - \frac{d}{dt} \frac{\partial \psi}{\partial X'} = 0 ;$$

(b) at any corner point (t_0, X_0) of the solution curve, the following relation must hold between the two slopes X'_{-0} and X'_{+0} :

(7) $$\psi(t_0 X_0 X'_{+0}) - \psi(t_0 X_0 X'_{-0}) = (X'_{+0} - X'_{-0}) \frac{\partial \psi}{\partial X'} (t_0 X_0 X'_{-0}) ;$$

(c) if the minimizing curve has an arc in common with a boundary S(t) (above which it is bound to lie), then the arc of the boundary must satisfy:[2/]

(8) $$\frac{\partial \psi}{\partial X} (tSS') - \frac{d}{dt} \frac{\partial \psi}{\partial X'} (tSS') \geq 0.$$

In our case:

2/ The proofs may be found in: Bolza, Lectures on the Calculus of Variations, N. Y. (1931), pp. 22, 35 & 43.

(6') extremals: $X'' f''(X') = \alpha$

(7') corner condition: $f(X'_{+0}) - f(X'_{-0}) = (X'_{+0} - X'_{-0})\, f'(X'_{-0})$

(8') boundary solution if: $S'' F''(S') \gtreqless \alpha$.

The general equation of the extremals contains two arbitrary constants, since it is the solution of a differential equation of the second order; a particular extremal has therefore to be determined by two conditions.

Examples:

(a) $\alpha = 0$ (no storage cost) : then $X'' = 0$ and the solutions are formed of <u>straight lines</u>.

(b) $\alpha \neq 0$; $f(X') = aX'^2 + b X' + c$ (marginal cost = $2aX' + b$ = linear function of output): the equation of the extremals becomes

$2aX'' = \alpha$, whose solution is:

$$X - \bar{X} = \frac{\alpha}{4a}(t - \bar{t})^2$$

with two arbitrary constant $\bar{X}$ and $\bar{t}$: the extremals are all the parabolas which are equal to $X = \frac{\alpha t^2}{4a}$.

(etc.) .

Since equation (6') does not contain explicitly X nor t, its solution must be of the form:

(9) $$X - \bar{X} = \Phi(t - \bar{t}) ,$$

which means that the set of extremals is a set of <u>equal curves</u>.

The corner condition (7') shows that the extremal must be tangent to S at any common point; that follows from assumption (f) of page 1: $f(X')$ is concave from above (figs. 1 and 2), hence $(X'_{+0} - X'_{-0})\, f'(X'_{-0})$ = AC can be equal

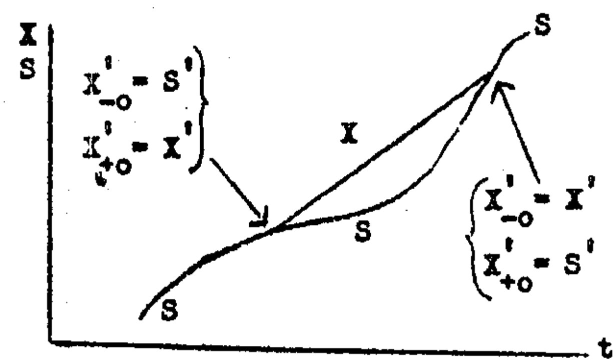

fig. 1

to $f(X_{+o}) - f(X_{-o}) = AB$ only if $X'_{+o} = X'_{-o}$,

that is: $S' = X'$.

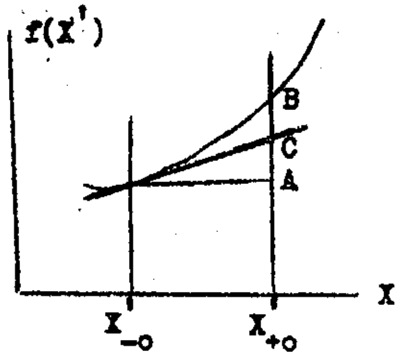

fig. 2

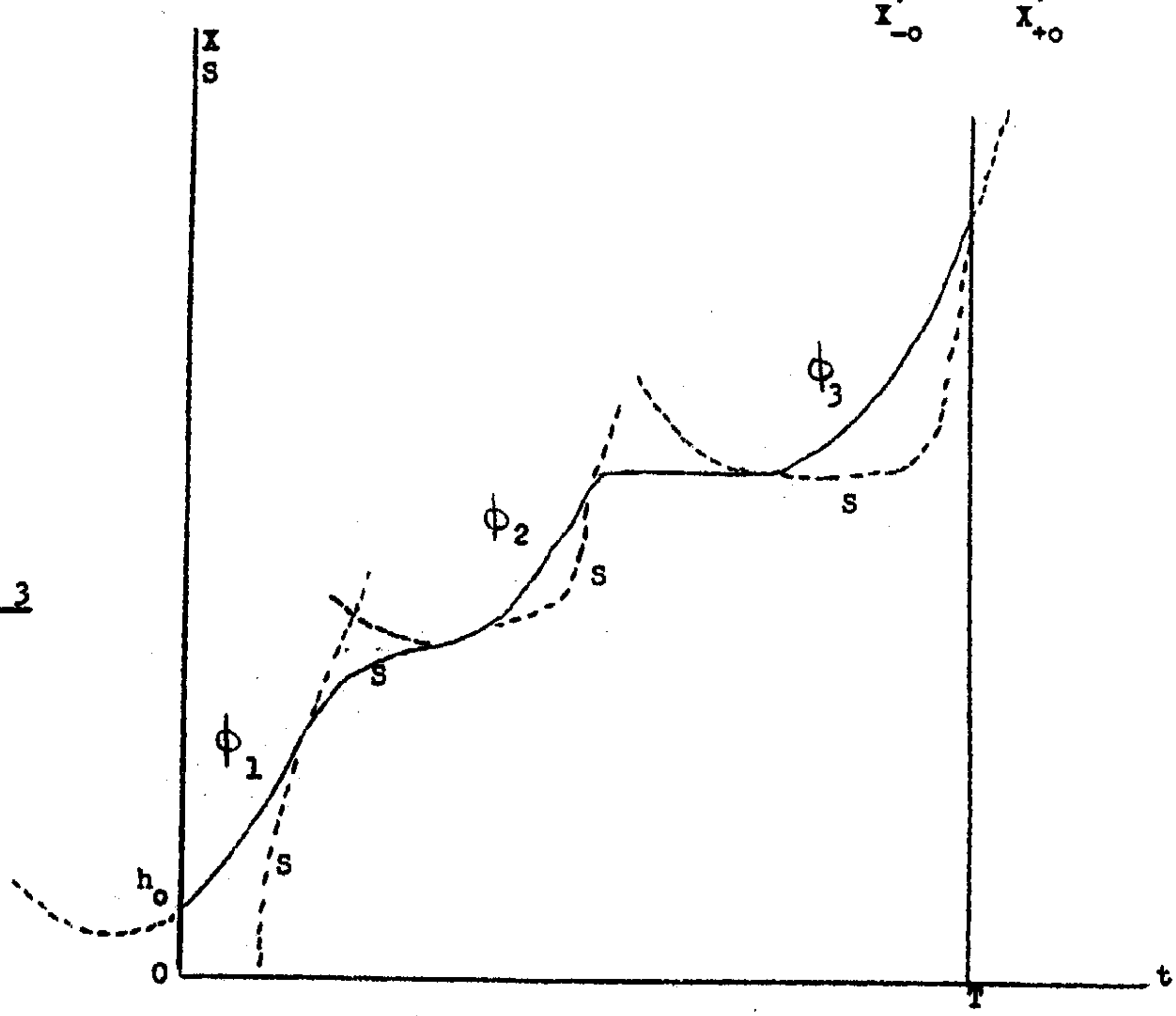

fig. 3

Finally, once the curve (9) is known from equation (6'), one has to move it all over the plane so as to meet the requirement that it should be:

- either tangent to S in 2 points,
- or tangent to S, and passing through one of the extreme points (0, h_o) and (T, S(T)),
- or passing through both extreme points.

All such possible positions of (9) are part of the solution (between the two contacts)[3]; condition (8') can be proved to show that in those intervals where no such curve can be found, the solution curve coincides with the boundary S(t).

Figure 3 shows a general example.

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1. In fact only the positively sloped arc of (9) is really a solution.

This remark may be relevant only for the first interval, and rules out a solution like Φ_1, (fig. 4). Of course the correct solution in that case is Φ_2.

fig. 4