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Note on Marschak's Cowles Commission Discussion Paper Economics 2034

Organized Decision-Making

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April 7, 1952

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This note to be inserted into C.C.D.P. 2034 after Section 15.

15a. Suppose now that the rule γ does not prescribe behavior with certainty but is in terms of randomized strategies. Denote by

$q_i(x_i)$	the probability that partner i commits the firm (i=1, 2)
$p_i(x_i)$	that he phones and
$o_i(x_i)$	that he does not do anything.

The integral (15.2) reads now explicitly as follows. Expected profits from commitments by partner i are

$$(15a.1)\qquad \int_0^1 \int_0^1 \left[(x_1 + x_2 - 1)\, q_i(x_i)\, [1 - p_j(x_j)]\right] dx_1\, dx_2 \quad (i \neq j)$$

and expected profits after communication are

$$(15a.2)\qquad \int_0^1 \int_0^1 [2 \max(x_1 + x_2 - 1, 0) - c]\, [p_1(x_1) + p_2(x_2) - p_1(x_1)\, p_2(x_2)\,]\, dx_1\, dx_2$$

If i does not do anything, no profits arise. Hence U is the sum of two expressions (15a.1) and (15a.2). The main observation to be made is that integration with respect to one variable, say x_j, will cause the integrand to become a linear function of $p_i(x_i)$ and $q_i(x_i)$, where $i \neq j$. We have to maximize with respect to the functions $q_i(\cdot)$, $p_i(\cdot)$, the expression

$$U = \int_0^1 [\lambda_i\,(x_i)\cdot p_i(x) + \mu_i\,(x_i)\cdot q_i(x_i) + \dots]\, dx_i,$$

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where ... denotes terms not containing $q_i(x_i)$ or $p_i(x_i)$ and therefore of no interest here. Since $q_i(x_i)$, $p_i(x_i)$ satisfy inequalities

$$0 \leqq p_i(x_i) \qquad\qquad 0 \leqq q_i(x_i)$$

$$p_i(x_i) + q_i(x_i) \leqq 1,$$

it is clear that U will be maximized by the following choices

Table 1

If $\mu_i(x_i)$	$\lambda_i(x_i)$	$q_i(x_i)$	$p_i(x_i)$	$x_i \in$
$\leqq 0$	$\leqq 0$	0	0	do nothing: $x_i \in X_i^0$
>0	$\lambda_i \leqq \mu_i$	1	0	commit: $x_i \in X_i^q$
$\mu_i \leqq \lambda_i$	0	0	1	phone: $x_i \in X_i^p$

This proves that the optimal strategies can be chosen among the pure strategies.

We proceed to calculate $\lambda_i(x_i)$ and $\mu_i(x_i)$. Write

$$\left.\begin{aligned} Q_i(x_i) &= \int_{1-x_i}^{1} q_j(t)\,dt \\ Q_i^*(x_i) &= \int_{1-x_i}^{1} t\cdot q_j(t)\,dt \\ P_i(x_i) &= \int_{1-x_i}^{1} p_j(t)\,dt \\ P_i^*(x_i) &= \int_{1-x_i}^{1} t\cdot p_j(t)\,dt \end{aligned}\right| \quad i \neq j$$

Hence $Q_i(0) = Q_i^*(0) = P_i(0) = P_i^*(0) = 0$

(15a.3) $\quad Q_i^*(1) \leq \frac{1}{2} \qquad P_i^*(1) \leq \frac{1}{2}$

$$Q_i(1) - Q_i^*(1) \leq \frac{1}{2} \qquad P_i(1) - P_i^*(1) \leq \frac{1}{2}$$

Then from the integral (15a.1)

$$\mu_i(x_i) = x_i\ [1 - P_i(1)] + P_i(1) - P_i^*(1) - \frac{1}{2}$$

and from both (15a.1) and (15a.2)

$$\lambda_i(x_i) = x_i^2 - x_i\ [Q_i(1) + 2\ P_i(x_i)] + 2\ P_i(x_i)$$

$$- 2\ P_i^*(x_i) + Q_i(1) - Q_i^*(1) - c\ [1 - P_i(1)]$$

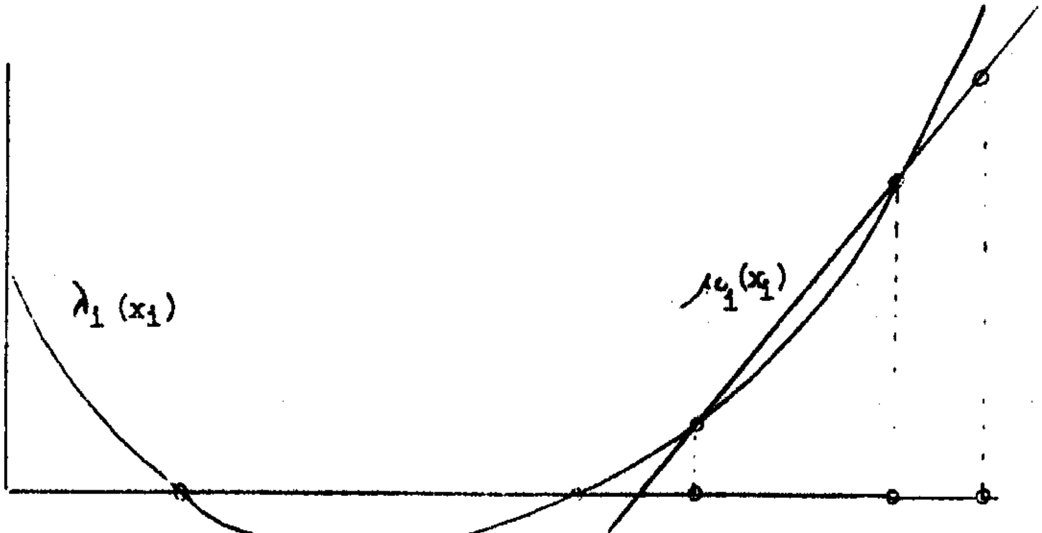
We first note that $\mu_i(x_i)$ is linear with non-negative slope coefficient. From the table it is seen now that an interval X_i^q can never be to the left of an interval X_i^o.

Secondly $\lambda_i(x_i)$ is convex since

$$\frac{d\ \lambda_i(x_i)}{dx_i} = 2\ x_i - 2\ P_i(x_i) - Q_i(1)$$

$$\frac{d^2\ \lambda_i(x_i)}{dx_i^2} = 2\ [1 - p_j(1 - x_i)\ ] \geq 0.$$

Thus the curve $y = \lambda_i(x_i)$ intersects the two lines $y = \mu_i(x_i)$ and $y = 0$ in at most two points each. These intersections on the other hand contain all the separation points between the sets X_i^k. Hence the sets X_i^k are intervals not exceeding 5 in number. Because of the restriction on the relative location of X_i^o and X_i^q the most general arrangement is p o p q p, and all other cases are obtained from it by omitting one or several symbols in the sequence. The 5 interval case is actualized by the following configuration, and any other one can be obtained by contraction of some of these intervals.



Let the strategy intervals now be labelled as in the following scheme

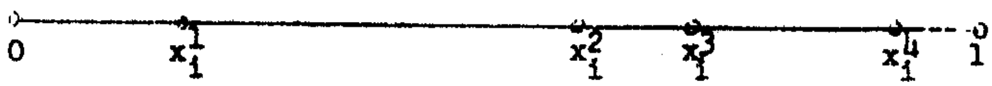


Table 2 lists the relations by which the interval endpoints x_i^1, x_i^2, x_i^3, x_i^4 are defined $(i = 1, 2)$.

Table 2

$(\cdot)$	$\lambda_i(\cdot)$	$\frac{d\lambda_i}{dx_i}(\cdot)$	$\lambda_i(\cdot)-\mu_i(\cdot)$	$\frac{d}{dx_i}[\lambda_i(\cdot)-\mu_i(\cdot)]$
x_i^1	$= 0$	≤ 0		
x_i^2	$= 0$	≥ 0		
x_i^3			$= 0$	≤ 0
x_i^4			$= 0$	≥ 0

for $i = 1, 2$.

Any set of numbers $x_i^1 \ldots x_i^4$, $x_j^1 \ldots x_j^4$ that solves the relations of Table 2 maximizes U and is thus a solution of our problem. For, the conditions given in Table 1 are sufficient and the intervals as defined in Table 2 are so derived as to satisfy the conditions of Table 1.

We conclude by giving a particular solution of the problem. Let S, t be parameters satisfying

$$0 \leq S; \quad S - 4c \leq t - 4c; \quad t \leq 1.$$

A family of solutions (and presumably the only solution) is then given by

$$x_i^1 = S \qquad x_j^1 = 1 - t$$

$$x_i^2 = S + 4c \qquad x_j^2 = 1 - t + 4c$$

$$x_i^3 = t - 4c \qquad x_j^3 = 1 - S - 4c$$

$$x_i^4 = t \qquad x_j^4 = 1 - S$$

(Note the anti-symmetric relationship $x_i^e = 1 - x_j^e$). One has

Table 3

$f(x_i)$ \ x_i	S	$S + 4c$	$t-4c$	t	1
$P_i(\cdot)$	S	S	$t-8c$	$t-8c$	$1-8c$
$P_i(\cdot)-P_i^*(\cdot)$	$\frac{1}{2}S^2$	$\frac{1}{2}S^2$	$\frac{t^2}{2} - 4cS - 4ct$	$\frac{t^2}{2} - 4cS - 4ct$	$\frac{1}{2} - 4cS - 4ct$
$Q_i(\cdot)$					$4c$
$Q_i(\cdot) - Q_i^*(\cdot)$					$4cS + 8c^2$

$$\lambda_i\ (x_i) = x_i^2 - x_i\ [Q_i(1) + 2P_i\ (x_i)] + 2[\ P_i\ (x_i) - P_i^*\ (x_i)]$$

$$+ Q_i\ (1) - Q_i^*\ (1) - c\ [1 - P_i\ (1)]$$

$$\lambda_i\ (S) = S^2 - S\ [4c + 2S] + S^2$$

$$+4cS + 8c^2 - c8c\ = 0$$

$$\lambda_i (S+4c) = (S+4c)^2 - (S+4c)\ [2S + 4c]$$

$$+ S^2 + 4cS + 8c^2 - 8c^2 = 0$$

$$\lambda_i(x_i) - \mu_i\ (x_i) = x_i^2 - x_i\ [Q_i\ (1) + 1 - P_i\ (1)$$

$$+ 2\ P_i\ (x_i)] + 2\ [P_i\ (x_i) - P_i^*\ (x_i)]$$

$$+ [Q_i\ (1) - Q_i^*\ (1)] - [P_i\ (1) - P_i^*\ (1)] + \frac{1}{2}$$

$$- c\ [1 - P_i\ (1)]$$

$$\lambda_i (t-4c) - \mu_i (t-4c) = (t-4c)^2 - (t-4c)\ [4c+1-1+8c+2t-16c] +t^2 -8cS-8ct$$

$$+ 4cS + 8s^2 - (\frac{1}{2} - 4cS - 4ct) + \frac{1}{2} - 8c^2 = 0$$

$$\lambda_i(t) - \quad _i\ (t) = t \quad - t\ [4c+1 - 1+8c + 2t - 16c]$$

$$+ t^2 - 8cS - 8ct + 4cS + 8c^2 - (\frac{1}{2} - 4cS - 4ct) + \frac{1}{2}$$

$$- 8c^2 = 0.$$

With

$S' = 1 - t$

$t' = 1 - S$

the conditions for the x_j follow. We may skip the verification of the inequalities for the derivatives of $\lambda_i\ (x_i)$ and $\lambda_i\ (x_i) - \mu_i\ (x_i)$ since our assumptions on S and t, which are preserved for S', t', ensure that the $x_i^1, \ldots, x_i^4$ are put in the right order. This completes the proof.

While we have not shown that this is the only solution, we have found a reasonable answer to the problem. For it is sufficient that the firm is in possession of one set of strategies ensuring the maximal profit.

The results of this paper are in accordance with those of K. Faxén in Cowles Commission Discussion Paper Economics 2037. In addition we have established that his relative optimum (for the case labelled II) is the absolute optimum.