Real Representation of a Preference Ordering [1/]

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Since the naive approach to utility has been abandoned it has generally been too readily assumed in economics that on a completely ordered space S (usually the finite Euclidean space of commodity bundles) it is possible to define a real-valued order-preserving function. The lexicographic ordering of the plane [2/] affords an immediate counter-example. [3/]

J. von Neumann and O. Morgenstern [5], and later J. Marschak [4], I. N. Herstein and J. Milnor [3] have given a rigorous treatment of this question in the particular case where there exists on S (the space of prospects) a certain algebra of combining (corresponding to the combination of probabilities).

The general problem is considered here. Among economists it seems to have received attention only from H. Wold [6] who however limits himself to a finite Euclidean space and uses very restrictive and non-intuitive axioms.

The most common preference ordering in economics is the ordering of n - commodity bundles i.e. of points of the n - dimensional Euclidean space. But more general ordered spaces (such as the spaces of prospects) have alrea[dy] occurred and it seems desirable to study as general a case as possible, dry[ly] abstract as it may be.

S will always denote a completely ordered space i.e. a set of elements (a, b, c, ...) endowed with a binary relation, denoted $\leqq$, satisfying the two conditions

(1) Given any two elements a, b; $a \leqq b$, or $b \leqq a$, or both

(2) Given three elements a, b, c such that $a \leqq b$ and $b \leqq c$, then $a \leqq c$.

The nature of the elements (a, b, c, ...) will be left unspecified; in terms of preference $a \leqq b$ can be read, for example, as b is at least as good as a. The conditions (1) (2) seem to many to be in full agreement with the notion of preference.

The two following concepts are derived immediately from the above binary relation:

$a \sim b$ (a indifferent to b) if $a \leqq b$ and $b \leqq a$.

$a < b$ (b better than a) if $a \leqq b$ and not $b \leqq a$.

A real-valued function f(a) (utility, satisfaction, ...) defined on S is said to be <u>order-preserving</u> if

$a \leqq b$ is <u>equivalent</u> to $f(a) \leqq f(b)$.

Our concern is the conditions to impose on the ordered space S for an order-preserving real function to exist on S .

Two standard concepts for a topological space are recalled:

S is <u>separable</u> if there is a countable subset of S dense in S (i.e. whose closure is S)

S is <u>connected</u> if there is no partition of S into two disjoint, non-empty, closed sets.

The main result is then

<u>Theorem. Let S be a completely ordered space such that</u>

(I) S <u>is separable and connected</u>

(II) <u>for every</u> $a_0 \in S$ <u>the sets</u> $\{a \in S \mid a \leqq a_0\}$, $\{a \in S \mid a_0 \leqq a\}$ <u>are closed.</u>

<u>There exists on S a continuous, real, order-preserving function.</u>

A finite Euclidean space is naturally separable, connected; so is the Hilbert space of all infinite sequences of complex numbers (x_i) such that $\sum_{i=1}^{\infty} |x_i|^2 < +\infty$. Axiom (I) is therefore not very restrictive in the present state of economic theory.

Axiom (II) can be interpreted as follows in the usual case of choice among bundles of n commodities. Such a bundle is represented by a point q in R^n. Let (q_i) be a sequence converging to q and imagine that there is a bundle $\bar{q}$ such that for all i $q_i \leq \bar{q}$ $(\bar{q} \leq q_i)$. (II) then means that $q \leq \bar{q}$ $(\bar{q} \leq q)$.

A Lemma shall first be proved

Lemma Let S be a completely ordered set satisfying

(III) there exists a countable subset $A = (a_0, a_1, \ldots, a_i, \ldots)$ of S such that that for any two elements b, c $b < c$ there is an element a_i of A between them: $b < a_i < c$.

Then there exists an order-preserving function from onto a dense subset of a real interval.

Postulate (III) has been used by G. Cantor in [2]. The trivial case where all elements of S are indifferent shall be excluded.

The function f(a) will first be defined on A and then extended to S .

Choose two arbitrary real numbers $\lambda_0 < \lambda_1$ (lower case greek letters shall always denote real numbers).

If A has a subset of smallest elements (a_m), we define for any one of them $f(a_m) = \lambda_0$.

If A has a subset of largest elements (a_M), we define for any one of them $f(a_M) = \lambda_1$.

Taking both subsets (a_m), (a_M) out of A we are left with A* which has no smallest element and no largest element (Condition III). Let Θ be the set of all rational numbers between λ_0 and λ_1: $\Theta = (\theta_0, \theta_1, \ldots, \theta_i \ldots)$ $\lambda_0 < \theta_i < \lambda_1$. An order preserving function from A* onto Θ is defined as follows

take $\alpha_0 = f(a_0) = \theta_0$ and then define $\alpha_{n+1} = f(a_{n+1})$ by induction as follows:

order $(a_0, a_1, \ldots, a_n)$ $\quad a_{i_0} \lesssim a_{i_1} \lesssim \ldots \lesssim a_{i_n}$

If $a_{n+1} \sim a_{i_p}$ $\qquad \alpha_{n+1} = \alpha_{i_p}$

If $a_{n+1} < a_{i_0}$ take for α_{n+1} the first $\theta_i < \alpha_{i_0}$

If $a_{i_n} < a_{n+1}$ $\qquad \theta_i > \alpha_{i_n}$

If $a_{i_q} < a_{n+1} < a_{i_{q+1}}$ $\qquad \theta_i$, $\alpha_{i_q} < \theta_i < \alpha_{i_{q+1}}$

Every rational number θ_i will eventually be taken.

The extension of f(a) from A to S is then immediate:

Consider $a \in S$ and define

$\alpha = f(a) =$ Sup $f(a_i)$ on the set $\{a_i \in A \mid a_i \lesssim a\}$ or, what is equivalent, $\alpha =$ Inf $f(a_i)$ on the set $\{a_i \in A \mid a \lesssim a_i\}$.

The function f(a) on S is indeed order-preserving:

1) If $b \sim c$ obviously $\beta = \gamma$

2) If $b < c$ there exist two elements a_p, a_q of A such that $b < a_p < a_q < c$ thus $\beta \lesssim \alpha_p$, $\alpha_q \lesssim \gamma$ i.e. $\beta < \gamma$.

To prove the theorem we will first show that (I) and (II) imply (III).

Since S is separable there is a countable dense subset A of S: $\bar{A} = S$.

If A does not satisfy (III) there are two elements b, c of S $b < c$ such that no element a of A lies between them ($b < a < c$). In other words the two sets $A_1 = \{a \in A \mid a \leq b\}$, $A_2 = \{a \in A \mid c \leq a\}$ form a partition of A: $A_1 \cup A_2 = A$. Moreover $A_1 \subset S_1 = \{a \in S \mid a \leq b\}$, $A_2 \subset S_2 = \{a \in S \mid c \leq a\}$. S_1 and S_2 are clearly non-empty, closed and $S_1 \cap S_2 = \emptyset$. Thus $\bar{A}_1 \subset S_1$, $\bar{A}_2 \subset S_2$ and the relation $\bar{A} = \bar{A}_1 \cup \bar{A}_2$ yields $S = S_1 \cup S_2$ contradicting the fact that S is connected.

A does therefore satisfy (III) and there exists an order preserving function f(a) from S onto a dense subset of the closed interval $[\lambda_0, \lambda_1]$. It remains only to show that f(a) is <u>continuous</u>.

For this, let us prove that f(a) takes on every value α, $\lambda_0 < \alpha < \lambda_1$. Denote $T_{a_0} = \{a \in S \mid a_0 \leq a\}$, $T^{a_0} = \{a \in S \mid a \leq a_0\}$; for any $a_0 \in S$, these two sets are closed (Axiom (II)). Let $T_\alpha = \{a \in S \mid \alpha \leq f(a)\}$, $T^\alpha = \{a \in S \mid f(a) \leq \alpha\}$. $T_\alpha = \bigcap_{a_0 \in T^\alpha} T_{a_0}$ and is therefore closed as an intersection of closed sets. Similarly T^α is closed. Since T_α and T^α are not empty and exhaust S, they must have a non-void intersection (by the connectedness of S) i.e. there exists $a \in S$ such that $f(a) = \alpha$.

It is then obvious that the inverse image by f(a) of any open interval (and therefore of any open set) of the real line is open i.e. that f(a) is continuous [1, ; 4].

FOOTNOTES

1/ I thank I. N. Herstein and M. Slater for the valuable discussions I had with them.

2/ A point of coordinates (a_2, b_2) is better than the point (a_1, b_1) if "$a_1 < a_2$" or if "$a_1 = a_2$ and $b_1 < b_2$".

3/ Suppose that there exists a real order-preserving function $\alpha(a, b)$. Take two fixed numbers $b_1 < b_2$. With a given number a, associate the two numbers $\alpha_1(a) = \alpha(a, b_1)$ and $\alpha_2(a) = \alpha(a, b_2)$. To two different numbers a and a' correspond two disjoint intervals $[\alpha_1(a), \alpha_2(a)]$ and $[\alpha_1(a'), \alpha_2(a')]$. One obtains therefore a one-to-one correspondence between the points of the real line and a set of disjoint intervals i.e. a countable set.

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