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An Axiomatic Approach to Measurable Utility 1/ 2/

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1. The concept of a measurable utility, that is, of a real valued function, appropriately linear with respect to probability distribution, measuring an individual's preference ratings is by no means a new one, tracing its origin as far back as Bernouilli and his "moral expectation". However, a completely rigorous treatment of the existence of such a utility, on the basis of a well-defined set of conditions or postulates, was completely lacking until the arrival of the von Neumann and Morgenstern book "The Theory of Games and Economic Behavior" [5]. In order to give a more acceptable (to some economists) set of axioms, and to have a simpler derivation mathematically from these axioms, Marschak [2] re-attacked the subject. However, in this paper, Marschak only considered the case of a finite number of sure prospects. Rubins [4] extended the Marschak system to the case of an infinite number of sure prospects. Herstein [1] and Milnor [3], gave quite different axiom sets for this problem and succeeded in simplifying and shortening the mathematical details considerably. However, topological considerations of the prospects' space entered into the axioms. In this paper we remove considerations of the topology of the prospect space itself, weaken the previous axioms

and allow an infinite number of sure prospects; in doing so the treatment actually becomes simpler and more transparent. On the basis of these axioms the existence of a measurable utility is established.

2. Notations

In the body of this paper

(a) capital script Latin letters will always denote sets

(b) lower case printed Latin letters will always denote elements of sets

(c) lower case Greek letters will denote real numbers whose values are between 0 and 1, end values also possible.

(d) $\mathcal{S} = \{x \mid P\}$ will denote the set of x having the property P.

3. The axioms

A set $\mathcal{S}$ is said to be a mixture set if for any $a, b \in \mathcal{S}$ and any μ we can associate another element, which we write as $\mu a + (1-\mu)b$, which is again in $\mathcal{S}$, and where

1. $1a + (1-1)b = a$

2. $\mu a + (1-\mu)b = (1-\mu)b + \mu a$

3. $\lambda(\mu a + (1-\mu)b) + (1-\lambda)b = (\lambda\mu)a + (1-\lambda\mu)b$

for all $a, b \in \mathcal{S}$ and all λ, μ.

A convex set in a real vector space, where we mean by $\mu a + (1-\mu)b$ the usual multiplication by scalars and the addition of elements of this vector space, is easily seen to be a mixture set.

The concept of a preference ordering arises naturally in certain phases of economics. We formalize the concept and define:

A binary relation, $\gtrsim$, defined on a set $\mathcal{S}$ is a complete ordering if

i. for any $a, b \in \mathcal{S}$ at least one of $a \gtrsim b$ or $b \gtrsim a$ must hold.

ii. if $a, b, c \in \mathcal{S}$ and $a \gtrsim b$, $b \gtrsim c$ then $a \gtrsim c$.

The simultaneous satisfaction of $a \gtrsim b$ and $b \gtrsim a$ need not imply that a is identical with b. This prompts us to make the following definition:

If $a, b \in \mathcal{S}$ then $a \sim b$ (read as a is indifferent or equivalent to b) if and only if both $a \gtrsim b$ and $b \gtrsim a$.

The following three properties then hold true:

A. $a \sim a$

B. $a \sim b$ implies $b \sim a$

C. $a \sim b$, $b \sim c$ imply $a \sim c$

for any $a, b, c \in \mathcal{S}$.

Let $I(a) = \left\{ x \in \mathcal{S} \mid x \sim a \right\}$. Properties A, B, C of $\sim$ imply that the distinct $I(a)$'s yield a decomposition of $\mathcal{S}$ into mutually disjoint subsets. We call $I(a)$ the <u>indifference set</u> of a.

If $a \gtrsim b$ but $a \notin I(b)$ we say that $a > b$.

A real valued function u defined on a completely ordered set $\mathcal{S}$ is called order preserving if for any $a, b \in \mathcal{S}$ $u(a) > u(b)$ if and only if $a > b$.

A real valued function v defined on a mixture set $\mathcal{M}$ is said to be <u>linear</u> if for all $a, b \in \mathcal{M}$ and any α

$$v(\alpha a + (1-\alpha)b) = \alpha v(a) + (1-\alpha)v(b).$$

The economist often considers a set $\mathcal{S}$ which is, at the same time, both a mixture set and a completely ordered set; the mixture operation being, in his case, probability mixture, the ordering being the preference ordering, and $\mathcal{S}$ the set of prospects. The economist is interested in finding a real valued, order-preserving, linear function on $\mathcal{S}$. This function is the so-called <u>measurable utility</u>. If there are no restrictions on $\mathcal{S}$ other than it be a completely-ordered mixture set, such a measurable utility need not exist. The problem, then, is to impose "natural" restrictions on the interrelation of the ordering and the mixing which give rise to a measurable utility.

Our aim has been many-fold; to use a system of axioms which was simple and seemingly not too restrictive, which seemed to approximate economic reality, which was transparent, and which led to the existence of a measurable utility with a minimum of mathematical difficulty or sophistication. Each of the axioms we use is true for $\mathcal{S}$ whenever a measurable utility exists on $\mathcal{S}$; so the axioms are at least necessary conditions. In this paper we prove that they are also sufficient to lead to the existence of the desired order-preserving, linear function.

Let $\mathcal{S}$ be a mixture set. We assume

<u>Axiom 1</u>. $\mathcal{S}$ <u>is completely ordered by</u> $\gtrsim$.

<u>Axiom 2</u>. <u>For any</u> $a, b, c \in \mathcal{S}$, the sets $\{\alpha \mid \alpha a + (1-\alpha)b \gtrsim c\}$ <u>and</u> $\{\alpha \mid c \gtrsim \alpha a + (1-\alpha)b\}$ <u>are closed</u>.

<u>Axiom 3</u>.[3/] <u>If</u> $a, a' \in \mathcal{S}$, $a \sim a'$, <u>then for any</u> $b \in \mathcal{S}$, $1/2\, a + 1/2\, b \sim 1/2\, a' + 1/2 b$.

In economic terms, axiom 2 approximately states that an individual's preference ordering is continuous with regard to probability distributions; namely that if $\lim_{i \to \infty} \mu_i = \mu$ and each $\mu_i a + (1-\mu_i)b \gtrsim c$ then

$\mu a + (1-\mu)b \gtrsim c$ (and similarly for $c \gtrsim \mu_i a + (1-\mu_i)b$).

Axiom 3 states that if an individual is indifferent as to a choice between a and a', then he is also indifferent to a choice between A and A', where A represents a 50-50 chance of getting a or b and A' a 50-50 chance of getting a' or b, for any prospect b.

4. The derivation of the measurable utility

Theorem 1. (Continuity theorem) *If* $a, b, c \in \mathcal{S}$ *and* $a \gtrsim b \gtrsim c$ *then there exists a* μ, *so that* $b \sim \mu a + (1-\mu)c$.

Let $\mathcal{T} = \{\mu \mid \mu a + (1-\mu)c \gtrsim b\}$. By axiom 2 $\mathcal{T}$ is a closed subset of the closed unit interval $[0,1]$. Since $a \gtrsim b$, $1 \in \mathcal{T}$, so $\mathcal{T}$ is not empty. Using axiom 2 we obtain that $\mathcal{W} = \{\lambda \mid b \gtrsim \lambda a + (1-\lambda)c\}$ is closed in $[0, 1]$; it is not empty since $0 \in \mathcal{W}$. $\mathcal{S}$ is a completely-ordered mixture set, so $\mathcal{T} \cup \mathcal{W} = [0, 1]$; the unit interval is connected, so can not be decomposed into a union of closed, disjoint subsets, thus $\mathcal{T} \cap \mathcal{W}$ is not empty. Let $\mu_0 \in \mathcal{T} \cap \mathcal{W}$; by the definitions of $\mathcal{T}$ and $\mathcal{W}$, $b \sim \mu_0 a + (1-\mu_0)c$, proving theorem 1. Clearly, if $a > b > c$ then $1 > \mu_0 > 0$.

Theorem 2.[4] *If* $a, a' \in \mathcal{S}$ *and* $a \sim a'$ *then for any* $b \in \mathcal{S}$ *and any* μ

$$\mu a + (1-\mu)b \sim \mu a' + (1-\mu)b$$

We will prove this result by first establishing several intermediate results. These will be of importance also for many subsequent proofs in this paper.

a) *If* $\mu_i a + (1-\mu_i)\, b \sim c$ *and* $\lim_{i \to \infty} \mu_i = \mu$ *then* $\mu a + (1-\mu)\, b \sim c$.

For $\mu_i a + (1-\mu_i)\, b \gtrsim c$ implies, by axiom 2 that $\mu a + (1-\mu)b \gtrsim c$. Similarly using axiom 2 we obtain $c \gtrsim \mu a + (1-\mu)b$.

b) *If* $a > b$ *then* $a > 1/2\, a + 1/2\, b > b$.

Suppose that $1/2\, a + 1/2\, b \gtrsim a > b$. Then by theorem 1, there exists a μ so that $a \sim \mu(1/2\, a + 1/2\, b) + (1-\mu)b = 1/2\,\mu a + (1-1/2\,\mu)b$.

Let $T = \{\mu \mid a \sim \frac{\mu}{2} a + (1-\frac{\mu}{2})b\}$. By a) T is a closed subset of $[0, 1]$, so has a least element μ_0, ($\mu_0 > 0$ since $a > b$). Now since

$a \sim 1/2\,\mu_0 a + (1-1/2\,\mu_0)b$, axiom 3 yields that $1/2a + 1/2b \sim 1/4\,\mu_0 a + (1-1/4\,\mu_0)b \gtrsim a > b$, so for some λ, $a \sim \lambda(1/4\,\mu_0 a + (1-1/4\,\mu_0)b) + (1-\lambda)b$ by theorem 1. But $\frac{\lambda\mu_0}{4} < \frac{\mu_0}{2}$, contradicting the choice of μ_0. Hence μ $a > 1/2a + 1/2b$. Similarly $1/2a + 1/2b > b$.

c) <u>If</u> $a > b$ <u>then</u> <u>for</u> <u>any</u> $0 < \mu < 1$

$a > \mu a + (1-\mu)b > b$.

In the rest of the proof of theorem 2, ρ will generically stand for a rational number of the form $\rho = \sum_{i=1}^{n(\rho)} \frac{\alpha_i}{2^i}$ where $\alpha_i = 0$ or 1.

Successive applications of remark b yields that if $\rho_2 > \rho_1$ then

$a > \rho_2 a + (1-\rho_2)b > \rho_1 a + (1-\rho_1)b > b$. For any $0 < \mu < 1$ pick ρ_2, ρ_1 so that $0 < \rho_1 < \mu < \rho_2 < 1$. Let $a' = \rho_2 a + (1-\rho_2)b$, $b' = \rho_1 a + (1-\rho_1)b$. So for any ρ_i, $\rho_2 \geqq \rho_i \geqq \rho_1$ $\quad a' \geq \rho_i a + (1-\rho_i)b \gtrsim b'$. Now $\mu = \lim \rho_i$ for suitable $\rho_2 \geqq \rho_i > \rho_1$ (use the binary expansion of μ to obtain such approximants). Thus axiom 2 yields that $a > a' \gtrsim \mu a + (1-\mu)b > b' \gtrsim b$, proving remark c.

d) If $a \sim a'$ then $\mu a + (1-\mu)a' \sim a$. Successive use of axiom 3 yields that $\rho_i a + (1-\rho_i)a' \sim a$. Pick ρ_i so $\lim \rho_i = \mu$. Then by remark a $\mu a + (1-\mu)a' \sim a$.

We are now in a position to prove theorem 2. Suppose $a \sim a'$. By remark d if $b \sim a$, $a \sim \mu a + (1-\mu)b \sim a' \sim \mu a' + (1-\mu)b$; so to prove theorem 2, suppose $a > b$. From axiom 3 for any ρ

$\rho a + (1-\rho)b \sim \rho a' + (1-\rho)b$. Given μ let $\Gamma = \{\lambda \mid \lambda a + (1-\lambda)b \gtrsim \mu a' + (1-\mu)b\}$. We pick a sequence ρ_i approaching μ so that $\rho_i \geqq \mu$ for each i. Then $\rho_i a + (1-\rho_i)b \sim \rho_i a' + (1-\rho_i)b \gtrsim \mu a' + (1-\mu)b$ by remark c.

So $\rho_i \in \mathcal{T}$, hence by axiom 2, $\mu = \lim \rho_i \in \mathcal{T}$. That is $\mu a + (1-\mu)b \gtrsim \mu a' + (1-\mu)b$.

This argument and construction is symmetric in a and a', so we also can obtain $\mu a' + (1-\mu)b \gtrsim \mu a + (1-\mu)b$. This leads to the desired result. If on the other hand, $b > a$, the same technique employed for $a > b$ gives the result.

Remark c of the proof of theorem 2 will be needed often so we exhibit it as

Theorem 3. <u>If</u> $a > b$ <u>then</u> <u>for</u> <u>every</u> $0 < \mu < 1$
$a > \mu a + (1-\mu)b > b$.

Suppose $a > b$ and $\lambda > \mu > 0$. Then $\lambda a + (1-\lambda)b > b$. Since $0 < \frac{\mu}{\lambda} < 1$, by theorem 3 $\lambda a + (1-\lambda)b > \frac{\mu}{\lambda}(\lambda a + (1-\lambda)b) + (1-\frac{\mu}{\lambda})b = \mu a + (1-\mu)b$. Similarly, if $a > b$ and $\lambda a + (1-\lambda)b > \mu a + (1-\mu)b$ then $\lambda > \mu$.

Summarizing we have

Theorem 4 <u>If</u> $a > b$ <u>then</u> $\lambda a + (1-\lambda)b > \mu a + (1-\mu)b$ <u>if</u> <u>and</u> <u>only</u> <u>if</u> $\lambda > \mu$.

As a consequence we obtain

Theorem 5. <u>There is only one indifference set or an infinite number of distinct indifference sets in</u> $\mathcal{S}$.

For if for some $a, b \in \mathcal{S}$, $a > b$, then theorem 4 yields a distinct indifference set for each λ.

If there were only one indifference set, the problem of finding a measurable utility on $\mathcal{S}$ would be trivial, namely, define $u(a) = 1$ for every $a \in \mathcal{S}$. So we assume, henceforth, that there is more than one indifference set, that is, in view of theorem 5, there are an infinite number of them.

In the presence of theorem 4, theorem 1 can now be sharpened to

Theorem 6. <u>If</u> $a > b > c$ <u>then there is a unique</u> μ <u>so that</u> $b \sim \mu a + (1-\mu)c$.

Given $a > b$, let

$$\mathcal{S}_{ab} = \left\{ x \in \mathcal{S} \mid a \gtrsim x \gtrsim b \right\}.$$

For $x \in \mathcal{S}_{ab}$ we define $\mu_{ab}(x)$ by:

$$x \sim \mu_{ab}(x)\, a + (1-\mu_{ab}(x))b.$$

In light of theorem 6 this definition is meaningful. By theorem 4 $\mu_{ab}(x) > \mu_{ab}(y)$ if and only if $x > y$. As a trivial consequence of theorem 2 $\mu_{ab}(\alpha x + (1-\alpha)y) = \alpha\mu_{ab}(x) + (1-\alpha)\mu_{ab}(y)$ for all $x, y \in \mathcal{S}_{ab}$ and all α. Thus μ_{ab} has the requisite properties of being both linear and order preserving. It is then natural to expect these to serve as the building stones for our final measurable utility. All we need is a mechanism to extend μ_{ab} from $\mathcal{S}_{ab}$ to all of $\mathcal{S}$.

Pick an $r_1 > r_o$ and consider these as fixed henceforth. In what follows we consider only a, b which satisfy

(I) $$r_o, r_1 \in \mathcal{S}_{ab}.$$

For any $x \in \mathcal{S}_{ab}$ we define:

$$M_{ab}(x) = \frac{\mu_{ab}(x) - \mu_{ab}(r_o)}{\mu_{ab}(r_1) - \mu_{ab}(r_o)}$$

Clearly $M_{ab}(r_o) = 0$ and $M_{ab}(r_1) = 1$.

We need

Theorem 7. <u>Let</u> f, g <u>be linear functions defined on</u> $\mathcal{S}_{ab}$ <u>so that</u> $f(r_o) = g(r_o)$, $f(r_1) = g(r_1)$. <u>Then</u> f <u>is identical with</u> g <u>on</u> $\mathcal{S}_{ab}$.

Proof. Let $x \in \mathcal{S}_{ab}$. If $r_1 \gtrsim x \gtrsim r_o$, then $x \sim \alpha r_1 + (1-\alpha)r_o$, so

$f(x) = \alpha f(r_1) + (1-\alpha) f(r_o) = \alpha g(r_1) + (1-\alpha) g(r_o) = g(\alpha r_1 + (1-\alpha)r_o) = g(x)$.

If $x \gtrsim r_1 \gtrsim r_o$, then $r_1 \sim \mu x + (1-\mu)r_o, \mu > 0$

so $f(r_1) = \mu f(x) + (1-\mu) f(r_o) = g(r_1) = \mu g(x) = (1-\mu) g(r_o)$,

so $f(x) = g(x)$ since $\mu > 0$. Similarly if $r_1 \gtrsim r_o \gtrsim x$.

We are now ready to define our measurable utility u. For any $x \in \mathcal{S}$, pick $a > b$ so that

(I) $r_1, r_o \in \mathcal{S}_{ab}$

(II) $x \in \mathcal{S}_{ab}$.

We then define u by $u(x) = M_{ab}(x)$.

By theorem 7, for all pairs a, b and c, d, each satisfying (I), (II)

$M_{ab}(x) = M_{cd}(x)$, since $M_{ab}(r_o) = M_{cd}(r_o) = 0$, $M_{ab}(r_1) = M_{cd}(r_1) = 1$,

so we need not worry about the particular choice of a, b used in defining u. Given any x, y, pick a, b so all of $x, y, r_o, r_1 \in \mathcal{S}_{ab}$. Then

$u(x) > u(y)$ if and only if $x > y$ since μ_{ab}, and so M_{ab}, are order preserving in $\mathcal{S}_{ab}$. Similarly $u(\alpha x + (1-\alpha)y) = \alpha u(x) + (1-\alpha)u(y)$. Thus u is the desired measurable utility.

We have proved

Theorem 8. If a mixture set $\mathcal{S}$ satisfies axioms 1, 2 and 3, a measurable utility can be defined on $\mathcal{S}$.

Footnotes

1. Research of first author supported by contract between Cowles Commission for Research in Economics and ONR.

2. The authors are deeply indebted to Gerard Debreu for his suggestions and helpful comments.

3. This form of the axiom, rather than the more restrictive form

$$\alpha a + (1-\alpha)b \sim \alpha a' + (1-\alpha)b$$

was suggested to us by Debreu.

4. If this theorem had been taken as axiom 3 instead of the revision of axiom 3 we used, the derivations would be even simpler than they are. However, we felt that the gain, both in generality and intuitive appeal, of our axiom 3, was worth the complication thereby brought about in the mathematics.

References

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