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An Economic Equilibrium Existence Theorem

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Economic theory no longer accepts the once standard implication that if the equilibrium of an economic system can be described by a set of equations whose number matches the number of unknowns, an equilibrium point actually exists. A proof of exacting rigor is now required. Until very recently Wald's papers (see in particular [11]) contained some of the most satisfactory solutions of this problem. A remarkable paper by Arrow [1] on this question reached me while I was trying another attack. I could then easily bridge the gaps remaining in my work.

The main differences between [1] and this paper are:

The general existence theorem given below where convexity is replaced by contractibility and (perhaps the most significant feature in this context) the variable constraints $A_\iota(\tilde{a}_\iota)$ are introduced.

The solution by Arrow for his competitive economy introduces m fictitious players in addition to the m consumers, the n firms and the market, and uses a theorem of Kuhn and Tucker. No such artificial device occurs in the treatment below.

Arrow has considered the case where the set of technological possibilities of each firm is a closed, convex set, whereas I was limiting myself to the case where it is a closed, convex cone. I present the latter model, here, as an illustration of the general theory, although it applies just as well to Arrow's case.

Finally, the proof given in [1] is not entirely correct since, in particular, the function $V_i(p)$ (p.17) is clearly not continuous at the point $p = 0$,

and the assumptions of Arrow do not seem sufficient to insure the existence of an equilibrium.

1. An abstract theory of equilibrium.

Only subsets of finite Euclidean spaces will be considered here. Let there be ν agents characterized by a subscript $\iota = 1, \ldots, \nu$.

The ι^{th} agent chooses an action a_ι in a set $\mathcal{A}_\iota$. The ν-tuple of actions $(a_1, \ldots, a_\nu)$, denoted by a, is an element of $\mathcal{A}$, the Cartesian product $\mathcal{A}_1 \times \ldots \times \mathcal{A}_\nu$. The payoff to the ι^{th} agent is a function $f_\iota(a)$ from $\mathcal{A}$ to the completed real line. (See a simple definition in [4] p. 6.) Denote further by $\bar{a}_\iota$ the $(\nu - 1)$-tuple $(a_1, \ldots, a_{\iota-1}, a_{\iota+1}, \ldots, a_\nu)$ and by $\bar{\mathcal{A}}_\iota$ the product $\mathcal{A}_1 \times \ldots \times \mathcal{A}_{\iota-1} \times \mathcal{A}_{\iota+1} \times \ldots \times \mathcal{A}_\nu$.

Consider then a sequence of time points ..., t-1, t, t+1, ... and the following rule:

If a^t has been chosen at time t, at time t+1 the choice of the ι^{th} agent is restricted to a non-empty, compact set function of $\bar{a}^t_\iota$ $A_\iota(\bar{a}^t_\iota) \subset \mathcal{A}_\iota$. We assume that $f_\iota(\bar{a}^t_\iota, a^{t+1}_\iota)$ is a continuous function in a^{t+1}_ι on $A_\iota(\bar{a}^t_\iota)$ and that a^{t+1}_ι is chosen so as to maximize this function on this set. If a^t_ι satisfies this condition then $a^{t+1}_\iota = a^t_\iota$.

a^t is an equilibrium point if $a^t = a^{t+1}$ (and it follows from the rule that all subsequent $a^{t+2}, \ldots$ also equal a^t). More formally

Definition a^* is an equilibrium point if for all $\iota = 1, \ldots, \nu$ $a^*_\iota \in A_\iota(\bar{a}^*_\iota)$ and $f_\iota(a^*) = \operatorname{Max}_{a_\iota \in A_\iota(\bar{a}^*_\iota)} f_\iota(\bar{a}^*_\iota, a_\iota)$.

This action was first formalized by Nash [7], although in a less general way, in a different context. Arrow's paper [1] fully uses the power of Nash's formalization.

The graph of the function $A_\iota(\bar{a}_\iota)$ is defined as the subset of $\bar{\mathcal{A}}_\iota \times \mathcal{A}_\iota$, $G_\iota = \left\{(\bar{a}_\iota, a_\iota) \mid a_\iota \in A_\iota(\bar{a}_\iota)\right\}$. $A_\iota(\bar{a}_\iota)$ is always understood to be void for no $\bar{a}_\iota$.

We can now state (simple definitions of a polyhedron and of a contractible set are given in [4] p. 5-6).

Theorem. Let for all $\iota = 1, \ldots, \nu$, $\mathcal{A}_\iota$ be a contractible polyhedron, $A_\iota(\bar{a}_\iota)$ a multi-valued function from $\bar{\mathcal{A}}_\iota$ to $\mathcal{A}_\iota$ whose graph G_ι is closed, f_ι a continuous function from G_ι to the completed real line such that $\varphi_\iota(\bar{a}_\iota) = \operatorname{Max}_{a_\iota \in A_\iota(\bar{a}_\iota)} f_\iota(\bar{a}_\iota, a_\iota)$ is continuous. If for every ι and $\bar{a}$, the set $M\,\bar{a}_\iota = \left\{a_\iota \in A_\iota(\bar{a}_\iota) \mid f_\iota(\bar{a}_\iota, a_\iota) = \varphi_\iota(\bar{a}_\iota)\right\}$ is contractible, then there exists an equilibrium point.

The proof uses as a lemma a particular case of a fixed point theorem due to Begle [2].

Let Z be a set and ϕ a function associating with each $z \in Z$ a subset $\phi(z)$ of Z. We have defined above the graph of ϕ as the subset of $Z \times Z$, $\left\{(z, z') \mid z' \in \phi(z)\right\}$. ϕ is said to be semi-continuous if its graph is closed. A fixed point of ϕ is a point z^* such that $z^* \in \phi(z^*)$.

Lemma. Let Z be a contractible polyhedron and $\phi: Z \to Z$ a semi-continuous multi-valued function such that for every $z \in Z$ the set $\phi(z)$ is contractible. Then ϕ has a fixed point.

$\mathbb{A}$, the product of ν contractible polyhedra, is a contractible polyhedron ([4],p. 6). Define on $\mathbb{A}$ the multi-valued function ϕ as follows:

$$\phi(a) = M\,\bar{a}_1 \times \ldots \times M\,\bar{a}_\nu .$$

Since $M\,\bar{a}_\iota$ is contratible for all ι and $\bar{a}_\iota$, $\phi(a)$ is contractible for all $a \in \mathbb{A}$ ([4] p. 6). To be able to apply the lemma it remains only to show that ϕ is semi-continuous.

For this first define in $\bar{\mathbb{A}}_\iota \times \mathbb{A}_\iota$ the set

$$M_\iota = \left\{ (\bar{a}_\iota , a_\iota) \,\middle|\, a_\iota \in M_{\bar{a}_\iota} \right\}.$$

The equivalent definition

$$M_\iota = \left\{ (\bar{a}_\iota , a_\iota) \in G_\iota \,\middle|\, f_\iota (\bar{a}_\iota , a_\iota) = \varphi_\iota (\bar{a}_\iota) \right\}$$

shows that M_ι is closed since G_ι is closed and f_ι and φ_ι are continuous.

The graph Γ of ϕ is the subset of $\mathbb{A} \times \mathbb{A}$

$$\Gamma = \left\{ (a,a') \,\middle|\, a' \in \phi(a) \right\} = \left\{ (a,a') \,\middle|\, a'_\iota \in M_{\bar{a}_\iota} \text{ for all } \iota \right\}$$
$$= \left\{ (a,a') \,\middle|\, (\bar{a}_\iota , a'_\iota) \in M_\iota \text{ for all } \iota \right\}.$$

Consider the subset of $\mathbb{A} \times \mathbb{A}$,

$$\mathbb{M}_\iota = \left\{ (a,a') \,\middle|\, (\bar{a}_\iota , a'_\iota) \in M_\iota \right\}$$; $\mathbb{M}_\iota$ is closed since M_ι is. As $\Gamma = \bigcap_\iota \mathbb{M}_\iota$, Γ is closed.

The conclusion of the lemma is then that there exists $a^* \in \mathbb{A}$ such that $a^* \in \phi(a^*)$ i.e. for all ι $a^*_\iota \in M_{\bar{a}^*_\iota}$; this is the definition of an equilibrium point a^*.

To conclude this section, we make a remark pointing out a wide class of cases in which $\varphi_\iota (\bar{a}_\iota)$ is continuous.

The function $A_\iota(\bar{a}_\iota)$ is said to be continuous at $\bar{a}^o_\iota$ if for any $a^o_\iota \in A_\iota(\bar{a}^o_\iota)$ and any sequence $(\bar{a}^n_\iota)$ converging to $\bar{a}^o_\iota$, there exists a sequence (a^n_ι) converging to a^o_ι such that for all n $a^n_\iota \in A_\iota(\bar{a}^n_\iota)$.

Remark. If $A_\iota(\bar{a}_\iota)$ has a compact graph G_ι and is continuous at $\bar{a}^o_\iota$, if f_ι is a continuous function from G_ι to the completed real line, then $\varphi_\iota(\bar{a}_\iota)$ is continuous at $\bar{a}^o_\iota$.

We drop everywhere the subscripts ι and reason as if f took its values in the real line (the isomorphism $\frac{e^f-1}{e^f+1}$ between the completed real line and the closed interval [-1, +1] immediately extends the results to the general case).

α) Using only the compactness of G and the continuity of f we first prove:

For any sequence $(\bar{a}^n)$ converging to $\bar{a}^o$ and any $\varepsilon > 0$, there is an N such that $n > N$ implies $\varphi(\bar{a}^n) < \varphi(\bar{a}^o) + \varepsilon$ (in other words, $\varphi(\bar{a})$ is upper semi-continuous at $\bar{a}^o$). For every n, choose $a^n \in A(\bar{a}^n)$ such that $f(\bar{a}^n, a^n) = \varphi(\bar{a}^n)$. Since G is compact it is possible to extract from the sequence $(\bar{a}^n, a^n)$ a sub-sequence $(\bar{a}^{n'}, a^{n'})$ converging to $(\bar{a}^o, a^o)$.

By continuity of f, $f(\bar{a}^{n'}, a^{n'})$ [which is $= \varphi(\bar{a}^{n'})$] tends to $f(\bar{a}^o, a^o)$ [which is $\leq \varphi(\bar{a}^o)$]. Therefore there exists N' such that $n' > N'$ implies $\varphi(\bar{a}^{n'}) < \varphi(\bar{a}^o) + \varepsilon$. Since from any sequence $(\bar{a}^n)$ converging to $\bar{a}^o$ it is possible to extract a subsequence $(\bar{a}^{n'})$ having the desired property, any sequence $(\bar{a}^n)$ converging to $\bar{a}^o$ has the property.

β) Using in addition the continuity of $A(\bar{a})$ at $\bar{a}^o$ we prove:

For any sequence $(\bar{a}^n)$ converging to $\bar{a}^o$ and any $\varepsilon > 0$, there is an N such that $n > N$ implies $\varphi(\bar{a}^n) > \varphi(\bar{a}^o) - \varepsilon$ (in other words, $\varphi(\bar{a})$ is lower semi-continuous at $\bar{a}^o$).

Choose $a^o \in A(\bar{a}^o)$ such that $f(\bar{a}^o, a^o) = \varphi(a^o)$. By continuity of $A(\bar{a})$ at $\bar{a}^o$, there is a sequence (a^n) converging to a^o such that for all n, $a^n \in A(\bar{a}^n)$. By continuity of f, $f(\bar{a}^n, a^n)$ [which is $\leq \varphi(\bar{a}^n)$] tends to $f(\bar{a}^o, a^o)$ [which is $= \varphi(\bar{a}^o)$]. Therefore there exists N such that $n > N$ implies $\varphi(\bar{a}^n) > \varphi(\bar{a}^o) - \varepsilon$.

α) and β) together naturally prove that $\varphi(\bar{a})$ is continuous at $\bar{a}^o$.

2. Application to a competitive economic system.

The general theorem will now be used to prove that the economy described below has an equilibrium point. This particular case obviously does not use all the power of the existence theorem.

Notations are those of [3] to which the reader is referred for a more detailed presentation of the concepts. I borrow from Arrow [1] the treatment of the limitation of the quantities of labor. Otherwise this model is the one I presented at the Washington symposium on linear inequalities (June 1951). Assumptions will be numbered with Roman numerals.

I There is a finite number of commodities.

Commodities are characterized by a subscript $h = 1, \ldots, \ell$. The subscripts corresponding to the different kinds of labor form a set H.

In R^{ℓ}, the commodity space, the i^{th} consumption-unit ($i = 1, \ldots, m$) chooses a commodity bundle x_i.

IIa If $h \notin H$, $x_{hi} \geqq 0$

IIb If $h \in H$, $x_{hi} \leqq 0$

(the opposite of the quantity of the h^{th} kind of labor produced by the i^{th} consumption-unit is taken); moreover if the elementary time interval is taken as the time unit and if labor services of any kind are measured in time units then

IIc $\sum_{h \in H} x_{hi} \geqq -1$.

The domain X to which x_i must belong is determined by IIa, IIb, IIc.

The points of X are completely ordered by the preferences of the i^{th} consumption-unit; it is assumed that this ordering can be represented by a satisfaction function $s_i(x_i)$ (defined but for a monotonic increasing increasing transformation).

IIIa $s_i(x_i)$ is a continuous function to the completed real line

IIIb "$x_i^2 \geqq x_i^1$" implies "$s_i(x_i^2) \geqq s_i(x_i^1)$"

IIIc "$s_i(x_i^2) > s_i(x_i^1)$" implies "$s_i[tx_i^2 + (1-t)\, x_i^1] > s_i(x_i^1)$ if $0 < t \leqq 1$."

IIIc and IIIa naturally imply that

(1) $$\left\{x_i \in X \;\middle|\; s_i(x_i) \geqq s_i^o\right\} \text{ is convex for all } s_i^o.$$

The resources of the economy are described by a vector $z^o \in R^\ell$ ($z_h^o = 0$ if $h \in H$); they are privately owned, the share of the i^{th} consumption-unit being z_i^o.

IVa $z_i^o \geqq 0$ is fixed

IVb $z_{hi}^o = 0$ if $h \in H$

IVc $z^o = \sum_i z_i^o$

There exists a price vector $p \geq 0$ in R^ℓ which every agent considers as a datum.

The i^{th} consumption-unit tries to maximize $s_i(x_i)$ by choosing x_i in X subject to a budgetary constraint[1].

V Given p, $s_i(x_i)$ is maximized subject to $p \cdot x_i = p \cdot z_i^o$.

The activity of the j^{th} production-unit ($j = 1, \ldots, n$) is described by an input-vector $y_j \in R^\ell$ (whose positive components are inputs, negative components opposites of outputs). It is restricted to belong to the set of technological possibilities $\mathcal{Y}_j$.

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1. One might think of imposing the constraint $p \cdot x_i \leqq p \cdot z_i^o$, but because of IIIb a consumption-unit is never harmed by restricting itself to $p \cdot x_i = p \cdot z_i^o$.

The positive orthant of R^{ℓ} (the set of all points of R^{ℓ} with non-negative coordinates) is denoted by Ω, the positive orthant translated from 0 to a point q by $\Omega_{(q)}$; similarly for the negative orthant $-\Omega$ and $-\Omega_{(q)}$.

It is then assumed that:

VIa $\mathcal{Y}_j$ is a closed, convex cone with vertex 0.

We postulate that $\mathcal{Y}_j \cap (-\Omega) = 0$, i.e., that it is impossible to produce without consuming anything ([6], Chapter III). This implies that $\mathcal{Y}_j$ has a polar[2] $\mathcal{Y}_j^+$ which intersects $\hat{\Omega}$ the interior of Ω.[3] We will further assume

VIb $(\bigcap_j \mathcal{Y}_j^+) \cap \hat{\Omega} \neq \emptyset$

Denote by $y = \sum_j y_j$, the global input vector; it belongs to $\mathcal{Y} = \sum_j \mathcal{Y}_j$. $\mathcal{Y}$ is clearly a convex cone with vertex 0. Since $\mathcal{Y}^+ = \bigcap_j \mathcal{Y}_j^+$, VIb means that $\mathcal{Y}$ has a positive normal $k > 0$ at 0; it is thus justified by the fact that the impossibility of producing without consuming which holds for each one of the $\mathcal{Y}_j$ must also hold for $\mathcal{Y}$.

Given p, the j^{th} production-unit chooses y_j in $\mathcal{Y}_j$ so as to maximize its profit $-p \cdot y_j$.

VII Given p, $-p \cdot y_j$ is maximized subject to $y_j \in \mathcal{Y}_j$.

If p does not belong to the polar of $\mathcal{Y}_j$, the y_j chosen is infinitely large and no equilibrium can occur in this case. If p belongs to the interior of the polar, the y_j chosen is 0 (the technology of this production-unit is not used). If p belongs to the boundary of the polar, all points of $\mathcal{Y}_j \cap p^{\perp}$, where $p^{\perp}$ is the plane through 0 normal to p,

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2. The polar Y^+ of a cone Y with vertex 0 is $\{\eta \mid \eta \cdot y \geq 0 \text{ for all } y \in Y\}$

3. The proof, almost trivial, uses the fact that $\mathcal{Y}_j^{++} = \mathcal{Y}_j$. (See Seminar on Convex Sets, The Institute for Advanced Study 1949-1950, Chapter I-II.)

maximize $-p \cdot y_j$ (all these points are equivalent and give a zero profit). Obviously an equilibrium p will turn out to be such that $p \in \mathbb{Y}_j^+$ for all j, i.e., $p \in \mathbb{Y}^+$.

Thus on the basis of p, m x_i's and n y_j's are chosen. Denoting $x = \sum_i x_i$, $z = x + y$ is the total net demand. Necessarily $z \leqq z^o$. It is the role of the market to realize this by varying p. Moreover when equilibrium is reached any commodity for which $z_h < z_h^o$ must have a zero price $p_h = 0$ (the owners of the h^{th} resource would otherwise start underbidding each other to sell their excess supply); such a commodity is a free good.

VIII $z = \sum_i x_i + \sum_j y_j$, $z \leqq z^o$, and $z_h < z_h^o$ implies $p_h = 0$.

Multiplying p by a positive scalar would not alter the decisions or the conditions described above; we will therefore assume from now on that

IX $p \in S = \left\{ p \in R^{\ell} \,\middle|\, p \geq 0, \sum_h p_h = 1 \right\}$.

To complete the description of the economy we now only need, for the market, a payoff function which "he" tries to maximize. Since we are not concerned with the dynamic evolution of the system moving toward equilibrium, we require only that this function satisfy the following condition: an equilibrium position reached on the basis of its maximization fulfills all the requirements listed above. $p \cdot (z - z^o)$ does this ($\frac{p \cdot z}{p \cdot z^o}$ [3] is maybe even preferable). No fully satisfactory justification for this choice can be given at this stage and only one remark will be made here. Maximize $p \cdot (z - z^o)$ for a given z means, for the market, select the commodities for which $z_h - z_h^o$ (the excess demand) is the largest; let $\mathbb{H}$ be the set of those commodities.

If $h \notin \mathbb{H}$, set $p_h = 0$.

If $h \in \mathbb{H}$, choose $p_h \geqq 0$, $\sum_{h \in \mathbb{H}} p_h = 1$.

It is clearly a very brutal adjustment process but, as we emphasized already, this is irrelevant for a proof of existence of equilibrium.

This model is almost ready for identification with the general theory. However X, the domain of the x_i, and the $\mathbb{Y}_j$, the domains of the y_j, are not compact. This can be easily obviated.

If $-\omega$ is the point with coordinates $\omega_h = -1$ if $h \in H$, $\omega_h = 0$ if $h \notin H$, $X \subset \Omega(-\omega)$, and therefore $x = \sum_{i=1}^{m} x_i$ belongs to $\Omega(-m\omega)$. Since for an equilibrium point (if it exists at all) $x + y \leqq z^o$, y must belong to $-\Omega(z^o + m\omega)$ and the domain of y to be considered need not be the whole cone $\mathbb{Y}$ but only $\hat{\mathbb{Y}} = \mathbb{Y} \cap -\Omega(z^o + m\omega)$. This set is obviously not empty since $z^o \geqq 0$ and $\omega \geqq 0$. It is <u>bounded</u> since, calling k^+ the polar of k (see VIb), $\mathbb{Y} \cap -\Omega(z^o + m\omega) \subset k^+ \cap -\Omega(z^o + m\omega)$ which is bounded since $k > 0$. As $y = \sum_j y_j$, y_j necessarily belongs to $\hat{\mathbb{Y}}_j$ a bounded subset of $\mathbb{Y}_j$; similarly $x \leqq z^o - y$ shows that x_i necessarily belongs to $\hat{X}$ a bounded subset of X, which contains all the z_i^o. Finally, take a cube $C = \left\{ x \;\middle|\; |x_h| \leqq b \text{ for all } h \right\}$ such that its interior contains $\hat{X}$ and all the $\hat{\mathbb{Y}}_j$ and take its intersections $\tilde{X}$ (resp. $\tilde{\mathbb{Y}}_j$) with X (resp. $\mathbb{Y}_j$). The domain of x_i (resp. y_j) will be $\tilde{X}$ (resp. $\tilde{\mathbb{Y}}_j$) compact, convex sets.

The identification with the general theory is finally as follows: There are $\nu = m + n + 1$ agents. For $\iota = 1, \ldots, m$ the agent is a consumption unit, its action is $a_i = x_i$; $\mathbb{A}_i = \tilde{X}$ which is compact, convex (therefore trivially a contractible polyhedron); $A_i(\bar{a}_i) = \left\{ x_i \in \tilde{X} \;\middle|\; p \cdot x_i = p \cdot z_i^o \right\}$, it is always non-empty since $z_i^o \in \tilde{X}$, its graph G_i is clearly closed; $f_i(a) = s_i(x_i)$ which is continuous on $\tilde{X}$; $\varphi_i(\bar{a}_i)$ is continuous (the proof will be postponed until the end of this section); finally for every $p \in S$, the set of x_i which maximize

$s_i(x_i)$ subject to $p \cdot x_i = p \cdot z_i^o$ is convex by (1) (therefore trivially contractible).

For $\iota = m + 1, \ldots, m + n$ the agent is a production unit, its action is $a_{m+j} = y_j$; $\mathbb{A}_{m+j} = \widetilde{\mathcal{Y}}_j$ which is compact, convex; $A_{m+j}(\bar{a}_{m+j}) = \widetilde{\mathcal{Y}}_j$ which satisfies trivially all the requirements; $f_{m+j}(a) = -p \cdot y_j$ which is continuous; finally for every $p \in S$ the set of y_j which maximize $-p \cdot y_j$ is convex.

For $\iota = m + n + 1 = \nu$ the agent is the market, its action is $a_\nu = p$; $\mathbb{A}_\nu = S$, compact, convex; $A_\nu(\bar{a}_\nu) = S$; $f_\nu(a) = p \cdot (z - z^o)$, continuous; finally for every z, the set of p which maximize $p \cdot (z - z^o)$ is convex.

All the conditions of the theorem are satisfied so there exists an equilibrium point x_i^*, y_j^*, p^*.

Since y_j^* maximizes $-p^* \cdot y_j$ over $\widetilde{\mathcal{Y}}_j$ which contains 0

(2) $$p^* \cdot y_j^* \leqq 0 \text{ for all } j \ .$$

On the other hand

(3) $$p^* \cdot x_i^* = p^* \cdot z_i^o \text{ for all } i \ .$$

Adding (2) and (3) one finds

(4) $$p^* \cdot z^* \leqq p^* \cdot z^o \quad \text{i.e.} \quad p^* \cdot (z^* - z^o) \leqq 0.$$

As p^* maximizes $p \cdot (z^* - z^o)$ in S, $z^* - z^o \leqq 0$.

y_j^* which maximizes $-p^* \cdot y_j$ over $\widetilde{\mathcal{Y}}_j$ also maximizes it over $\mathcal{Y}_j$ (we must make sure that the artificial limitation introduced on the domain of y_j has not changed the problem in any respect). Suppose that there is in $\mathcal{Y}_j$ a y_j' such that $-p^* \cdot y_j' > -p^* \cdot y_j^*$ then for all t, $0 < t \leqq 1$ $-p^* \cdot [ty_j' + (1-t)y_j^*] > -p^* \cdot y_j^*$.

Since $\widetilde{\mathcal{Y}}_j$ contains a neighborhood of y_j^* (in $\mathcal{Y}_j$), by choosing t small enough, one has a contradiction.

As an immediate consequence, $p^* \cdot y_j^* = 0$ for all j and (4) is actually $p^* \cdot (z^* - z^o) = 0$. This implies that if for any commodity $z_h^* < z_h^o$, then $p_h = 0$. This is the complete justification of the choice of $p \cdot (z - z^o)$ as a payoff function for the market.

Finally x_i^* which maximizes $s_i(x_i)$ in $\widetilde{X} \cap \left\{x_i \mid p^* \cdot x_i = p^* \cdot z_i^o\right\}$ also maximizes it in $X \cap \left\{x_i \mid p^* \cdot x_i = p^* \cdot z_i^o\right\}$. This follows from IIIc; the proof is the same as above.

One fact, the continuity of $\varphi_i(\bar{a}_i)$, has been left unproved, we come to it now.

Let us call P the plane of equation $p \cdot x_i = p \cdot z_i^o$ normal to $p \in S$ through z_i^o.

1) If P is not supporting for $\widetilde{X}$, it is easily shown that $A_i(\bar{a}_i)$ is continuous (and therefore by the remark of Section 1 $\varphi_i(\bar{a}_i)$ continuous) for this value of p.

Take q^o arbitrary in $\widetilde{X}$ such that $p \cdot q^o = p \cdot z_i^o$,

take $q' \in X$ such that $p \cdot q' < p \cdot z_i^o$,

$q'' \in X$ such that $p \cdot q'' > p \cdot z_i^o$.

If (p^n) is a sequence converging to p, q^n, the point where P^n intersects the broken line $q'q^oq''$, tends to q^o.

2) If P is supporting for $\widetilde{X}$, denote by v the point with coordinates $v_h = b$ if $h \notin H$, $v_h = 0$ if h H. By definition of $\widetilde{X}$, $v \in X$, $-\Omega(v) \supset \widetilde{X}$, it can even be said that in a small neighborhood of v, $\widetilde{X}$ is identical with $-\Omega(v)$.

From IIIb follows

(5) $\quad$ for any $x_i \in \tilde{X}$, $s_i(v) \gtrless s_i(x_i)$.

Since $v \gtrless z_i^o$ and $p \geq 0$, $v \in P$. When p^n tends to p (in S), there are points q^n of $P^n \cap \tilde{X}$ arbitrarily close to v , the inequalities

$$s_i(q^n) \leqq \operatorname*{Max}_{x_i \in P^n \cap \tilde{X}} s_i(x_i) \leqq s_i(v)$$

then prove that $\operatorname*{Max}_{x_i \in P^n \cap \tilde{X}} s_i(x_i)$ is continuous for this

value of p.

3. Equilibrium points and saddle points.

By taking particular cases of the existence theorem one obtains all known equilibrium point and saddle point existence theorems.

In all these cases, for all ι and $\bar{a}_\iota$ $A_\iota(\bar{a}_\iota) = \mathbb{A}_\iota$. Taking for the $\mathbb{A}_\iota$ compact, convex sets, for f_ι a real, continuous function quasi-concave in a_ι, one obtains Arrow's Lemma 2 [1].

Restricting further the $\mathbb{A}_\iota$ and the f_ι one obtains Nash's theorem [7]. A saddle point for a continuous real function $f(x_1, x_2)$ of two variables $x_1 \in X_1$, $x_2 \in X_2$ (X_1 and X_2 compact) is a point (x_1^o, x_2^o) such that

$$\operatorname*{Max}_{x_1 \in X_1} f(x_1, x_2^o) = f(x_1^o, x_2^o) = \operatorname*{Min}_{x_2 \in X_2} f(x_1^o, x_2).$$

It is obviously an equilibrium point for two agents with payoff functions $f_1 = f$ and $f_2 = -f$. One obtains therefore immediately the theorem of [4], and as more and more particular cases Kakutani's [5], von Neumann's [13], [12], and von Neumann and Morgenstern's theorems [14].

References

[1] Arrow, K. J., "On the Existence of Solutions to the Equations of General Equilibrium under Conditions of Perfect Competition." (Unpublished — in Cowles Commission library).

[2] Begle, E. G., "A Fixed Point Theorem," Annals of Mathematics, Vol. 51, No. 3, May, 1950, p. 544-550.

[3] Debreu, G., "The Coefficient of Resource Utilization," Econometrica, July, 1951.

[4] Debreu, G., "Saddle Point Existence Theorems," C.C.D.P. Mathematics 412.

[5] Kakutani, S., "A Generalization of Brouwer's Fixed Point Theorem," Duke Mathematical Journal, Vol. 8, No. 3, Sept. 1941, p. 457-459.

[6] Koopmans, T. C., ed., Activity Analysis of Production and Allocation, Cowles Commission Monograph No. 13, J. Wiley and Sons, New York, 1951.

[7] Nash, J. F., "Equilibrium Points in N-Person Games," Proceedings of the National Academy of Science, U.S.A., Vol. 36, 1950, p. 48-49.

[8] Neumann, J. von, "Zur Theorie der Gesellschaftsspiele," Mathematische Annalen, Vol. 100, 1928, p. 295-320.

[9] ______________, "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes," Ergebnisse eines Mathematischen Kolloquiums, Vol. 8 (1937), p. 73-83 (Translated in The Review of Economic Studies, Vol. XIII, No. 33, (1945-46), p. 1-9.

[10] ______________, and Morgenstern, O., Theory of Games and Economic Behavior, Princeton, Princeton University Press, 2nd edition, 1947, (1st edition, 1944).

[11] Wald, A., "On Some Systems of Equations of Mathematical Economics," Econometrica, October 1951.

Errata and Corrections

for

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Gerard Debreu

March 14, 1952

Page 3, Line 5 from top:	"notion" instead of "action".
11 " bottom:	"$M_{\bar{a}_i}$" instead of "$M\bar{a}_i$" and everywhere this symbol occurs).
Page 4, Line 4 from top:	"contractible" instead of "contratible".
Page 5, Line 10 " " :	new line after "... at $\bar{a}^o$)."
4 " bottom:	"$\bar{a}^o$" instead of "a^o".
Page 7, Line 4 " top:	"$s_i(x_i^2) \geqq s_i(x_i^1)$" instead of "$s_i(x_i^2) \neq s_i(x_i^1)$".
10 " top:	$z_{hi}^o > 0$ if $h \notin H$ instead of $z_i^o \gneqq 0$.
Page 8, Line 1 " bottom:	"$p^{\perp}$ " instead of "p^{1} " in both places.
Page 10, Line 13 " top:	$z^o \ngeq 0$ instead of $z^o \gneqq 0$.
7 " top:	new line after "agents."
Page 12, Line 7 " " :	symbol $\cap$ missing after "$\tilde{X}$" and before brace.
Lines 7 & 8 from bottom:	$\tilde{X}$ instead of X.
Line 3 from " :	$v_h = 0$ if $h \in H$, ..., $v \in \tilde{X}$.