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# On the Interpretation of Professor Leontief's System

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Leontief's input-output tables are said to yield a set of technical coefficients showing how physical input is transformed into physical output. In the customary exposition[1] we have:

$x_i$ = output of the i-th sector

$x_{ik}$ = output of the k-th sector used in the i-th sector.

The technical coefficients are defined as

$$(1) \qquad a_{ik} = \frac{x_{ik}}{x_i} .$$

The $a_{ik}$ are the elements, assumed constant, of the input-output table.

One interpretation of (1) is that it is a technological production function, with the condition that inputs are used in fixed proportions; that is to say

$$(2) \qquad a_{ij} = \frac{x_{ij}}{x_i} \qquad j \neq k$$

is an equally valid production function for output $x_i$. In the framework of this type of Leontief model, Samuelson[2] has shown that the same set of observed technical constants could have been derived as elements of the input-

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1. See e.g., W. Leontief, Structure of the American Economy, Cambridge, Mass., 1941.

2. P. A. Samuelson, Ch. VII, Activity Analysis of Production and Allocation, Wiley, 1951.

output matrix in a competitive economy if the technological production function

$$(3) \qquad F(x_i, x_{i1}, \ldots, x_{in}) = 0$$

were homogeneous of degree zero, permitting substitution among different inputs contrary to the assumptions of (1) or (2). Samuelson's treatment explicitly assumes that each sector produces only a single output. He rules out joint production.

Even if input-output tables are refined to 1000 x 1000 classifications, the problem of joint production cannot be avoided. It is simply true, in most cases at least, that the $a_{ik}$ are computed as the ratios of two values—the value of output of the k-th sector used in the i-th sector divided by the value of the output of the i-th sector. Joint production is the rule and not a special case.

The purpose of this note is to show that even in the case of joint production in a competitive economy, there exists a production function permitting substitution among inputs and yielding a set of constant elements of an input-output table. The approach used here and the results obtained are not strictly parallel to those of Samuelson in the one-commodity-per-industry case, but they do have a point in common, namely, to show that substitutibility as an alternative to fixed proportions is consistent with Leontief's empirical findings and theoretical model.

Let $x_i^{(s)}$ = s-th output of sector i

$x_{ik}^{(r)}$ = r-th output of sector k used in i.

The elements of an input-output table are

$$(4) \qquad a_{ik} = \text{const.} = \frac{\sum_r p_k^{(r)} x_{ik}^{(r)}}{\sum_s p_i^{(s)} x_i^{(s)}} .$$

The problem is to determine the characteristics of a technological relation, independent of market phenomena like prices,

$$(5) \quad F_i(x_i^{(1)} \ldots x_i^{(S)}, x_{i1}^{(1)} \ldots x_{i1}^{(R_1)}, \ldots, x_{in}^{(1)}, \ldots, x_{in}^{(R_n)} = 0, \; i=1,2,\ldots, n,$$

such that (4) holds. In order to solve this problem some assumptions must be made to connect the pricing system to the technology. In a competitive market

$$(6) \qquad \frac{\partial F_i}{\partial x_i^{(s)}} = - \lambda_i \, p_i^{(s)} ,$$

$$(7) \qquad \frac{\partial F_i}{\partial x_{ik}^{(r)}} = \lambda_i \, p_k^{(r)} \quad ;$$

hence the ratio in (4) can be written as

$$(8) \qquad \frac{\sum_r \frac{\partial F_i}{\partial x_{ik}^{(r)}} x_{ik}^{(r)}}{- \sum_r \frac{\partial F_i}{x_i^{(s)}} x_i^{(s)}} = a_{ik}; \quad k = 1, 2, \ldots, i-1, i+1, \ldots, n.$$

Equations (8) provide a set of partial differential equations whose solution is the technological function in (5).

Multiply both sides of (8) by the denominator of the left hand side and sum over input sectors k,

$$(9) \qquad \sum_k \sum_r \frac{\partial F_i}{\partial x_{ik}^{(r)}} x_{ik}^{(r)} = - \sum_k a_{ik} \sum_s \frac{\partial F_i}{\partial x_i^{(s)}} x_i^{(s)} .$$

If

$$(10) \qquad \sum_k a_{ik} = 1,$$

equation (9) is immediately recognized as Euler's equation for homogeneous functions of degree zero. Euler's equation is both necessary and sufficient for the existence of an $F_i$ - function with the stated homogeneity property. Condition (10) is a condition of zero profits, and it is, of course, obvious that a competitive economy paying factors their marginal productivities and distributing the entire value of production will imply a production function, $F_i$, homogeneous of degree zero in all inputs and outputs.[3]

For the case

(11) $$\sum_k a_{ik} \neq 1,$$

especially where this sum is less than unity, H. Rubin suggests the following transformation in order to put (9) in the form of Euler's equation for homogeneous functions:

(12) $$y_i^{(s)} = [x_i^{(s)}]^{\frac{1}{\Sigma a_{ik}}} .$$

Substitution of (12) into (9) yields

(13) $$\sum_k \sum_r \frac{\partial F_i}{\partial x_{ik}^{(r)}} x_{ik}^{(r)} = - \sum_s \frac{\partial F_i}{\partial y_i^{(s)}} y_i^{(s)},$$

which, according to Euler's theorem, is satisfied by a function $F_i$ homogeneous of zero degree in the $x_{ik}^{(r)}$ and $[x_i^{(s)}]^{\frac{1}{\Sigma a_{ik}}}$. If (10) holds, the homogeneity is in $x_{ik}^{(r)}$ and $x_i^{(s)}$.

The elements of Leontief's input-output table may thus be interpreted as parameters of some production functions all of which permit substitution among factors of production and types of output. Moreover, the $a_{ik}$ can be interpreted as technological parameters.

This interpretation is applicable only if equations (4), (6), and (7) hold. Insofar as the $a_{ik}$ are determined purely from engineering information

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3. The classical difficulty remains that second order conditions for profit maximization are not satisfied if the $F_i$ are homogeneous of zero degree.

without any assumptions that different products are combined in proportion to relative prices, equations (4) do not hold and the above results about substitutability do not apply. In earlier publications of Leontief, equations of the type in (4) were used to determine the $a_{ik}$. Even engineering information, however, must make assumptions about the proportions in which different types of product or input are combined. In many or most cases, these proportions are current relative prices or time averages of them. For example in studying railroad traffic, one might use

traffic units = freight ton miles + 2.4 passenger miles

as a composite output variable since it is not possible to make a complete separation of operations into those dealing exclusively with freight and exclusively with passenger service. The coefficient 2.4 is not a technological parameter; it is the mean price ratio of the two types of service averaged over many years.

In the event that the market structure is not competitive, equations (6) and (7) do not hold. Corresponding equations for a noncompetitive market might be

$$\frac{\partial F_i}{\partial x_i^{(s)}} = -\lambda_i p_i^{(s)} \left(1 - \frac{1}{\eta_i^{(s)}}\right), \tag{14}$$

$$\frac{\partial F_i}{\partial x_{ik}^{(r)}} = \lambda_i p_k^{(r)} \left(1 + \frac{1}{\varepsilon_k^{(r)}}\right), \tag{15}$$

in which $\eta_i^{(s)}$ and $\varepsilon_k^{(r)}$ are demand and supply elasticities respectively. In this case the equation system consists of (4), (14), (15), the demand equations for outputs, and the supply equations of inputs. Except under very special circumstances, the $a_{ik}$ will not be parameters of the technological equations alone; they will also occur as parameters in equations of economic behavior. This is perhaps the most realistic interpretation of Leontief's system.