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On Optimal Communication Rules for Teams[1]

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1. General

1.1. We call a team a group of persons/(members) acting for a common goal and bound by certain rules. The team problem is one of finding the rules best suited to achieve the goal. The team concept is related to the more general concept of organization; but this will not be discussed in this paper (see 2.7.7).

1.2. More precisely, consider a team of n members. At a certain point of time the i-th member finds himself in a certain "situation" or "state of information," s_i, an element of the set S_i of all possible states of his information. Let A_i be the set of all possible actions a_i that the i-th person can undertake upon receiving the information. A rule for the i-th person, r_i establishes a correspondence (one-to-one or many-to-one) between the elements of S_i and those of A_i:

$$(1.1) \qquad r_i: \quad S_i \rightarrow A_i \qquad i=1, \ldots, n$$

The Cartesian products $S=S_1 \times S_2 \times \ldots \times S_n$ and $A=A_1 \times A_2 \times \ldots \times A$ represent, respectively, the set of joint situations and the set of joint actions of the team members. The set $R \equiv (r_1, \ldots, r_n)$ is the team rule; and (1.1) can be

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rewritten as the mapping of S into A :

(1.2) $$R : \quad S \rightarrow A \text{ ; or } A = R(S).$$

1.3. To every combination of a joint situation $s \in S$ and a joint action $a \in A$ corresponds a <u>prospect</u>, π : a probability distribution of physical results. We have

(1.3) $$\pi = \pi(S,A) = \pi(S,R(S)) = \phi(S,R),$$

say: the prospect depends on the joint situation and on the team rule; the function ϕ can be regarded as a "random production function."

The existence of a <u>goal</u> for the team means that prospects are completely ordered. (The case of partial ordering of prospects would, however, deserve a separate study). We assume, even more stringently, that the other "Neumann-Morgenstern" axioms (or some variants of those) are satisfied, so that there exists a utility function (unique up to linear transformation) exists, to be denoted by

(1.4) $$w(p) = w[\phi(S,R)] = u(S,R) = u,$$

say. We shall also write

(1.5) $$u = u(S,R;\ w)$$

to emphasize that utility can be regarded as a function on the sets S and R, and a functional of the utility function of prospects.

1.4. Uncertainty enters our problem twice. First, because the physical result of a situation and an action is random: this is the aspect we have just dealt with in 1.3. Second, uncertainty arises inasmuch as one does not know which element of the set S will actually occur. We shall assume that one knows the probability distribution F on set S. The axioms mentioned in 1.3 will imply that the organizer of the team will choose the rule $R = \hat{R}$ so as to maximize the expectation

(1.6) $$Eu = \int_S u(S,R;\ w)dF(S) \equiv U(R,F;\ w),$$

say. An "efficient rule" $\hat{R}$ is thus defined by

(1.7) $$\underset{R}{\text{Max}}\ U(R,F;\ w) = U(\hat{R},\ F;\ w);$$

hence $\hat{R}$ depends only on F, the distribution of situations, and on w, the utility function of prospects:

(1.8) $$\hat{R} = \hat{R}(F,\ w),$$

say.

1.5. Our discussion has been confined (and will remain thus confined in this paper) to the "static" or "one-interval" case. In general, a temporal sequence of situations, actions, situations, actions... should be considered; but we do not go into this now.

2. An Example

2.1. Consider a firm which consists of two partners: one of them specializes in the market of the firm's (unique) product; the other, in the market of the (unique) raw material used by the firm. Let x be the price of the product; let y' be the price of the raw material. For the purposes of our example, we normalize the price scales as follows:

(2.1) $$x_{min} = 0\ ;\ \ x_{max} = 1\ ;\ \ y_{min} = 0\ ;\ \ y'_{max} = 1\ ;\ \ 1 - y' = y\ .$$

In the notation of Section 1,

$x \in [0,\ 1]$ corresponds to $s_1 \in S_1$

$y \in [0,\ 1]$ " " $s_2 \in S_2$.

The unit square corresponds to S .

2.2. Suppose each partner can commit the firm only in the following way: upon his order, the firm purchases one unit of raw material and at the same time sells one unit of product. (We imagine the firm to be able to draw on an infinite inventory; or consider it as a member of a commodity exchange). Let the set A_i of actions a_i open to the i-th partner $(i=1,2)$ be:

$$(2.2)\qquad \begin{cases} a_i' : \text{commit the firm;} \\ a_i'' : \text{phone the other partner, at a cost } 2c \text{ to the firm;} \\ a_i''' : \text{don't do anything.} \end{cases}$$

The team rule R will consist of elements r_1 and r_2 where r_1 is a partition of the set $S_1 = [0,1]$ into three subsets such that

$$(2.2a)\qquad a_1 = \begin{cases} a_1' & \text{if } x \in S_1' \\ a_1'' & \text{if } x \in S_1'' \\ a_1''' & \text{if } x \in S_1''' \,; \end{cases}$$

and similarly with r_2 (respecting y). The problem consists in selecting the best rule $\hat{R} = (\hat{r}_1, \hat{r}_2)$ so as to maximize the expected value U of the firm's profit, u. (It will be seen in 2.5 that this agrees exactly with the notation of Section 1.)

2.3. Let the amount added to the firm's profit as the result of a commitment (i.e., of action a_i') on the part of any one of the partners be

$$(2.3)\qquad x - ky' = x - k(1-y)\ ; \quad 0 < k < 1 \ .$$

The coefficient k gives the proportion of the raw material price to the cost of the finished product. If k is small, the information obtained by the second partner is "relatively unimportant" and this fact will presumably affect the choice of the rule. In the example worked out in detail we have assumed $k=1$. This is roughly the case of a firm speculating in a single commodity, possibly bought and sold in two different places; so that x=selling price, and $y'=1-y$=buying price including transportation. Then the profit on the commitment is

(2.3a) $$x + y - 1,$$

a symmetrical function of x and y.

2.4. We can now proceed to evaluating the firm's profit u for all possible pairs (x,y). By (2.3a)

(2.4a)
$$\begin{aligned}
&(x \in S_1', \; y \in S_2') &&\text{implies } u = 2(x+y-1)\\
&(x \in S_1''', \; y \in S_2''') &&\quad " \quad u = 0\\
&(x \in S_1', \; y \in S_2''') &&\quad " \quad u = x+y-1\\
&(x \in S_1''', \; y \in S_2') &&\quad " \quad u = x+y-1
\end{aligned}$$

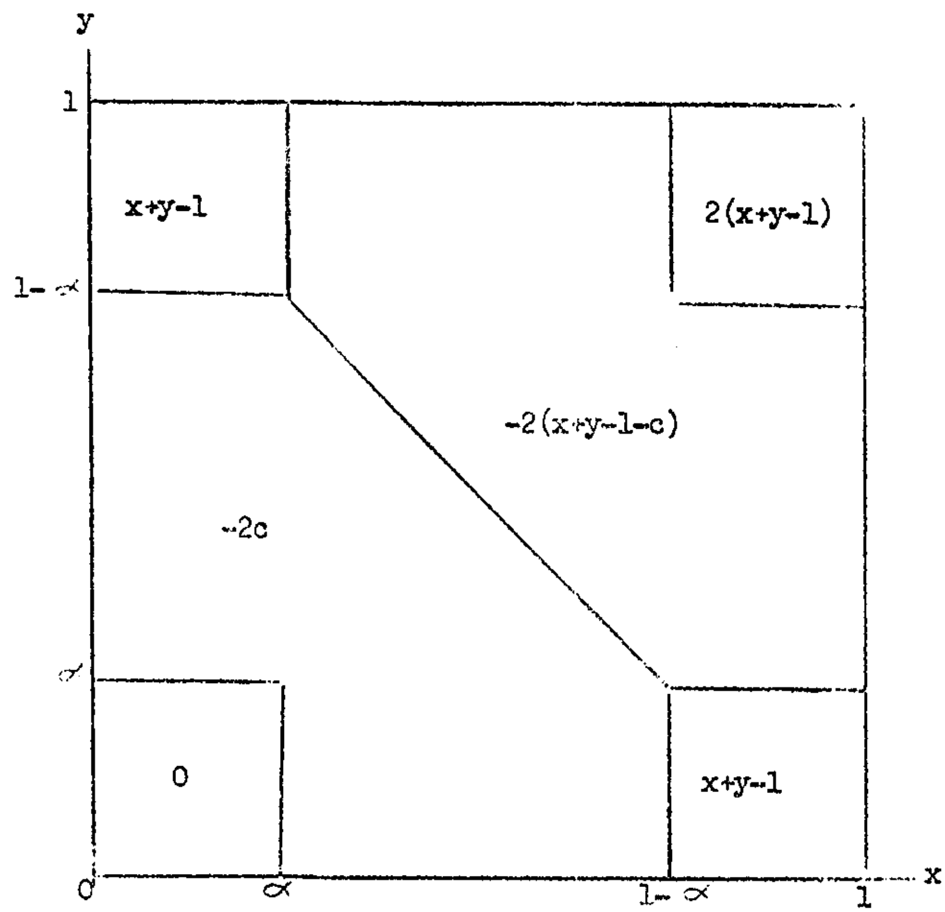
There remain the cases when $x \in S_1''$ or $y \in S_2''$, i.e., the cases when at least one partner takes up the phone. On exchanging information, the two partners will find it advantageous that each of them should commit the firm, if $x+y-1 > 0$; the firm's profit is, in this case, $2(x+y-1-c)$. If, on the other hand, $x+y-1 \leq 0$, then they will abstain from commitment, and the firm's profit profit $= -2c$. Thus

(2.4b)
$$\begin{aligned}
&(x \in S_1'' \text{ or } y \in S_2'') \text{ and } (x+y > 1) &&\text{imply } u = 2(x+y-1-c)\\
&(x \in S_1'' \text{ or } y \in S_2'') \text{ and } (x+y \leq 1) &&\quad " \quad u = -2c .
\end{aligned}$$

To illustrate geometrically, suppose the rule R partitions the sets S_1, S_2 into the following subsets, with properties defined in (2.2a):

(2.5) $$S_i''' = [0, \alpha); \quad S_i'' = [\alpha, 1-\alpha]; \quad S_i' = (1-\alpha, 1], \text{ for } i=1,2.$$

Then the conditions (2.4a), (2.4b) will be represented by Graph I where the values of $u=u(x,y)$ are inserted in the appropriate subsets of the unit square S:



GRAPH I

We wish to emphasize that the graph gives the function $u(x,y;\ R)$ for a possible rule R, viz., one that (1) partitions the sets S_1, S_2 into intervals, rather than into some other kinds of subsets; (2) associates low values of x or y with "doing nothing," high values with "committing," and intermediate values with "communicating"; (3) defines the intervals in a doubly-symmetrical fashion, using a single parameter α. We do not assert at this place that such a rule will belong to the set of optimal ones (in the sense to be defined presently). In fact, the optimality of properties (1) and (2), while appealing to intuition, will not be proved at all; and the question of (3) will be discussed in Section 3.

2.5. Conditions (2.4a), (2.4b) correspond to equations (1.3)-(1.5) of Section 1, which have degenerated as follows: A prospect is, in the present case, not a distribution but a single physical result (a "sure prospect") called profit; this is because we have assumed that the profit is exactly determined, given the prices and the team action. Moreover, the utility of profit is assumed to be proportional to it (or linear in it). This justifies the use of the same letter u for the prospect, the profit, and the utility of profit.

However (cf. subsection 1.4 above), while we have, in our Example, no randomness of the physical result (given the situation and the action), the situation itself is random. Let the joint distribution of x,y be $F(x,y)$; then the expected utility is -- corresponding to equation (1.6) --

$$U = \int_0^1 \int_0^1 u(x,y;\ R;\ c)\ dF(x,y), \tag{2.6}$$

where $u(x,y)$ is defined by (2.4a), (2.4b). U is to be maximized with respect to R.

2.6. In Section 3, the solution of this problem is discussed under two assumptions:

(1) that the rule R is defined by four numbers, α, $\bar{\alpha}$, β, $\bar{\beta}$, each contained in $[0,1]$ and such that

$$(2.7)\qquad \begin{cases} S'_1 = [0, \alpha), & S''_1 = [\alpha, 1-\bar{\alpha}], \quad S'''_1 = [1-\bar{\alpha}, 1] \\ S'_2 = [0, \beta), & S''_2 = [\beta, 1-\bar{\beta}], \quad S'''_2 = [1-\bar{\beta}, 1]; \end{cases}$$

(2) that x and y are independent and have each a rectangular distribution.

Accordingly, (2.6) becomes

$$(2.8)\qquad U = \int_0^1\int_0^1 u(x,y;\alpha,\bar{\alpha},\beta,\bar{\beta};c)\,dx\,dy = U(\alpha,\bar{\alpha},\beta,\bar{\beta},c).$$

A team rule $R = (\alpha, \bar{\alpha}; \beta, \bar{\beta})$ that will maximize U belongs to the set of optimal team rules. It will depend on c only.

2.7. The next natural extension of the problem would probably lie in the following directions: (see 2.7.1-2.7.7).

2.7.1. The assumption (2) of 2.6 should be replaced by one that would allow correlation between x and y. Buying and selling prices are likely to be correlated; and, even regardless of the particular economic interpretation of the model, it is interesting to check upon the following intuitive conjecture: the stronger the correlation between the situations s_1 and s_2 the less need is there for communication -- i.e., the smaller will be the sets S''_i. It may be mathematically advantageous to replace, for this problem, the rectangular distributions by a joint normal distribution, defined over the set of all real numbers x and y (with 0-means and 1-variances), and fully described by its correlation coefficient.

2.7.2. The economic model can be generalized to permit variations of quantities bought and sold (in the present model they can be only 0 or 1).

2.7.3. As mentioned in subsection 2.3, we might make the u-function (for a given subset of S) non-symmetrical in x,y. This may be interpreted as a transition from a team of equal partners to a team involving <u>subordination</u>. Presumably, when k is small, the "expert on raw materials" will frequently consult the other team member while the latter will mostly act on his own: the interval S_2'' will be small, and the interval S_1'' will be large.

2.7.4. One may prescribe the pattern of the "<u>communication network</u>". In the 2-member team, we can replace our assumption of 2-way communication by assuming that only the first partner has the power to initiate communication; this can be assumed to consist either of (1) exchange of information or of (2) one-sided messages (orders) from the first partner. The action a_2'' will now mean: "postpone decision till the time scheduled for the senior's call." A new function u(x,y) will replace equations (2.4a), (2.4b).

2.7.5. One can extend the model to the case of $n > 2$ team members. This can be combined with the suggestions made in 2.7.3 and 2.7.4, as follows: the commitment by any member adds to the firm's profit the amount

$$u = \sum_{i=1}^{n} k_i x_i - 1; \quad (0 \leq x_i \leq 1, \quad k_i > 0, \quad \Sigma k_i = 1);$$

and the communication network of the team is described by a matrix $[c_{ij}]$, where c_{ij} is the cost of sending a message from the i-th to the j-th member and can be equal either to c or to infinity. The expected utility becomes a function of the matrix $[c_{ij}]$ <u>and</u> of the rule R, and is to be maximized with respect to both of them.

2.7.6. As stated in 1.5, one should proceed to "many-interval" (or "dynamic") models. The problem of decentralized inventories belongs here.

2.7.7. Finally, one should proceed from the study of teams to that of organizations, i.e., groups of n persons whose goals (the n utility functions) are not identical and the results of whose joint action affect a $n+1$-th utility function, that of the "organization as a whole." This will superimpose upon our problem of best communication rules the problem of "shadow prices."

3. A Method of Solution

Let us denote by $\left\{u(x, y; \alpha, \bar{\alpha}, \beta, \bar{\beta}, c)\right\}$ a five parameter family of functions defined on the unit square. We seek that function which maximizes [see equation (2.8)]

$$U(\alpha, \bar{\alpha}, \beta, \bar{\beta}, c) = \iint_{\square} u \quad dxdy$$

We restric our parameters as follows:

(3.1) $$\tfrac{1}{2} \geq \alpha, \bar{\alpha}, \beta, \bar{\beta} \geq 0, \quad c \geq 0.$$

Our functions $\{u\}$ are then defined,

(3.2.1) $u = 0$ for $0 \leq x \leq \alpha$, $0 \leq y \leq \beta$

(3.2.2) $u = x + y - 1$ for $0 \leq x \leq \alpha$, $1 - \bar{\beta} \leq y \leq 1$

(3.2.3) $u = x + y - 1$ for $1 - \bar{\alpha} \leq x \leq 1$, $0 \leq y \leq \beta$

(3.2.4) $u = 2(x + y - 1)$ for $1 - \bar{\alpha} \leq x \leq 1$, $1 - \bar{\beta} \leq y \leq 1$

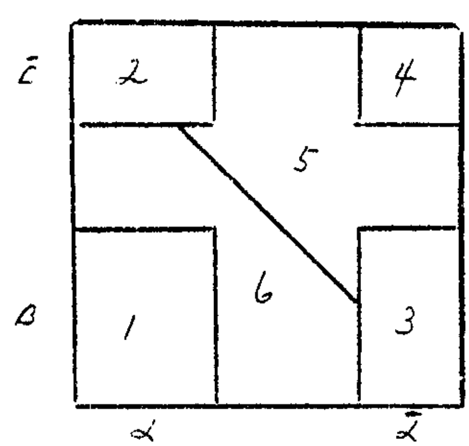
(3.2.5) $u = 2(x + y - 1 - c)$ for $x + y - 1 \geq 0$, $(x, y) \notin 2 \cup 3 \cup 4$

(3.2.6) $u = -2c$ elsewhere.

The function u is thus defined by a choice of c and a partition of the unit square into 6 regions. We will distinguish between four cases:

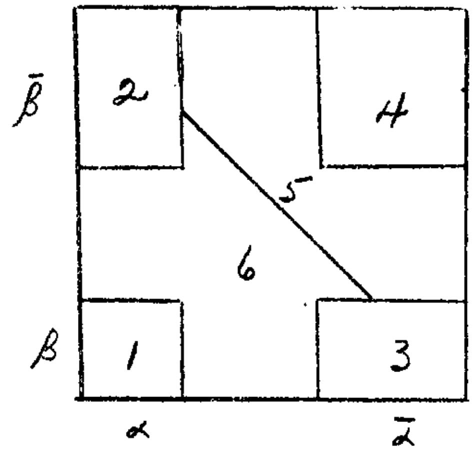
(3.3) Case I $\quad \bar{\alpha} \leq \beta$

$\bar{\beta} \leq \alpha$



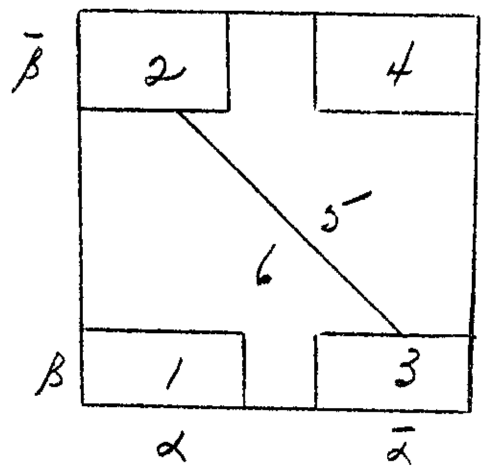
Case II $\alpha \leq \bar{\beta}$

$\beta \leq \bar{\alpha}$



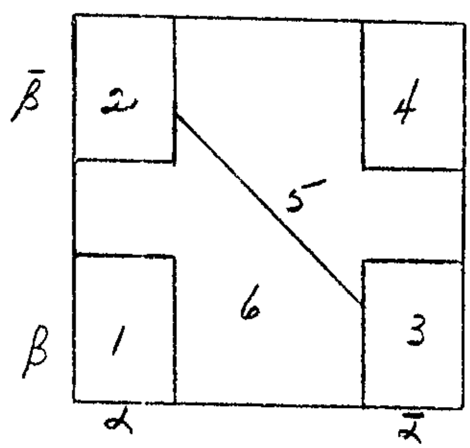
Case III $\bar{\beta} \leq \alpha$

$\beta \leq \bar{\alpha}$



Case IV $\alpha \leq \bar{\beta}$

$\bar{\alpha} \leq \beta$



Integrating the function u over the unit square and equating the first partials with respect to α, $\bar{\alpha}$, β, $\bar{\beta}$ to zero, we obtain 4 sets of 4 equations.

If we set

$$f_c(\beta, \bar{\beta}, \alpha) = 2c\beta + 2c\bar{\beta} - \bar{\beta}\alpha + \tfrac{1}{2}\beta^2 \tag{3.3.1}$$

$$g_c(\beta, \bar{\beta}, \bar{\alpha}) = 2c\beta + 2c\bar{\beta} + \tfrac{1}{2}\beta^2 + \bar{\alpha}\beta - \bar{\alpha}^2, \tag{3.3.2}$$

we may write the four four-equation systems as,

(3.4)

I	II
$f_c(\beta, \bar{\beta}, \alpha) = 0$	$f_c(\bar{\beta}, \beta, \bar{\alpha}) = 0$
$f_c(\alpha, \bar{\alpha}, \beta) = 0$	$f_c(\bar{\alpha}, \alpha, \bar{\beta}) = 0$
$g_c(\beta, \bar{\beta}, \bar{\alpha}) = 0$	$g_c(\bar{\alpha}, \alpha, \beta) = 0$
$g_c(\alpha, \bar{\alpha}, \bar{\beta}) = 0$	$g_c(\bar{\beta}, \bar{\beta}, \bar{\alpha}) = 0$

III	IV
$f_c(\beta, \bar{\beta}, \alpha) = 0$	$f_c(\alpha, \bar{\alpha}, \beta) = 0$
$f_c(\bar{\beta}, \beta, \bar{\alpha}) = 0$	$f_c(\bar{\alpha}, \alpha, \bar{\beta}) = 0$
$g_c(\bar{\alpha}, \alpha, \beta) = 0$	$g_c(\bar{\beta}, \beta, \alpha) = 0$
$g_c(\alpha, \bar{\alpha}, \bar{\beta}) = 0$	$g_c(\beta, \bar{\beta}, \bar{\alpha}) = 0.$

Clearly I and II are equivalent in the sense that the interchange

$$\begin{aligned} \alpha &\longleftrightarrow \bar{\alpha} \\ \beta &\longleftrightarrow \bar{\beta} \end{aligned}$$

or the interchange

$$\begin{aligned} \alpha &\longleftrightarrow \bar{\beta} \\ \beta &\longleftrightarrow \bar{\alpha} \end{aligned}$$

effect a transformation of system I into II and vice versa.

Similarly III and IV are equivalent under the interchanges

$$\begin{aligned} \alpha &\longleftrightarrow \beta \\ \bar{\alpha} &\longleftrightarrow \bar{\beta} \end{aligned} \qquad\qquad \begin{aligned} \alpha &\longleftrightarrow \bar{\beta} \\ \beta &\longleftrightarrow \bar{\alpha} \end{aligned}$$

We note further that I and II are invariant under the interchange

$$\alpha \longleftrightarrow \beta$$
$$\bar{\alpha} \longleftrightarrow \bar{\beta}$$

and III and IV invariant under the interchange

$$\alpha \longleftrightarrow \bar{\alpha}$$
$$\beta \longleftrightarrow \bar{\beta}$$

If we then had a solution of I, which we will indicate in the first column headed I we may write another solution of I and two corresponding solutions of II as below. A similar comment may be made on III and IV.

	I	I	II	II
α	ρ	σ	$\bar{\rho}$	$\bar{\sigma}$
$\bar{\alpha}$	$\bar{\sigma}$	$\bar{\sigma}$	ρ	σ
β	σ	ρ	$\bar{\sigma}$	$\bar{\rho}$
$\bar{\beta}$	$\bar{\sigma}$	$\bar{\rho}$	σ	ρ

Let us suppose that $\alpha = \bar{\alpha} = \beta = \bar{\beta}$ (see Section 2, Graph I). Then our system of equations reduces to

$$\frac{dU}{d\alpha} = \alpha^2 - 8\alpha c = 0 \tag{3.5}$$

and so

$$\alpha = 0 \qquad \text{or} \qquad \alpha = 8c. \tag{3.6}$$

The second derivative is positive at zero, negative at 8c, and so zero is a relative minima, 8c a relative maxima.

α, however, is restricted to the interval $(0, \frac{1}{2})$. When $c > \frac{1}{16}$, $U' > 0$ for all admissible α. Hence the largest value of U occurs at $\alpha = \frac{1}{2}$. When $c = 0$, $U' < 0$ for all $\alpha > 0$. Hence zero is the absolute maximae. If $\hat{\alpha}$ denotes our absolute maxima we may write then

(3.7) $\hat{\alpha} = \min\left(\frac{1}{2}, 8c\right).$

In the two general cases we have no conclusive result at this time. We have confined our investigation to Case I. Let us set

$$\alpha = c\,\alpha', \quad \beta = c\,\beta', \quad \bar{\alpha} = c\,\bar{\alpha}' \qquad \bar{\beta} = c\,\bar{\beta}'.$$

We may then divide by c^2 and we see that in effect we might have set $c = 1$ for the purposes of this study, and so henceforth we will replace α', β', $\bar{\alpha}'$, $\bar{\beta}'$ by α, β, $\bar{\alpha}$, $\bar{\beta}$. By equating linear terms in (3.4), (Case I) we can show that

(3.8) $(\alpha - \bar{\beta})^2 + (\beta - \bar{\alpha})^2 = \alpha^2 - \bar{\alpha}^2 + \beta^2 - \bar{\beta}^2 .$

Hence

(3.9) $\bar{\alpha} \leq \begin{cases} \alpha \\ \beta \end{cases} , \quad \bar{\beta} \leq \begin{cases} \alpha \\ \beta \end{cases}$

Assuming that $\alpha = \beta$ we find that either

(3.10.1) $\alpha = \beta = \bar{\alpha} = \bar{\beta}$

or

(3.10.2) $\alpha = \beta = \frac{48}{5}, \quad \bar{\alpha} = \bar{\beta} = \frac{10}{5} .$

The first of these has been studied above, and examination of the second order conditions shows it to be a maxima; by examining the second derivatives we can show that (3.10.2) is a saddle point.

Assuming $\alpha = \bar{\beta}$, $\bar{\alpha} = \bar{\beta}$, or $\bar{\alpha} = \beta$ yields the previous results. Similarly assuming $\bar{\alpha} = \eta\,\beta$, $\bar{\beta} = \eta\,\alpha$, or more generally the vanishing of any 2x2 determinant of the variables give the same results.

We make now the change of variable

(3.11)
$$\begin{aligned} \alpha &= \alpha \\ \beta &= \beta \\ \bar{\alpha} &= \lambda\,\beta \\ \bar{\beta} &= \mu\,\alpha \end{aligned}$$

and pass to an equivalent system of equations in $\alpha, \beta, \lambda, \mu$. We may eliminate successively and obtain a polynomial in μ of degree 14,

$$
\begin{aligned}
(3.12) \qquad & 63\mu^{14} - 648\mu^{13} + 3134\mu^{12} - 9048\mu^{11} \\
& + 17547\mu^{10} - 24644\mu^{9} + 26238\mu^{8} \\
& - 21714\mu^{7} + 14151\mu^{6} - 7290\mu^{6} + 2946\mu^{4} - 914\mu^{3} + 207\mu^{2} \\
& - 30\mu + 2 = h(\mu).
\end{aligned}
$$

We may factor $h(\mu)$,

$$
h(\mu) = (\mu - 1)^3 \quad (3\mu - 1) \quad h^*(\mu),
$$

where $h^*(\mu)$ is a polynomial of degree 10 with alternating signs. Thus all roots of $h^*(\mu)$ are positive or complex and by Gauss' theorem it may be shown to have no rational roots.

Our second order conditions show that

$$
(3.13) \qquad \mu \geq \frac{1}{2}
$$

and the inequalities between $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ show that

$$
(3.14) \qquad \mu \leq 1.
$$

The problem is thus reduced to a determination of the existence and value of roots of $h(\mu)$ between $\frac{1}{2}$ and 1. The existence of such roots can be determined by Sturm's method and this calculation is in progress at this time.

ERRATA for ECONOMICS No. 2029

1. Page 3, equation (2.1) for $y_{min} = 0$ read $y'_{min} = 0$.

2. Page 6, Graph I for $-2(x+y-1-c)$ read $2(x+y-1-c)$.

3. Page 3' equation (3.4) II for $g_c(\bar{\beta}, \bar{\beta}, \bar{\alpha}) = 0$ read $g_c(\bar{\beta}, \beta, \alpha) = 0$.

I

4. Page 4'

$$\begin{array}{c|c|} & \\ \alpha & \rho \\ \hline \bar{\alpha} & \bar{\sigma} \end{array} \quad \text{to read} \quad \begin{array}{c|c|} & \\ \alpha & \rho \\ \hline \bar{\alpha} & \bar{\rho} \end{array}$$

5. Page 4' equation (3.5) for $\frac{dU}{d}$ to read $-\frac{1}{2}\frac{dU}{d}$

6. Page 5' equation (3.10.2) for $\bar{\alpha} = \bar{\beta} = \frac{10}{5}$ to read $\bar{\alpha} = \bar{\beta} = \frac{16}{5}$

7. Page 6' line ten for Gauss' read Gauss's