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Alternative Conditions for Social Orderings

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In a recent monograph, Arrow [1] has considered methods for obtaining group preferences among alternative situations given the preferences of the individual members of the group. After developing a set of conditions to be imposed on such a method, he obtained the result that no method could be found to satisfy these conditions and considered possible modifications of the conditions.

In Section II of the present paper, an alternative set of conditions for group preferences is presented and it is shown that many methods for obtaining group preferences satisfy these conditions. Many of the concepts and symbols used are taken from Arrow's study. Section III contains some discussion of the difference between the conditions of Section II and those suggested by Arrow. In the last section, a proof needed in the development of Section II is given.

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1. The author is indebted to I. N. Herstein, Morton Slater, and Dan Waterman for mathematical advice and to Leonid Hurwicz and Milton Friedman for criticism of an earlier paper.

The remainder of this section contains some general remarks on the relation of the present discussion to other work in welfare economics. The interpretations presented are not regarded as original contributions[2] or as final dicta but do summarize the author's current views and are intended to help explain the motivation for the material which follows.

Welfare economics may be regarded as the study of the implications of ethical propositions for the choice of economic policies. In what is called "old" welfare economics the implications of the precept that total utility should be maximized were studied. Much of the "new" welfare economics consists of seeking implications of the proposition that one situation is to be regarded as better than an alternative if in the former situation someone is more satisfied and no one is less satisfied than in the alternative. Let us call a set of ethical propositions together with its economic implications a system of welfare economics. It is clear that the economist's interest in any system of welfare economics will be strongly influenced by--(1) the acceptability of the ethical propositions used, (2) the extent to which the propositions can be related to observable or potentially observable phenomena, and (3) the range of policy questions to which the system gives answers.

Criticisms of proposed systems of welfare economics reflect these desiderata. The principal criticisms of old welfare economics were either that utility was nonmeasurable (nonobservable) or that, if measurable, the measure was arbitrary within wide limits and could therefore not have the ethical significance attributed to it. The new welfare economics has been primarily criticized for the narrowness of the range of questions to which it provides

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2. In fact, these introductory remarks are mainly a restatement of some previous observations of Samuelson [2].

answers. In general, the range of application of a system can be extended by adding to the set of ethical propositions on which it is based. However, this will typically involve more controversial propositions. Construction of a useful system of welfare economics thus involves a rather delicate compromise between the desire to keep the underlying ethical judgments generally acceptable and the desire to provide criteria for a wide range of policy choices.

From the point of view outlined above, the contribution of Bergson [3] is not a new system of welfare economics but an analytical and expository device for ascertaining and explaining the implications of various sets of ethical propositions. As an indicator of social preferences he postulated a social welfare function--a function of the amounts of commodities used and contributed by various individuals in society. He did not suggest a specific form for the function but showed how various welfare propositions could be interpreted as restrictions on the form of the function and their implications studied in this fashion. While Arrow's approach is fundamentally similar to that of Bergson, there are several differences.[3/] Both consider methods for ordering social situations. Arrow makes the social ordering dependent on individual preferences in the first instance, Bergson treats this dependence as one of a number of alternative assumptions to be analyzed.[4/] Bergson's ordering is given by the relative magnitudes assigned to different situations by a real-valued function; Arrow does not assume the existence of such a real valued function and in this respect his approach is more general.

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3. See [1], pp. 22, 23.

4. See [3], pp. 311-319.

The latter difference makes the mathematical techniques employed by Bergson inappropriate for Arrow's study and leads to the use of symbolic logic rather than calculus.

The approach used in this paper is similar to that of Arrow. It is assumed at the outset that social choices are to be based on individual preferences.[5] The alternative situations among which a choice is to be made will be called social states and will be defined more precisely, along with other concepts, in the next section.

II

Consider a group of n individuals. Let X_i ($i = 1, 2, \ldots, n$) be the prospect[6] of the ith individual in social state X. A prospect is either a specification of amounts of commodities to be received and furnished by an individual over a given period of time or a probability combination of such specifications. $Z_i = pX_i + (1-p)\,Y_i$ is the prospect that specification X_i will be realized by the ith individual if a chance event with probability p occurs and Y_i will be realized otherwise. Z_i is then a probability combination of X_i and Y_i.

A social state is an assignment of a prospect to each individual. $Z = pX + (1-p)\,Y$ means $Z_i = pX_i + (1-p)\,Y_i$ for every i. If this holds, the state Z is said to be a probability combination of states X and Y. The set of all conceivable social states is denoted by $\mathcal{S}$.

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5. From the observations of Knight [7] and Clark [8] on the nature of individual preferences, one might easily question the desirability of this aspect of the approach. While most writers on welfare economics have used a similar approach, this question should be recognized as one of the unsettled problems beyond the scope of the present paper.

6. For a fuller discussion of individual prospects, see Marschak [4], pp. 113, 114.

It is assumed that social states are ordered according to the preferences of each individual. The following notation (taken from Arrow [1]) will be used in discussing the individual orderings --

(2.1) XR_iY means social state X is preferred or indifferent to state Y in the preferences of the ith individual.

(2.2) XP_iY means XR_iY, $Y\bar{R}_iX$ where $\bar{R}_i$ means "is not preferred or indifferent to" XP_iY can be read, "X is preferred to Y in the preferences of the ith individual."

(2.3) XI_iY means XR_iY, YR_iX.

This can be read, "X is indifferent to Y in the preferences of the ith individual."

Let $\mathcal{R}_i(\mathcal{S})$ be the ordering of elements of $\mathcal{S}$ according to the preferences of the ith individual. We are interested in methods for obtaining social orderings of $\mathcal{S}$ that will satisfy certain conditions. Let $\mathcal{R}(\mathcal{S})$ be a social ordering of $\mathcal{S}$. XRY means X is preferred or indifferent to Y in the social ordering $\mathcal{R}(\mathcal{S})$; XPY and XIY express, respectively, preference and indifference in the social ordering.

It was indicated in the previous section that the social ordering was to depend on individual orderings. We may indicate this symbolically as follows --

$$(2.4) \qquad \mathcal{R}(\mathcal{S}) = M\left\{\mathcal{R}_1(\mathcal{S}), \mathcal{R}_2(\mathcal{S}), \ldots, \mathcal{R}_n(\mathcal{S})\right\}$$

where M stands for the method of obtaining a social ordering from a set of individual orderings. The interesting aspects of any discussion using this approach will lie in the assumptions made about the $\mathcal{R}_i(\mathcal{S})$ and the conditions imposed on $\mathcal{R}(\mathcal{S})$. M may be regarded as a function in a very general sense. Assumptions about the $\mathcal{R}_i(\mathcal{S})$ limit the domain of M, conditions on

$\mathcal{R}(\mathcal{S})$ restrict its range. Together the assumptions and conditions impose requirements on M.

The conditions to be imposed in this section will be required to hold under the following assumptions about the individual orderings. These are assumed to hold for all i.

A1. Each $\mathcal{R}_i(\mathcal{S})$ is a complete ordering. This means --

A1a. For all X, Y either XR_iY or YR_iX

A1b. $XR_iY,\ YR_iZ \Longrightarrow XR_iZ$.

A2. $XP_iY \Longrightarrow XP_i\left\{pX + (1-p)\ Y\right\}\ P_iY$ for $0 < p < 1$.

A3. If XP_iYP_iZ then there exists a p $(0 < p < 1)$ such that

$YI_i\left\{pX + (1-p)\ Z\right\}$.

A4. There exist two states, say $\breve{X}$, $\breve{Y}$, for which the following hold--

A4a. $\breve{X}_i = \breve{X}_j,\ \breve{Y}_i = \breve{Y}_j$ all i, j.

A4b. $\breve{X}P_i\breve{Y}$ all i.

A1-A3 are just a restatement of the axioms of von Neumann and Morgenstern [5]. A4 assumes at least a certain amount of similarity of individual preferences--i.e., there are two individual prospects such that everyone agrees that the state which assigns one of the prospects to every individual is preferable to the state that assigns the other prospect to every individual.

The symbol S will be used to denote a subset of $\mathcal{S}$ consisting of the social states that are regarded as achievable when technical, economic, and possibly other limitations are taken into account.[7/] Much of economic analysis is concerned with finding out which social states are achievable and which states will result from particular policies. In considering methods

7. Possible other limitations are briefly discussed on p. 16.

of obtaining social orderings we are concerned with the problem of choosing from an achievable set that is regarded as given. The achievable set may vary from one application to another. It will be left unspecified in the present discussion except that it will be assumed that any achievable set with which we may be concerned will satisfy Assumptions A5 and A6 below.

A5. $X \in S,\ Y \in S \Longrightarrow \left\{ pX + (1-p)\ Y \right\} \in S.$

Let be a subset (null set not excluded) of the indices that identify individuals. For any $Y \in S$, let $S_{\alpha Y}$ be the set of elements, X, in S for which XI_iY for all $i \in \alpha$.

A6. Given any $S_{\ \ Y}$ and any j not in α there exists an $X^{(j)}$ in $S_{\alpha Y}$ such that $X^{(j)}R_jX$ for all $X \in S_{\alpha Y}$.

The role of A5 and A6 is somewhat different from that of the previous assumptions. A1-A4 state the nature of the individual orderings to which the analysis applies and may be regarded as restrictions on the domain of M in (2.4). Since the concept of the achievable set S is used later in the statement of conditions on the social ordering, A5 and A6 help specify these conditions. A5 says that if two states are achievable, probability combinations of the two states are also achievable. A6 says that if we consider only achievable states that assign given levels of satisfaction to certain individuals, there exists an optimal state (or states) in the preferences of any other individual. The motive for this assumption is to rule out sets of achievable states in which, whatever state is chosen, it is possible to offer a preferred prospect to some individual without forcing any of a previously selected set of individuals to take a less desirable prospect. Suppose for example, that we had a technical restriction which stated that, with given resources, any amount of some commodity, say beef, less than a given upper limit, say 100 thousand tons, could be produced. If the decision as to allocation of resources is regarded as already determined, there is clearly no best amount of beef to be produced. For any stated amount that is achievable, there is a larger (and presumably better) amount that is also achievable. It seems reasonable to rule out this kimd of indeterminancy and while A6 seems to the author to be an acceptable way it is possible that a simpler and more convenient assumption could be formulated.

One more concept should be defined before the conditions to be imposed on $\mathcal{R}$ are stated. Let $X^{[ij]}$ be the social state obtained from X by interchanging the ith and jth elements of X--i.e., by having individuals i and j trade prospects. i and j will be said to be similar individuals if--

(2.5) $XR_iY \Longleftrightarrow X^{[ij]}\ R_jY^{[ij]}$ for all X, Y

(2.6) $XI_kX^{[ij]}$ for all X and all $k \neq i$ or j

(2.7) $X \in S \Longrightarrow X^{[ij]} \in S$. for all X.

(2.5) means that the two individuals have similar preferences. (2.6) states that they are similar with respect to the preferences of others. When (2.7) holds they may be said to have similar possibilities.

The conditions on the social ordering can now be specified.

C1. $\mathcal{R}(\mathcal{S})$ is a complete ordering.

C1a. For all X, Y either XRY or YRX

C1b. XRY, YRZ $\Longrightarrow$ XRZ.

C2. For any S satisfying A5, A6 there exists an element X^S in S such that X^SRX for all X in S.

C3. $\mathcal{R}$ agrees with the partial ordering of the new welfare economics.

C3a. If XR_iY for all i then XRY

C3b. If XR_iY for all i and XP_jY for some j then XPY.

C4. The social ordering is independent of the way in which indices are assigned to individuals.[8/]

C5. If the ith and jth individuals are similar then $X^S I_i X^{[ij]}$.

C1 and C2 might be called operational conditions. They are not motivated by ethical considerations but by the desire for power and convenience of application. C1 guarantees that any two states can be compared and that the comparisons will be transitive.[9/] C2 requires that the ordering specify an optimal state (or a set of optimal states) for each achievable set considered. C3 expresses the value judgment underlying the new welfare economics—that satisfying individual preferences insofar as they do not conflict

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8. This condition can be stated more simply in words than in the symbols employed in this section. I believe the verbal statement is clear but, for the sake of formal completeness, it may be worthwhile to state an equivalent condition symbolically.

C4'. If there are two sets of individual orderings $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n$ and $\mathcal{R}_1^*, \mathcal{R}_2^*, \ldots, \mathcal{R}_n^*$ such that

(1) $XR_iY \Longleftrightarrow X^{[ij]} R_j^* Y^{[ij]}$ and $XR_jY \Longleftrightarrow X^{[ij]} R_i Y^{[ij]}$ for some i, j $i \neq j$ and for all X, Y.

(2) $XR_kY \Longleftrightarrow X^{[ij]} R_k^* Y^{[ij]}$ all $k \neq i$ or j.

then (3) $XRY \Longleftrightarrow X^{[ij]} R^* Y^{[ij]}$ where

$\mathcal{R}, \mathcal{R}^*$ are the social orderings corresponding to $\mathcal{R}_1, \ldots \mathcal{R}_n, \mathcal{R}_1^* \ldots \mathcal{R}_n^*$ respectively. (1) and (2) hold if the orderings $\mathcal{R}_1^* \ldots \mathcal{R}_n^*$ are obtained from $\mathcal{R}_1 \ldots \mathcal{R}_n$ by interchanging the subscripts identifying the ith and jth individuals with no change in individual preferences. Since any permutation of subscripts can be obtained by a finite number of such interchanges, $C4' \Longrightarrow C4$. The converse is obvious, hence the two are equivalent.

9. While the requirements of a complete ordering seem to me to have intuitive desirability, one might consider relaxing this condition if it were found to conflict with the realization of strongly held ethical values. It might suffice to require a method for making a choice from any of a suitably general class of achievable sets. The formulation of the bargaining problem by Nash [6] is an example of the latter approach.

with each other is socially desirable. C4 and C5 prescribe, in addition, a kind of equality of treatment of individuals. C4 insures that no accidental elements in arranging individuals or assigning them numbers will affect the social ordering. In the present context, this means that the social ordering depends only on the set of individual orderings from which it is derived. If one wished to make the social ordering a function of other variables as well (education, ancestry, or other characteristics) it would probably be convenient to introduce these explicitly (alter the domain of M) and still retain a condition analogous to C4. C5 might be described as requiring similar treatment for similar people, similar treatment being realized when the individuals would be indifferent to an exchange of prospects.

A class of methods which yield orderings satisfying the above conditions is described below. There is no reason to believe that these are the only methods; though others would probably be more difficult to investigate. The fact that a large class of methods exists means that value judgments in addition to those embodied in the above conditions would be needed to choose a unique social ordering.

Let

(2.8) $\quad u_i = f_i(X)$

be a numerical utility function for the ith individual. Assumptions A1 to A3 imply that individual utility functions exist which have the property--

(2.9) $\quad$ For $Z = pX + (1-p)\,Y$, $f_i(Z) = pf_i(X) + (1-p)\,f_i(Y)$.

Functions that satisfy (2.9) are unique up to a linear transformation. A unique function is found for each individual by imposing (2.9) and--

(2.10) $\quad f_i(\tilde{X}) = a$

$f_i(\tilde{Y}) = b \qquad a, b \text{ constants} \qquad a > b$

where $\tilde{X}$, $\tilde{Y}$ are the states assumed to exist in A4. Let $g(u_i)$ be continuous, monotonic increasing, and convex in the domain $-\infty < u_i < \infty$. We now define--

(2.11) $$v = \sum_{i=1}^{n} g(u_i) = \sum_{i=1}^{n} g[f_i(X)] = h(X).$$

The social ordering given by--

(2.12) $$XRY \Longleftrightarrow h(X) = h(Y)$$

will satisfy C1-C5.

There are, of course, many ways to define v. g can be chosen from a wide class of functions, a and b are arbitrary except for order, and in general there will be alternative pairs $\tilde{X}$, $\tilde{Y}$ to choose from. It is bbvious that any of these definitions leads to an ordering which satisfies C1, C3, C4. The proof that C2 is satisfied is rather long and will be given in Section IV. C5 presupposes C2 and is proved below on the assumption that C2 is satisfied.

Suppose i, j represent similar individuals and that X^S is a maximal element of S, i.e., $X^S RX$, $h(X^S) \geq h(X)$ for all $X \in S$. To avoid lengthy superscripts let $X^{S[ij]} = X^*$. C5 requires that $X^S I_i X^*$ or in terms of the utility measure $f_i(X^S) = f_i(X^*)$. We shall show that to suppose the contrary leads to a contradiction. Let $Y = \frac{1}{2} X^S + \frac{1}{2} X^*$. $X^* \in S$ by (2.7), $Y \in S$ by A5.

From similarity of i, j and (2.9) we have --

(2.13) $$f_i(X^S) = f_j(X^*)$$

$$f_j(X^S) = f_i(X^*)$$

$$f_i(Y) = f_j(Y) = \frac{f_i(X^S) + f_i(X^*)}{2}$$

$$f_k(X^S) = f_k(X^*) = f_k(Y) \qquad \text{for all } k \neq i \text{ or } j.$$

And from (2.11) --

(2.14) $$h(Y) - h(X^S) = 2g \frac{f_i(X^S) + f_i(X^*)}{2} - g[f_i(X^S)] - g[f_i(X^*)].$$

If $f_1(X^S) \neq f(X^*)$, the quantity on the right is positive by convexity of g and $h(Y) > h(X^S)$ which contradicts the assertion that X^S is maximal.

III

The reason that the difficulty encountered in Arrow's treatment does not arise in the previous section can be found by asking which of Arrow's conditions would be violated by methods satisfying C1-C5. A method satisfying the conditions of Section II will also satisfy all of the Arrow conditions except his Condition 3.[10] For reference Condition 3 is quoted--[11]

> Condition 3: Let $\mathcal{R}_1, \ldots, \mathcal{R}_n$ and $\mathcal{R}_1', \ldots, \mathcal{R}_n'$ be two sets of individual orderings and let C(S) and C'(S) be the corresponding social choice functions. If, for all individuals i and all X and Y in a given environment S, XR_iY if and only if $XR_i'Y$, then C(S) and C'(S) are the same (independence of irrelevant alternatives).

C(S) in this statement stands for the set of all $X \in S$ such that XRY for all $Y \in S$.[12] In the notation of Section II it could be defined as the subset of S that includes X^S and all X for which XIX^S. In Arrow's context this condition requires that the social ordering of alternatives in an achievable set, S, depend only on the individual orderings of alternatives in S. If this were to hold (2.4) could be rewritten --

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10. It is perhaps worth noting that Condition 2 would be vacuously satisfied. Assumptions A2 and A5 of Section II exclude the possibility of two sets of orderings bearing the relationship postulated in Arrow's Condition 2.

11. See Arrow [1], p. 27. Minor changes in the notation have been made to agree with that used in Section II. The reader should also note that S in Arrow's book is not subject to A5, A6.

12. Arrow [1], p. 15.

$$(3.1) \quad \mathcal{R}(S) = M\left\{\mathcal{R}_1(S), \mathcal{R}_2(S), \ldots, \mathcal{R}_n(S)\right\}.$$

This seems to me to be rather extreme and, on the whole, undesirable. Consider two cases involving two individuals i, j and two states X, Y. Suppose that in Case I, i barely prefers X to Y and j desperately prefers Y to X. In Case II let i desperately prefer X to Y and j barely prefer Y to X. By Condition 3, the social ordering between X and Y must be the same in Case I as in Case II. Admittedly our facilities for distinguishing bare preferences from desperate preferences may often be questionable, but we have to decide whether or not this justifies us in excluding all variations in degrees of preference from consideration.[13]

It may be noted that, while the methods described in Section II do not satisfy Condition 3, the difficulties cited by Arrow (see [1], pp. 26, 27] as evidence for the need of such a condition would not arise with these methods. That is, a change in the achievable set S would not affect the relations among those elements of S not involved in the change. In the approach of Section II this is achieved by defining a basic ordering for an all-inclusive set $\mathcal{S}$ and maintaining this ordering regardless of which elements are or are not achievable in any particular instance. This is analogous to the way in which the matter is treated in the theory of individual choice (see Arrow [1], pp. 11, 12).

Condition 3 was undoubtedly motivated by Arrow's determination to exclude any possibility of inter-personal comparison of utilities.[14] It is one thing

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13. L. J. Savage has made the interesting observation that many of the comparisons of degrees of preference which people are commonly willing to make contain an implicit assumption that certain corporal states have similar significance for different individuals. Marshall's observation that a clerk with a lower salary will generally choose to walk to business through a heavier rain than a clerk with a higher salary (Marshall [9], p. 19, p. 95) seems to involve a comparison of this kind. Willingness to risk extreme physical hardships such as hunger, exposure, torture or death for an objective is almost universally accepted as evidence of the subjective importance of the objective.

14. [1], pp. 9-11.

to admit that no intuitively acceptable basis for inter-personal comparisons exists and quite a different thing to say that all such comparisons are fundamentally impossible. If we interpret utility as an ordinal preference indicator and if social orderings are to be based on individual preferences, then the latter position really excludes the possibility of ordering social states any more completely than is done by the new welfare economics. For as soon as we say that state X is socially preferred to state Y for two states such that some individuals prefer X to Y and others prefer Y to X, we are thereby saying that the gains to those who prefer X are socially more important than the losses of those who prefer Y. This implies that we have some basis for comparing the relevant gains and losses; and such a comparison is fundamentally an inter-personal comparison of utilities. For example, Arrow has shown ([1], Ch. VII) that under suitable assumptions about individual preferences (which involve modifying his Condition 1) the method of majority decision satisfies his conditions. Does the method of majority decisions involve inter-personal comparisons of utilities? I believe so; it chooses between X and Y by comparing the number who would gain utility in passing from X to Y with the number who would lose. Thus anyone's gain is exactly the equivalent of any other's gain and exactly the inverse of anyone's loss.

Of course it should not be supposed that any ethical priority has been established for the methods of making inter-personal comparisons that are described in Section II--i.e., inter-personal comparisons based on some function of a von Neumann-Morgenstern utility measure. The measure served as a mathematical device for showing that Conditions C1-C5 are not mutually inconsistent. I have not attempted to find policy applications for this particular set of conditions, and my current conjecture would be that the conditions are too weak to give answers to many interesting policy questions. They seem to me to serve two purposes—the analysis of these conditions may help explain the nature of the difficulty encountered in Arrow's initial formulation, and to the extent that the conditions are regarded as reasonable they may serve as a start for the development of stronger and more interesting sets. This development would presumably take place through the formulation of new conditions reflecting additional or stronger value judgments.

While this paper is primarily concerned with formal aspects of the problem, it may be worth noting that it is possible to vary somewhat the interpretations of the concepts employed without essentially altering the formal analysis. For example, one could include noneconomic factors in the definition of a social state without changing the formal analysis if he were willing to retain unchanged the assumptions about individual preferences and about sets of achievable states. It has sometimes been assumed that each individual is indifferent to the prospects of others and evaluates each social state solely with reference to his own prospect in that state. If this assumption had been made in the previous section, a few simplifications would have been possible but the results would have been unchanged.

It is possible that some ethical values which should be recognized in making social choices apply independently of individual preferences. In the present approach at least some of these values might be expressed as restrictions on the set of achievable alternatives and entered along with technical restrictions in the determination of S. Thus if certain social states were judged to be ethically undesirable independently of individual preferences, these states could be regarded as not achievable and could be excluded before the ordering based on individual preferences was applied.

IV

In this section we show that C2 is satisfied by a method given by (2.12).[15] Let $F = (f_1, f_2, \ldots, f_n)$ represent the individual utility functions of (2.8). Let $U = (u_1, u_2, \ldots, u_n)$ be a point in an n-dimensional Euclidean space E. The n equations (2.8) can also be indicated by--

(4.1) $\quad U = F(X)$.

Let T be the image in E of the achievable set S--i.e.,

(4.2) $\quad T = F(S)$

and define $\varphi(U)$ by

(4.3) $\quad \varphi(U) = \sum_{i=1}^{n} g(u_i)$ where

g is continuous, monotonic increasing and convex as in (2.11).

From A5 and (2.9)--

(4.4) $\quad$ T is convex.

As in Section II let α be a subset of the indices that represent individuals. Equivalently, denotes a subset of the coordinates of E. For any $U^o \in T$, let $T_{\alpha U^o}$ contain those element of T that agree with U^o in the coordinates denoted by α. From A6 and the continuity of g--

(4.5) $\quad$ Given any $T_{\alpha U^o}$ and any j not in α, there exists a $U^{(j)} \in T_{\alpha U^o}$ such that $u_j^{(j)} \geq u_j$ for all $U \in T_{\alpha U^o}$.

To show that C2 holds we must prove the following assertion--

(4.6) $\quad$ For any T satisfying (4.4) and (4.5) there exists a U^T such that $\varphi(U^T) \geq \varphi(U)$ for all $U \in T$.

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15. Several suggestions of I.N. Herstein have been used in this proof.

Let $\bar{T}$ be the closure of T. When α is null, $T_{\alpha U^o} = T$ and (4.5) says that there exists a maximum for the values of any particular coordinate of points in T. Let u_i^m $i = 1, 2, \ldots, n$ be the maximum of u_i for points of T. To prove (4.6) it is convenient to first prove the following lemma--

(4.7) If $\bar{T}$ contains an element in which some r coordinate values ($r \leq n$) attain their maxima, then T contains an element in which the coordinate maxima are attained in the same r coordinates.

There is clearly no loss of generality in taking the maximum coordinate values in the first r coordinates. The above can then be restated--

(4.7a) If there exists an element $U^+ \varepsilon \bar{T}$ such that $u_i^+ = u_i^m$ $i = 1, 2, \ldots, r$, then there exists a $U^* \varepsilon T$ such that $u_i^* = u_i^m$ $i = 1, 2, \ldots, r$.

The lemma will be proved by induction. From (4.5) the lemma clearly holds for $r = 1$. We shall show that its validity for any r follows from the assumption that it holds for $r - 1$. Given a U^+ and the assumption that the lemma holds for $r - 1$, there will be points of T, say $U^{[2]}, U^{[3]}, \ldots, U^{[r]}$ with the following components--

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\begin{array}{llllll}
(4.8) & U^{[2]}: & u_1^{[2]} = u_1^m, & u_2^{[2]} = u_2^m - \eta_2, & u_3^{[2]} = u_3^m & , \ldots, u_r^{[2]} = u_r^m \\
 & U^{[3]}: & u_1^{[3]} = u_1^m, & u_2^{[3]} = u_2^m & , u_3^{[3]} = u_3^m - \eta_3, & \ldots, u_r^{[3]} = u_r^m \\
 & & & \vdots & & \\
 & U^{[r]}: & u_1^{[r]} = u_1^m, & u_2^{[r]} = u_2^m & , u_3^{[r]} = u_3^m & , \ldots, u_r^{[r]} = u_r^m - \eta_r.
\end{array}
$$

Components beyond the first r are unspecified. The η 's are all non-negative; if one happens to be zero, the lemma is established. If not, consider $r - 1$ arbitrarily small δ's which satisfy--

$$(4.9) \quad 0 < \delta_i < \frac{\eta_i}{r} \qquad i = 2, 3, \ldots, r.$$

We shall show that there exists an element of T, say U^{δ} with the following restrictions on components--

(4.10) $u_i^{\delta} = u_i^m - \delta_i \qquad i = 2, 3, \ldots, r$

(4.11) $u_1^m - \mathcal{E} < u_1^{\delta} = u_1^m \qquad$ where $\mathcal{E}$ is arbitrarily small and

(4.12) $0 < \mathcal{E} < \min \delta_i$.

Consider a point $U^{\mathcal{E}}$ of T in a neighborhood of radius $\mathcal{E}$ about U^{+}. For the first r components of $U^{\mathcal{E}}$ we have--

(4.13) $u_i^{\mathcal{E}} = u_i^m - \mathcal{E}_i, \qquad 0 \leqq \mathcal{E}_i < \mathcal{E} \qquad i = 1, 2, \ldots, r.$

The convex body generated by $U^{\mathcal{E}}$, $U^{[2]}$, $U^{[3]}$, ..., $U^{[r]}$ is contained in T and it contains an element U^{δ} whose first r components satisfy (4.10), (4.11). This is equivalent to saying that there exist $p_1, p_2, \ldots, p_r$ such that--

(4.14) $p_i \geqq 0 \qquad i = 1, 2, \ldots, r$

(4.15) $\sum_{i=1}^{r} p_i = 1$

(4.16) $p_1 U^{\mathcal{E}} + \sum_{j=2}^{r} p_j U^{[j]} = U^{\delta}$.

For given $U^{\mathcal{E}}$ and δ_i, (4.16) gives r - 1 restrictions on the p_i in the form of equalities. (4.15) gives an rth equality. We shall show that the r equalities have a solution and that inequalities (4.11), (4.14) are satisfied by this solution. Using (4.8) and (4.13) the r - 1 equalities given by (4.16) can be written--

(4.17) $p_1(u_j^m - \mathcal{E}_j) + u_j^m \sum_{k=2}^{r} p_k - p_j \eta_j = u_j^m - \delta_j \qquad j = 2, 3, \ldots, r$

or more simply, using (4.15)--

(4.18) $p_1 \mathcal{E}_j + p_j \eta_j = \delta_j \qquad j = 2, 3, \ldots, r.$

These, when combined with (4.15), can be shown to have the unique solution--

(4.19) $$p_1 = \frac{\lambda_1}{\lambda_2}$$

(4.20) $$p_j = \frac{\delta_j}{\eta_j} - \frac{\varepsilon_j}{\eta_j}\frac{\lambda_1}{\lambda_2} \qquad j = 2, 3, \ldots, r$$

where

(4.21) $$\lambda_1 = 1 - \sum_{k=2}^{r} \frac{\delta_k}{\eta_k}, \qquad \eta_2 = 1 - \sum_{k=2}^{r} \frac{\varepsilon_k}{\eta_k}$$

and from (4.9), (4.12)

(4.22) $$0 < \lambda_1 < \lambda_2 < 1$$

hence (4.14) holds. From (4.16) we know that

(4.23) $$u_1^{\delta} = p_1(u_1^m - \varepsilon_1) + \sum_{j=2}^{r} p_j u_1^m = u_1^m - p_1 \varepsilon_1$$

which shows that (4.11) is satisfied.

Next we wish to show that there exists an element of T, say $U^{(1)}$, such that--

(4.24) $$u_1^{(1)} = u_1^m$$

(4.25) $$u_j^{(1)} = u_j^m - \delta_j \qquad j = 2, 3, \ldots, r.$$

If ε_1 happens to be zero, we may set $U^{\delta} = U^{(1)}$. If ε_1 is positive we define a $T_{\propto U^o}$ by letting $\propto$ include components 2, 3, ..., r and choosing U^o so that

(4.26) $$u_j^o = u_j^m - \delta_j \qquad j = 2, 3, \ldots, r.$$

It follows from (4.5) that u_1 has a maximum value for $U \,\varepsilon\, T_{\propto U^o}$ and since ε_1 can be made arbitrarily small, this maximum cannot be less than u_1^m. Thus T contains an element $U^{(1)}$ satisfying (4.24), (4.25).

Now let $\propto$ include components 1, 3, 4, ..., r and choose U^o so that

(4.27) $$u_1^o = u_1^m$$

$$u_j^o = u_j^m - \delta_j \qquad j = 3, 4, \ldots, r.$$

There is a point in $T_{\alpha U^o}$ where u_2 is maximized for $U \in T_{\alpha U^o}$. But $U^{(1)}$ above is in $T_{\alpha U^o}$ and $u_2^{(1)} = u_2^m - \delta_2$ where δ_2 can be made arbitrarily small. Thus $\max_{U \in T_{\alpha U^o}} u_2$ cannot be less than u_2^m and there exists $U^{(2)} \in T$ for which

$$(4.28) \qquad \begin{aligned} u_1^{(2)} &= u_1^m \\ u_2^{(2)} &= u_2^m \\ u_j^{(2)} &= u_j^m - \delta_j \quad \text{for } j = 3, 4, \ldots, r. \end{aligned}$$

Since all the δ_j can be made arbitrarily small, this process can be repeated until at the rth stage we have $U^{(r)}$ with components—

$$(4.29) \qquad u_i^r = u_i^m \qquad i = 1, 2, \ldots, r.$$

Setting $U^* = U^{(r)}$ then proves (4.7a).

(4.6) can now be proved by induction. It clearly holds for n = 1. Its validity for any n follows from the assumption that it is valid for n - 1. Take $U^+ \in \overline{T}$ such that—

$$(4.30) \qquad \varphi(U^+) \geq \varphi(U) \quad \text{for all } U \in \overline{T}.$$

Suppose that there are r components of U^+ equal to their respective coordinate maxima for $U \in T$. For convenience we again suppose that they are the first r components—i.e.,

$$(4.31) \qquad u_i^+ = u_i^m \qquad i = 1, 2, \ldots, r.$$

By (4.7a) there exists a $U^* \in T$ for which—

$$(4.32) \qquad u_i^* = u_i^m \qquad i = 1, 2, \ldots, r.$$

Since $\varphi(U)$ is monotonic in every component, one of the remaining n - r components of U^* must be smaller than the corresponding component of U^+ or else U^* must be identical with U^+. If the latter is true we may set $U^T = U^*$

and (4.6) holds. If not, we have

(4.33) $\quad u_k^* < u_k^+ < u_k^m \qquad$ for some $k > r$.

Let $U^{(k)}$ be an element of T such that

(4.34) $\quad u_k^{(k)} = u_k^m$.

Let E^+ be the $n - 1$ dimensional space given by setting $u_k = u_k^+$ and let T^+ be $T \cap E^+$. T^+ is convex and a condition analagous to (4.5) holds so by our induction hypothesis there exists an element in T^+, say U^M, that maximizes $\Psi(U)$ for $U \in T^+$.

We shall show that--

(4.35) There exist elements of T^+ whose components are arbitrarily close to the components of U^+.

Thus there are elements U of T^+ such that $\Psi(U)$ is arbitrarily close to $\Psi(U^+)$ and $\Psi(U^M)$ must equal $\Psi(U^+)$.

Take $U^\varepsilon \in T$ in a neighborhood of radius ε about U^+. Consider three cases--

(1) $u_k^\varepsilon = u_k^+$.

In this case $U^\varepsilon \in T^+$ and (4.35) holds.

(2) $u_k^\varepsilon > u_k^+$.

In this case we consider the straight line joining U^ε to U^*. Let U^a be the point of intersection of this line with E^+. By convexity of T, $U^a \in T^+$. By the construction of U^a--

(4.36) $$\left|u_j^+ - u_j^a\right| \leq \left|u_j^+ - u_j^\varepsilon\right| + \frac{\left|u_j^\varepsilon - u_j^*\right|}{\left|u_k^\varepsilon - u_k^*\right|}\left|u_k^+ - u_k^\varepsilon\right| \qquad \text{all } j \neq k.$$

Since U^ε is in a neighborhood of U^+, $\left|u_j^+ - u_j^\varepsilon\right|$ and $\left|u_k^+ - u_k^\varepsilon\right|$ are both less than ε and can be made arbitrarily small. Thus (4.35) holds for case (2).

(3) $u_k^+ > u_k^\varepsilon$.

Let U^a be the intersection of E^+ with the line joining U and $U^{(k)}$. We then have--

$$(4.37) \qquad \left|u_j^+ - u_j^a\right| \leq \left|u_j^+ - u_j^\varepsilon\right| + \frac{\left|u_j^\varepsilon - u_j^{(k)}\right|}{\left|u_k - u_k^{(k)}\right|} \left|u_k^+ - u_k^\varepsilon\right| \qquad \text{all } j \neq k.$$

Again, $\left|u_j^+ - u_j^\varepsilon\right|$ and $\left|u_k^+ - u_k^\varepsilon\right|$ can be made arbitrarily small and (4.35) holds.

If we now take $U^T = U^M$, (4.6) is proved.

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