COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 2027

IRREVERSIBLE DEMAND FUNCTIONS*

Michael Farrell

November 9, 1951

It is pointed out that in certain recent analyses of demand the explanation achieved is overwhelmingly in terms of various functions of time (or "trends") and very little part is played by the economic variables used. It is suggested that it may be possible to explain part of these "trends" in terms of economic factors by using irreversible demand functions, and a simple form of such a function is discussed. This function is then fitted to data for tobacco, beer, and spirits, and the results compared with those obtained by using the orthodox functions.

I. The Importance of Trends in Demand Analysis

This paper originated in an attempt to explain in terms of economic variables the "trends" which seem to play so overwhelming a part in demand analysis. Perhaps the most striking examples of this tendency are to be found in Prest's analyses of demand for consumer goods in the United Kingdom for the period 1870-1938.[1] For instance, in the case of beer, spirits, tobacco, and tea, Prest's principal models were of the form:

(1) $$C_t = kY_t^a P_t^b e^{ct+dt^2+fx},$$

* This paper, which was written by the author while at the Department of Applied Economics, Cambridge, was read at the Louvain conference of the Econometric Society, and will be published in a forthcoming issue of *Econometrica*. The author is at present a Commonwealth Fund Fellow, working as a guest at the Cowles Commission.

where C_t represents consumption per head of the commodity in year t; Y_t, real income per head in year t; P_t, ratio of the price of the commodity to the cost-of-living index in year t; t represents time in years, from t = 0 in 1870; x = 0 for the period 1870-1914; x = 1 for the period 1920-1938; and k, a, b, c, d, and f are parameters to be estimated. The variable x, which Prest used only in the cases of tobacco and tea, is a "discontinuity variable" designed to allow for the very considerable changes in tastes which undoubtedly took place during the Great War and for the fact that the United Kingdom included Southern Ireland before the War but not afterwards.

The "economic variables," Y_t and P_t, explained unequivocally less than 1 percent of the variance of the consumption of tea and tobacco, while for spirits and beer the corresponding figures were 9 percent and 17-1/2 percent.[2/] Thus, in every case, the economic variables were quite unimportant in determining demand, as compared with the "social variables"--the time-trends and the discontinuity variables. In so far as these results give a true picture of the situation, the outlook for demand analysis is a dismal one. For, although it is possible to "explain" any series of consumption figures by fitting sufficiently complicated functions of time, this can hardly be regarded as establishing an economic law. Moreover, while both methods rely on induction, prediction which consists simply of the extrapolation of a function of time is a very different matter from prediction based on the assumption that an observed, and presumably causal, relation between economic variables will continue to hold.

The situation is not really as bad as this. For one thing, we are concerned with the sort of commodity whose consumption one would expect to be strongly influenced by social factors; and, in fact, results have been obtained for other commodities which attribute much more importance to economic factors.[3/] In addition, the analyses cover very long periods, and one would expect social changes to be relatively more important in the long than in the short-run; while in matters of prediction, we are usually concerned with fairly short periods. A better measure of the usefulness of this model for short-period prediction can be obtained by splitting the data up into short periods, and considering the variances within these periods. The data were split, for reasons of convenience, into a mixture of five- and six-year periods. Then, using, of course, the estimates of the parameters obtained by fitting the equations to the whole series, the total variance and the variance explained by the economic factors within each short period were computed. These two quantities were summed over all periods, to give what we may call the proportion of total variances within short periods explained by economic factors. We should expect this to be slightly larger, while at the same time, we should expect the variance between short periods to be even more exclusively explained by social factors.

The results of these calculations are given in Table 1, and it will be seen that the percentages rise, for tobacco to 25 percent, for beer to 44 percent, and for spirits to 75 percent; these figures are not strictly comparable, because they are based, not on Prest's calculations, but on recalculations using the revised data (see p. 11). However, Prest's calculations were repeated on the revised data, and these results are also shown in Table 1. It will be seen that the effect of merely using the revised data is small, but that in the case of beer and spirits, where we also introduced discontinuity variables, a considerable increase in the proportion of variance explained in the long period was obtained. This is an interesting phenomenon, in that the introduction of another "social" factor increases the proportion of the variance explained by the economic variables. But it will be noticed that, while for the spirits the increase was at the expense of the variance attributed to the social variables, for beer it was at the expense of the covariance term. However, the table shows that, for all three commodities, the economic variables explained a much larger proportion of the short-period variation then they did of the long-period variation even in the most favorable case.

That is to say, as we had suspected, economic factors are relatively much more important in the short run. But even so, and while admitting that much of the "trend" is probably due to genuine social changes, which it would be difficult or impossible to represent by economic variables, or indeed, to represent numerically at all, it still seems desirable that we should try to explain at least some of it in economic terms.

II. Irreversible Demand Functions as an Explanation of Trends

Two ways in which economic factors can produce changes in tastes are through fashion, where the individual's demand function depends on other people's consumption behavior, and through habit, where it depends on the individual's own past behaviour. It can plausibly be argued that both these phenomena will be of some importance in determining the demand for the commodities we are considering. Many people take up smoking or drinking because their friends and acquaintances smoke or drink and, in Britain at least, tea is often drunk purely as a matter of social convention; while all four commodities are to a greater or lesser extent habit forming.

In this paper we shall be concerned with the effect of habit, because it seems to be easier to investigate statistically. The idea is, quite simply, that a man who has been induced by a rise in income, or a fall in the price of tobacco, to take up smoking, or to smoke more heavily, will form a habit, and will not, when price or income returns to its former level, cut his consumption to its former level, though he will curtail it to some extent. That is to say, his demand function will be irreversible. The idea is, of course, not a novel one. Marshall[4/] mentions the possibility and Haavelmo[5/] discusses the problem theoretically, but, so far as I know, it has not been investigated empirically with reference to any particular commodity, although Duesenberry and Modigliani[6/] use irreversible consumption functions.

The phenomenon need not be confined to commodities which are habit forming in a medical sense; a consumer may become attached to any aspect of a higher standard of living, once he has experienced it. Further, the argument above is that people will increase their consumption of some commodities more readily than they will curtail it, thus giving a positive trend to the demand for those commodities. But, since income must equal expenditure plus saving, if consumption of some commodities is more sensitive to upward than to downward movements of income, then the reverse must be the case for at least one commodity. In general, we should expect this effect to show itself principally in savings, but there is no reason why it should not extend to other commodities. This is probably the main way in which irreversibility works to produce a negative trend, though it is possible to think of commodities (e.g., hats and suspenders) which the individual prefers to do without, when once he has been induced to go without them.

III. A Simple Form of Irreversible Demand Function

The most general form of irreversible demand function would make the individual's demand a function of all his past price-income-consumption positions, but it would obviously be impossible to estimate such a function statistically from the data available. A simpler form,

$$C_t = f(Y_t, P_t, C_{t-1}, Y_{t-1}, P_{t-1}),$$

would make current demand depend only on current price and income, and on the individual's price-income-consumption position in the previous period. This involves some loss of generality, although it should be noted that, since C_{t-1} depends on C_{t-2}, and so on, the individual's history in the more distant past is allowed some influence on current consumption. It is a specific function of this kind that we consider, i.e.,

$$(2) \qquad \frac{C_t}{C_{t-1}} = e^c \left(\frac{Y_t^+}{Y_{t-1}}\right)^{a^+} \left(\frac{Y_t^-}{Y_{t-1}}\right)^{a^-} \left(\frac{P_t^+}{P_{t-1}}\right)^{b^+} \left(\frac{P_t^-}{P_{t-1}}\right)^{b^-},$$

where

$$\frac{Y_t^+}{Y_{t-1}} = \frac{Y_t}{Y_{t-1}}, \quad \frac{Y_t^-}{Y_{t-1}} = 1, \quad \text{for } \frac{Y_t}{Y_{t-1}} \geq 1;$$

$$\frac{Y_t^+}{Y_{t-1}} = 1, \quad \frac{Y_t^-}{Y_{t-1}} = \frac{Y_t}{Y_{t-1}} \quad \text{for } \frac{Y_t}{Y_{t-1}} < 1;$$

$$\frac{P_t^+}{P_{t-1}} = \frac{P_t}{P_{t-1}}, \quad \frac{P_t^-}{P_{t-1}} = 1, \quad \text{for } \frac{P_t}{P_{t-1}} \geq 1;$$

$$\frac{P_t^+}{P_{t-1}} = 1, \quad \frac{P_t^-}{P_{t-1}} = \frac{P_t}{P_{t-1}} \quad \text{for } \frac{P_t}{P_{t-1}} < 1;$$

and where c, a^+, a^-, b^+, and b^- are parameters to be estimated.

We thus assume that there is a constant elasticity of demand with respect to upward movements of income, and a constant elasticity with respect to downward movements, although the two elasticities may differ; and similarly for price elasticity. This is clearly a highly simplified model, and, in particular, it is open to almost all the criticisms that can be made of reversible models which assume constant elasticities. Indeed, there is one respect in which its weakness is much more apparent: an individual whose income fluctuates for a long enough period must be assumed to increase his consumption of the commodity indefinitely, without any necessary increase in the average level of his income. This is partly because the assumption of constant elasticities is unrealistic. Strictly, our model should make provision for some sort of saturation level of demand,[7/] as should any reversible model, but the practical difficulties of fitting such a function to the available data become even more formidable as soon as the possibility of irreversibility is allowed for; we must just accept this as an approximation dictated by practical considerations, and hope that the error involved is not too serious. The other reason for our implausible result is that, if a man expects his income to fluctuate, he may tend to align his expenditure in any particular period to what he regards as his normal or average income, rather than to the income he actually receives in that period. The period we use is a year, so that we rule out seasonal and other very short-run variations, but we are left with year-to-year fluctuations, which are likely to be important for quite a number of people. In a reversible model, this effect will show itself, perhaps in some sort of lag, but more generally in deviations from the regression plane. It will mean that, when elasticities are estimated from aggregate data, they will be some sort of average of the "genuine" elasticities (where consumption is fully adjusted to income, because the change in income is regarded as permanent) and the "quasi-elasticities" (where consumption is not fully adjusted, because the change in income is regarded as only temporary).[8/] In our case, it will mean, in addition, that since all changes in income do not exhibit the effect of irreversibility, the difference between the upward and downward elasticities, which measures that effect, will be a similar sort of average.

Further complications arise when we consider the distinctions between long- and short-run, and between individual and community, demand functions. In the first place, the individual's demand function may be reversible in the long run and yet irreversible in the short run. For example, an individual with a reversible long-run demand function may yet react more quickly to upward than to downward changes in income. Unless the lags involved are small compared with our period (one year)--and it seems difficult to argue that they are--we shall be likely to find the difference between these lags showing up as irreversibility in our demand function. Similarly, if the long-run function is irreversible, differential lags may either exaggerate or diminish this irreversibility, though the former seems the more likely.

The individual demand function can be aggregated over any group of individuals to give a community demand function of the same form, on the assumption that the proportional change in income from year to year is the same for each individual.[9/] This assumption, of course, will not be fulfilled in reality, and the effect of this on consumption is in general rather complicated, but in so far as some people's incomes fall while others' rise, we must expect to underestimate the effects of irreversibility, and be prepared to leave, perhaps a large part of them, as "trend." This is an inherent defect of the method--in so far as the irreversibility effect depends on variations of individual incomes which are not reflected in aggregate income, its effect cannot be brought out in an analysis based on aggregate data. Similarly, the irreversibility effect in respect of variations in prices and incomes within the period we are considering will not show up; but there is little short-period fluctuation in price for the commodities we consider, and, as we mentioned above, in so far as changes in income are regarded as purely temporary, and consumption is not adjusted, it is all to the good that our analysis should ignore such changes.

As this aggregation is over a specific group of individuals, and the adult population of the United Kingdom almost completely replaced itself between 1870 and 1938, we must consider how far we are justified in applying such an aggregate model to a changing population. Strictly speaking we are not justified in doing so; the long-run demand function for the community might be reversible, even though the individual demand functions (and hence our aggregate function) were irreversible. For instance, a plausible argument might be made that the higher post-war taxation of tobacco has had more effect in preventing young people from taking up smoking than in cutting down the consumption of smokers, so that the full effect on demand will only be seen when the adult population has completely replaced itself. But, in this case, the "long period" is very long indeed--seventy years or so--and if we bring in the effect of fashion, we might make it run into hundreds of years; for the consumption of each new generation will be influenced by that of the existing adult population, which will be unduly high owing to the irreversibility effect, so that the long-period equilibrium position is only approached after many generations. In practice, we must console ourselves with the fact that the population only changes slowly, and that, as pointed out above, fashion will tend to reduce its short-term effect.

The above arguments have been conducted mainly in terms of changes in income, but in general they hold, <u>mutatis mutandis</u>, for price-changes too.

One important problem which the presence of irreversibility raises is the difficulty of using data from cross-section studies to help in determining the demand function. Analyses combining budget-study with the time-series data have considerable advantages over those based on time-series data alone,[10] and it would be a pity if the presence of irreversibility were to deny us the use of this additional information. So far as irreversibility is simply a matter of differential lags, the elasticity observed in budget-studies will be numerically greater than either of the time-series elasticities, thus providing an upper limit to the time-series estimate; or we might extend Tobin's assumption about lagged income, and apply the restriction that the sum of the upward and the sum of the downward elasticities should both equal the budget elasticity. But for genuine irreversibility the position is more complicated. For instance, if we assume that the proportional change in income from year to year is the same for each individual, the cross-section elasticity will remain the same from year to year, irrespective of the values of the time-series elasticities, which is not a helpful result! Success here would seem to depend on producing a plausible hypothesis as to the way in which the income distribution of a given budget-study had been reached. However, in applying our model to Prest's data the question did not arise, as there is no satisfactory cross-section data available.

IV. Application to Prest's Data

In this application, the sources cited by Prest were used, but as these figures had been revised since Prest made his analyses, his calculations were repeated with the revised data.[11] This was done for the "Basic Equation" (1), and for the "First Differences Equation"

$$\text{(3)} \qquad \frac{C_t}{C_{t-1}} = \left(\frac{Y_t}{Y_{t-1}}\right)^a \left(\frac{P_t}{P_{t-1}}\right)^b e^{c + d(2t-1)}$$

which is obtained from (1) by taking logarithms and then first differences. The results for tobacco, beer, and spirits[12] are given in Tables 2-4, together with the results of fitting the Irreversible Equation (2).

Apart from using the revised figures, two departures were made from Prest's procedure. First, the figures for 1920 were excluded from the analysis, as consumption still showed the effects of rationing. Secondly, the experiment was made of including discontinuity variables in the analyses for beer and spirits. This produced significant increases in the correlation coefficients, and the bunch-maps showed some improvement; it also led to increased estimates of the price- and income-elasticities, and to an increase in the proportion of the variation explained by the economic variables, as noted on page 3.

However, our main interest in Tables 2-4 is in comparing Prest's equations (1) and (3) with the Irreversible Equation (2). We should expect the result of introducing the possibility of irreversibility to decrease the trend, thus indicating that part of the trend was due to the irreversibility effect, and part to other social changes. Comparing the results for equations (2) and (3), we see that this is so in the case of beer, although the irreversibility effect is quite insigificant; but tobacco and spirits show an increased trend, offset by an irreversibility effect in the opposite direction. For spirits, where the trend is negative, this is quite plausible; we could easily argue that the habit forming properties of spirit-drinking lead to a positive irreversibility effect, which only partially offsets a strong negative trend due to social factors. But it is very hard to believe that the irreversibility effect for tobacco is negative, and in all three cases there seems to be considerable intercorrelation between the trend and irreversibility effects. The results of fitting the Irreversible Equation without trend [i.e., putting $c = 0$ in (2)] are also given in Tables 2-4. They show, in each case, that the irreversibility effect can be made to explain most of the trend. This is not, of course, a very satisfactory state of affairs—either a time trend or the irreversibility effect can be made to explain the trend, but, owing to intercorrelation, we cannot fit both together and so obtain an explanation in which the trend is attributed in part to each. However, while bearing in mind that the irreversibility effect is probably taking the credit for a good deal of trend which is not due to irreversibility at all, it is interesting to consider how good an explanation can be obtained in terms of irreversibility alone, and to compare this with the explanation by time-trends. Table 5 gives the size of the irreversibility effect (i.e., the difference between the upward and downward elasticities) and its standard error, for what it is worth. It is difficult to regard the "significance" of the effect in the case of tobacco as proving anything, because we have omitted the time-trend from the equation. Nor can its lack of "significance" in the case of beer and spirits be held to show the absence of the irreversibility effect; for, in the First Differences Equation, the standard errors vary between 10 percent and 50 percent of their elasticities, so that one could hardly expect a difference of 10 percent or 20 percent between two such elasticities to be "significant." These qualifications are quite apart from possible bias in the estimates of standard errors, due, for example, to serial correlation of the errors in the regression equations. In short, we must regard the standard errors as providing merely a rough guide as to the relative orders of magnitude of the effects we are investigating and the errors in our estimates.

In comparing the multiple correlation coefficients (R) obtained by the two methods, we must compare the Irreversible Equations only with the First Differences Equations, and not with the Basic Equations. For, quite apart from serial correlation of errors, if the predetermined variables are positively serially correlated, the multiple correlation coefficient for a First Difference

Equation will tend to be appreciably less than that for the corresponding Basic Equation,[13] as a glance at Tables 2-4 will verify. On this basis, it seems that irreversibility alone gives a slightly less good explanation of the trend than time-trend alone.[14]

It is, however, possible to compare the Irreversible with the Basic Equation, and at the same time to see how far the irreversibility effect can be used to explain the change in tastes during the war. If we write z_t for the estimated value of $\log C_t/C_{t-1}$ obtained from the Irreversible Equation, then $M + \Sigma_{i=1}^{t} z_i$ gives us an estimate of $\log C_t$, where M is a constant to be determined. If we let M be the mean of $\log C_t - \Sigma_{i=1}^{t} z_i$, then $\Sigma_t(\log C_t - M - \Sigma_{i=1}^{t} z_i)^2$ gives us a residual sum of squares from which we can obtain an estimate of R^2 comparable with that for the Basic Equation. In fact, by varying our assumptions slightly we can obtain several estimates, and these we can use to throw some light on the change of tastes during the war. First, if we put $z_i = 0$ for the war years, we get an estimate of the correlation obtained making no allowance at all for the change in tastes during the war. If we give z_i its calculated value for the war years, we are allowing for such change as is explicable by irreversibility. Finally, if we allow M to take two different values, for the pre-war years being equal to the mean of $\log C_t - \Sigma_{i=1}^{t} z_i$ over the pre-war years, and for the post-war years to the mean over the post-war years, we get an estimate allowing for all changes--what we have done is equivalent to inserting a discontinuity variable.

The results of these calculations are shown in Table 6, where it will be seen that, on the third assumption the explanation obtained was again slightly less good than that of the time-trend analysis. Perhaps more interesting is the comparison of the first three columns of Table 6, where it will be seen that, for tobacco, 39 percent of the variation in consumption is due to changes during the war, and 27 percent to war-time changes explicable by the irreversibility effect. The corresponding figures for spirits are 56 percent and 48 percent, and for beer 25 percent and 22 percent; while, if time-trend is included in the beer analysis, the figures become 20 percent and 17 percent.[15] The position is complicated by the fact that the definition of the United Kingdom changed during the war too, so that the wartime changes reflect the difference in habits between the United Kingdom and Southern Ireland as well as social changes. However, rough calculations show that the differences are very small for spirits and tobacco, while for beer they lead to an underestimate of the social change during the war of about 3 percent.

These figures are all rather striking, but the most noteworthy point is that the introduction of time-trend into the beer analysis, which reduces the irreversibility effect to apparently negligible proportions, does not prevent it from explaining a large part of the wartime change. Thus, while our results are generally inconclusive--and it must be emphasized that they are inconclusive rather than negative--they do suggest that irreversibility may be quite an important factor in the change of tastes that occurs in the general upheaval of a war.

However, it may be well to conclude by emphasizing what we have not done. We have shown that an explanation of the phenomena can be given in terms of irreversibility, but even so, it does not explain the facts so closely as the models using complicated functions of time, and intercorrelation prevented us from attempting to decide how far irreversibility was "really" responsible for the trend. We have not investigated systems using complicated lags, or the prices of numerous commodities which could plausibly be considered to be "related," and we must expect that these systems, and many others, could also be made to explain the data. In short, we have achieved an explanation, but the data are too scanty for us to make a claim that it is the explanation.

TABLE I

Commodity	Analysis	Percent of Variance Attributed to:		
		Economic Variables	Covariance Term	Social Variables
Tobacco	Prest's	0.6	-10	109
	Revised Data	1.4	-17	115
	Short Periods	25		
Beer	Prest's	17.5	39	44
	Revised Data	14	31	55
	+ Discontinuity Variables	22	1	73
	Short Periods	44		
Spirits	Prest's	9	31	60
	Revised Data	8	27	65
	+ Discontinuity Variables	25	29	45
	Short Periods	75		

Note: - The "revised data" analysis consists simply in applying Prest's equations to the revised data. For beer and spirits, we also show the results of inserting discontinuity variables into Prest's equation, and applying these new equations to the revised data; while the "short periods" analysis gives the results of the process described on page 3, of splitting the data into 5 and 6 year periods.

TABLE 2

TOBACCO

Parameter	Basic Equation (1)	First Differences (3)
k	2.54	-
a	0.12 ± 0.07	0.13 ± 0.07
b	-0.39 ± 0.04	-0.33 ± 0.06
c	0.0075 ± 0.0005	0.017
d	0.000076 ± 0.000006	0.00015 ± 0.00006
f	0.04 ± 0.01	-
R^2	0.995	0.40

Parameter	Irreversible Equation (2)	
	With Trend	Without Trend
c	0.019	≡ 0
a^+	0.08 ± 0.12	0.35 ± 0.11
a^-	0.15 ± 0.13	0
b^+	-0.35 ± 0.09	-0.23 ± 0.08
b^-	-0.21 ± 0.19	-0.67 ± 0.17
R^2	0.36	0.24

TABLE 3

BEER

Parameter	Basic Equation (1)		First Differences (3)
	With Discontinuity Variable	Without Discontinuity Variable	
k	2.08	2.34	-
a	0.70 ± 0.13	0.40 ± 0.11	0.26 ± 0.13
b	-0.76 ± 0.08	-0.59 ± 0.07	-0.70 ± 0.08
c	-0.012 ± 0.002	-0.0072 ± 0.0014	-0.0033
d	-0.00015 ± 0.00002	-0.00012 ± 0.00002	-0.00014 ± 0.00010
f	0.08 ± 0.02	≡ 0	-
R^2	0.96	0.95	0.58

Parameter	Irreversible Equation (2)	
	With Trend	Without Trend
c	-0.0020	≡ 0
a^+	0.25 ± 0.20	0.22 ± 0.15
a^-	0.29 ± 0.43	0.34 ± 0.37
b^+	0.71 ± 0.11	-0.72 ± 0.11
b^-	-0.66 ± 0.23	-0.63 ± 0.20
R^2	0.57	0.53

TABLE 4

SPIRITS

Parameter	Basic Equation (1)		First Differences (3)
	With Discontinuity Variable	Without Discontinuity Variable	
k	-0.53	+0.38	-
a	1.22 ± 0.18	0.77 ± 0.16	0.66 ± 0.18
b	-1.00 ± 0.12	-0.59 ± 0.06	-0.60 ± 0.13
c	-0.013 ± 0.001	-0.011 ± 0.001	-0.021
d	-0.00013 ± 0.00001	-0.00012 ± 0.00002	-0.00032 ± 0.00013
f	0.20 ± 0.05	≡ 0	-
R^2	0.992	0.989	0.40

Parameter	Irreversible Equation (2)	
	With Trend	Without Trend
c	-0.041	≡ 0
a^+	0.99 ± 0.24	0.35 ± 0.22
a^-	0	0.91 ± 0.52
b^+	0.51 ± 0.16	-0.68 ± 0.17
b^-	-1.23 ± 0.39	-0.47 ± 0.37
R^2	0.39	0.27

TABLE 5

The Significance of the Irreversibility Effect

Commodity	Size of Irreversibility Effect and Standard Error	
	For Income	For Price
Tobacco	0.35 ± 0.11	0.44 ± 0.20
Beer (With Trend)	-0.04 ± 0.56	-0.06 ± 0.29
Beer (Without Trend)	-0.12 ± 0.43	-0.08 ± 0.25
Spirits	-0.56 ± 0.60	-0.21 ± 0.44

TABLE 6

Comparison of Irreversible With Basic Equations

Commodity	R^2 For Irreversible Equation			R^2 For Basic Equation
	Not Allowing For any Change	Allowing for Irreversibility Effect	With Discontinuity Variable	
Tobacco	0.58	0.85	0.97	0.995
Beer (With Trend)	0.72	0.89	0.92	0.96
Beer (Without Trend)	0.67	0.89	0.92	
Spirits	0.40	0.88	0.96	0.99

FOOTNOTES

1. A. R. Prest, "Some Experiments in Demand Analysis," The Review of Economics and Statistics, Vol. 31, February, 1949, pp. 33-49.

2. The regression equation was written in the form

$$\log C_t = \log K + a \log Y_t + b \log P_t + ct + dt^2 + fx$$

and fitted by least squares. Then, for the sample,

$$\mathrm{var}(\log C_t) = \mathrm{var}(a \log Y_t + b \log P_t) + \mathrm{cov}(a \log Y_t + b \log P_t, ct+dt^2+fx) + \mathrm{var}(ct+dt^2+fx) + \mathrm{var}(u_t)$$

where u_t is the difference between the observed and calculated values of $\log C_t$. Thus we may regard $\mathrm{var}(a \log Y_t + b \log P_t)$ as that part of $\mathrm{var}(\log C_t)$ which is unequivocally explained by Y_t and P_t, although any conclusions we may draw from its size must be qualified by the existence of the covariance term, which cannot be unequivocally attributed to either economic or social factors. Table 1 gives the relative magnitudes of these terms.

It will be seen that the results are, strictly, the proportions of variance of $\log C_t$, rather than of C_t, explained by the economic variables. In Prest's analyses, and in my own work on his data, it was convenient to work in terms of logarithms of variables, so that, throughout, variances and correlation of coefficients refer to the logarithms of variables. If we were comparing correlation coefficients from logarithmic with those from nonlogarithmic equations, this procedure would not be justifiable, as it tends to overestimate the former. However, as all the comparisons we make are between logarithmic equations, it seems unlikely that the errors introduced will be important, to the order of accuracy to which we are working, and a considerable amount of computing is saved.

3. Cf., J. R. N. Stone, Consumers' Expenditure in the United Kingdom, 1920-38.

4. A. Marshall, Principles of Economics, Appendix H, 3.

5. T. Haavelmo, "The Probability Approach in Econometrics," Econometrica, Vol. 17, Supplement, July, 1944.

6. J. S. Duesenberry, Income, Saving and the Theory of Consumer Behaviour, Chapter 5. F. Modigliani, "Fluctuations in the Saving-Income Ratio: A Problem in Economic Forecasting," Studies in Income and Wealth, Vol. XI.

I also found that Prest had himself, carried out an (unpublished) analysis for tobacco in which he split his data into two groups, according as tobacco money price was rising or falling, and fitted his function (1) to each separately. This is equivalent to a rather odd function, in which the elasticities of relative price and real income can have different values according as money price is rising or falling.

7. It is possible to think of functions which would give C as a monotonic increasing function of Y, approaching asymptotically a minimum and a maximum

value as Y tends to 0 and ∞, thus allowing for the saturation phenomenon. If such a function is made irreversible, as it easily can be, we get something looking very like the hysteresis curves which occur in physics, and indeed, the two phenomena are at least superficially analogous.

8. This phenomenon is, of course, quite distinct from the straight-forward lag which may occur when a family adjusts itself to a new income-level which it regards as quite permanent; although, of course, the two effects may become confused when a function is fitted to actual data.

9. Let y_{it}, c_{it} be the income and consumption of the i^{th} of a group of n individuals in period t, and $Y_t = \Sigma_{i=1}^{n} y_{it}$, $C_t = \Sigma_{i=1}^{n} c_{it}$ the income and consumption of the group.

Then if

$$\frac{y_{it}}{y_{it-1}} = k_t \qquad (i = 1, \ldots, n)$$

$$\frac{Y_t}{Y_{t-1}} = \frac{\sum_{i=1}^{n} y_{it}}{\sum_{i=1}^{n} y_{it-1}} = k_t = \frac{y_{it}}{y_{it-1}} \qquad (i = 1, \ldots, n)$$

so that $$\frac{c_{it}}{c_{it-1}} = e^c \left(\frac{y_{it}^+}{y_{it-1}}\right)^{a^+} \left(\frac{y_{it}^-}{y_{it-1}}\right)^{a^-} \left(\frac{P_t^+}{P_{t-1}}\right)^{b^+} \left(\frac{P_t^-}{P_{t-1}}\right)^{b^-}$$

implies $$\frac{c_{it}}{c_{it-1}} = e^c \left(\frac{Y_t^+}{Y_{t-1}}\right)^{a^+} \left(\frac{Y_t^-}{Y_{t-1}}\right)^{a^-} \left(\frac{P_t^+}{P_{t-1}}\right)^{b^+} \left(\frac{P_t^-}{P_{t-1}}\right)^{b^-}$$

and hence $$\frac{C_t}{C_{t-1}} = e^c \left(\frac{Y_t^+}{Y_{t-1}}\right)^{a^+} \left(\frac{Y_t^-}{Y_{t-1}}\right)^{a^-} \left(\frac{P_t^+}{P_{t-1}}\right)^{b^+} \left(\frac{P_t^-}{P_{t-1}}\right)^{b^-}.$$

10. See, for example, J. Tobin, "A Statistical Demand Function for Food in The United States," Journal of the Royal Statistical Society, 1950. Perhaps the most important advantage is that the cross-section data provide independent variation of the explanatory variables, which is so notoriously lacking in time-series data.

11. I should like to express my gratitude to Dr. Prest for his kindness in letting me use his data, much of it unpublished, and to his assistant, Mr. A. A. Adams, for help on many points of detail.

12. The tea analysis is omitted, because the results are unsatisfactory. For the First Difference Equation, R^2 was only .22, and the income variable quite insignificant, while the Irreversible Equation gave even worse results. H. S. Houthakker's work on family budgets suggests that part of the trouble

is a wide difference in income elasticities for different income levels.

We may note here that as the prices of beer, spirits and tobacco are largely determined by the level of excise duties, we can be fairly confident that it is the demand, and not the supply functions, that we are estimating.

13. $R^2 = \frac{A}{A+B}$, where A is the variance explained, B the residual variance. If the errors are serially uncorrelated but the predetermined variables have a positive first serial correlation, taking first differences will double the expected value of B, but less than double that of A.

14. Here we have not tested whether this difference is "significant" or not. Even if it proved not to be significant we should be over-stating our case if we claimed that, therefore, irreversibility provided "as good" an explanation.

15. 0.3 percent is explained by the operation of the time trend during the war years.

GAMES AGAINST NATURE

John Milnor

The following is a statment of some results in the study of games against nature. They will be treated in more detail in a RAND paper which I am working on.

Consider a matrix in which the player chooses a row and nature chooses a column. Assume that there is a rule which assigns to each such matrix a relation $\gtrsim$ between pairs of rows or probability combinations of rows, and which satisfies some of the following axioms:

1. $\gtrsim$ is a complete ordering.
2. The order of two rows is not changed by the adjunction of a new row.
3. It is not changed by any permutation of the rows or columns.
4. If each component of r is greater than the corresponding component or r' then $r > r'$.
5. The order of the rows is not changed if a constant is added to a column.
6. It is not changed if a new column is added which is a probability combination of the others.
7. It is not changed by a transformation $u' = \lambda u + \mu$, $\lambda > 0$, of the utility.
8. It is continuous. That is if the matrices $M_i \to M$ and if for each i, $r_i \gtrsim r_i'$, then $r \gtrsim r'$.
9. The set of maximal rows (optimal strategies) is convex.

The following assertions are proved.

10. Axioms 1, 2, 3, 4, 5 imply the Laplace criterion.
11. Axioms 1, 2, 3, 4, 6, 7, 8 imply the Hurwicz criterion.
12. Axioms 1, 2, 3, 4, 6, 8, 9 imply the minimax criterion.

AN ALGORITHM FOR THE DETERMINATION OF ALL SOLUTIONS OF A TWO-PERSON ZERO-SUM GAME

H. Raiffa

Step 1: Compute the initial set of critical points consisting of the m corner points of $\tilde{C}_1$, and initial index set $K_1 = \{1, 2, \ldots, m\}$.

Step j + 1: Assume the j^{th} step ($j = 1, 2, \ldots, n-1$) to be completed with critical points $P_1, \ldots, P_{k_j}$ and index set K_j obtained.

1. For each critical point $P_i = (\xi^{(i)}; x^{(i)})$, $i \in K_j$ compute $d_i^{(j+1)} = P_i \cdot C'_{j+1}$ where $C'_{j+1} = (a_{1,j+1}, \ldots, a_{m,j+1}, -1)^T$.
2. Add the symbol $\tilde{C}_{j+1}$ to the label of all points P_i such that $d_i^{(j+1)} = 0$.
3. Find the piercing point of the line between P_{i_1} and P_{i_2} and the plane $\tilde{C}_{j+1}$ if
   (a) $d_{i_1}^{(j+1)} \quad d_{i_2}^{(j+1)} < 0$
   (b) m-1 of the planes in each of the labelled points $\tilde{P}_{i_1}$ and $\tilde{P}_{i_2}$ are the same, and the vertices P_{i_1} and P_{i_2} are adjacent.

The new point to be added is

$$\frac{-d_{i_1}^{(j+1)} \; P_{i_1} + d_{i_2}^{(j+1)} \; P_{i_2}}{-d_{i_1} + d_{i_2}} \, .$$

This point is to be added to the set of critical points and the planes in its symbol are the planes which the two points above have in common together with $\tilde{C}_{j+1}$.

4. Add to the set of critical points all corner points of $\tilde{C}_{j+1}$ which are relative row minima i.e., whose x values satisfy

$$a_{i,j+1} < \min\left\{a_{i_1}, a_{i_2}, \ldots, a_{i_j}\right\}.$$

5. Eliminate all critical points P_i with $d_i^{(j+1)} < 0$.

6. On the basis of rules 3, 4, and 5 compute the index set K_{j+1} from the set K_j.

At the conclusion of n such steps a finite set of critical points which completely describe $f(\xi)$ is obtained.

Example illustrating computational technique.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 5/2 \\ 3 & 2 & 1 \end{bmatrix}$$

	P	ξ			x	$\tilde{C}_1$	$\tilde{C}_2$	$\tilde{C}_3$	π_1	π_2	π_3	d^2	d^3
	1	1	0	0	1	x				x	x	2	1
	2	0	1	0	2	x			x		x	-2	
	3	0	0	1	3	x			x	x		-1	
$1/2\ P_1 + 1/2\ P_2$	4	1/2	1/2	0	3/2	x	x				x		3/4
$1/3\ P_1 + 2/3\ P_3$	5	1/3	0	2/3	7/3	x	x			x			-1
	6	0	0	1	2		x		x	x			-1
	7	0	1	0	0		x		x		x		2.5
$5/7\ P_6 + 2/7\ P_7$	8	0	2/7	5/7	10/7		x	x	x				
$1/2\ P_1 + 1/2\ P_5$	9	2/3	0	1/3	5/3	x		x		x			
$4/7\ P_4 + 3/7\ P_5$	10	3/7	2/7	2/7	13/7	x	x	x					
	11	0	0	1	1			x	x	x			