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An Axiomatic Approach to Measurable Utility

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We consider a closed bounded and convex subset A of E_n, the n-dimensional Euclidian space. A will be the set of prospects. In A we consider a closed, convex subset F, the set of feasible prospects. Given any two elements, a and b, in F we assume a relation $\geq$ exists ($a \geq b$ to be read as: a is at least as good as b). Moreover, we usually assume that given two feasible prospects a and b at least one of the two statements $a \geq b$ or $b \geq a$ is true. Moreover, the reasonable assumption $a \geq b$, $b \geq c$ imply $a \geq c$ is usually made. We should like to introduce axioms on the set F and this relation in order to reduce this ordering to an ordering by real numbers attached to these prospects.

Definition. If $a \geq b$ but not $b \geq a$, then $a > b$ (read as a is preferred to b)

Definition. If $a \geq b$ and $b \geq a$ then a i b (read as a is indifferent to b)

Definition. A relation R defined on a set S is a complete ordering if:

1. $a, b \in S$ then at least one of aRb or bRa is true.
2. $a, b, c \in S$ aRb, bRc imply that aRc.

Whenever we write $ta + (1-t)b$ $a,b \in F$, t a real number with $0 \leq t \leq 1$ we mean the usual scalar multiplications, and additions of vectors in the n-dimensional Euclidian space.

Axiom 1. F is completely ordered under the relation $\geq$.

If $a \in F$ we define two subsets of F related to a.

Definition. $F_a = \{x \in F \mid x \geq a\}$

Definition. $F^a = \{x \in F \mid a \geq x\}$

Our next axiom will deal with the nature of these sets F_a and F^a.

Axiom 2. For any $a \in F$ the sets F^a and F_a are closed.

On the basis of these two axioms we prove

Theorem 1. There exists at least one element $M \in F$ and one element $m \in F$ so that for any $x \in F$

(1) $M \geq x$

(2) $x \geq m$

Proof. Consider the collection of subsets F_a where a ranges over F. Let $F_{a_1}, F_{a_2}, \ldots, F_{a_m}$ be any finite subcollection of this. Without loss of generality we suppose $a_m \geq a_{m-1} \geq \ldots \geq a_2 \geq a_1$. From the definitions involved it is clear that if $a \geq b$ then $F_a \subseteq F_b$. Thus $F_{a_m} \subseteq F_{a_i}$ for each $i = 1, 2, \ldots, m$. Hence $\bigcap_{i=1}^{m} F_{a_i}$ is not empty since it contains F_{a_m}. Since F is closed and bounded in E_n it is compact; moreover since the F_a are closed and any finite number of them have non-void intersection all the F_a's have a non-zero intersection T since F is compact. Let $M \in T$. Suppose $x \in F$, $x > M$. Then $M \notin F_x$, and so $M \notin T$, a contradiction. Similarly we obtain the existence of minimal elements in F. We shall use these maximal and minimal elements to construct our utility functions.

The next few axioms will involve the interrelation of the ordering and the vector properties of our space F. We give these axioms with no comments nor justification (as these can be found in Marschak's paper).

Axiom 3. If $a,b,c,d \in F$ and a i b, c i d then for any t, $0 \leq t \leq 1$ $[ta + (1-t)c]$ i $[tb + (1-t)d]$.

Axiom 4. (Continuity Axiom) If $a > b > c$ then there exists a t, $0 < t < 1$ so that b i $(ta + (1-t)c)$.

Definition. The indifference set of a, $J(a) = F_a \cap F^a$; i.e. the set of elements b with b i a. J (a) being the intersection of two closed sets is itself closed. Clearly if $b \notin J(a)$, then $J(a) \cap J(b)$ = null set.

Definition. $A \in F$ is a maximal element of F if for any $x \in F$, $A \geq x$.

Sublemma. If A and M are maximal elements then A i M.

Proof. Trivial.

We likewise define minimal and have a corresponding trivial sublemma. We pick a maximal element M and a minimal element m and consider them fixed henceforth.

Lemma 1. If $a \in F$ then there exists a t, $0 \leq t \leq 1$ so that a i $[tM + (1-t)m]$.

Proof. If a i M, put t = 1; if a i m put t = 0. If $M > a > m$ the lemma is just axiom 4.

Using the proof as in Marschak (Rationality, Uncertainty, Utility, lemma 1) we have

Lemma 2. If a i a' and p, an interior point, is collinear with a and a', then p i a.

However we can prove the stronger

Theorem 2. If $a \; i \; a'$ and $p \in F$ is collinear with a and a' then $p \; i \; a$.

Proof. If p is interior we are done. So suppose p is a boundary point. Then p is the limit of interior points on the line aa'; i.e. p is the limit of points in $J(a)$. Since $J(a)$ is closed (remark above) $p \in J(a)$; i.e. $p \; i \; a$.

Theorem 3. There is either only one indifference set or an infinite number of distinct indifference sets.

Proof. Suppose there are more than one indifference sets. We claim then that $M > m$. For if $m \; i \; M$ then if $a \in F$, $a \; i \; tM + (1-t)m$ by axiom 4 for some $0 \leq t \leq 1$ and $[tM + (1-t)m] \; i \; [tM + (1-t)M] = M$ by axiom 3 and so there would only be one indifference set. Consider the line joining M and m. Pick any two distinct points P and Q on the line segment Mm. If $P \; i \; Q$, then by theorem 2 all points on the segment Mm are indifferent, so $M \; i \; m$ and there is only one indifference set. Thus P is not indifferent to Q and the theorem is proved.

What we are aiming at is to prove the existence of a real valued function, u, defined on F which is linear and has the property that $u(a) > u(b)$ if and only if $a > b$. Now if there were only one indifference set we construct the function u by defining $u(a) = 1$ for every $a \in F$. This would be seen to possess the requisite properties. So henceforth we assume there is more than one indifference set; i.e. there are an infinite number of distinct indifference sets.

Under these conditions in proving theorem 3 we have also proved

Theorem 4. If $0 \leq t \leq 1$, $0 \leq s \leq 1$, then $[tM + (1-t)m] \; i \; [sM + (1-s)m]$

if and only if $t = s$.

Actually in proving theorem 3 we proved more, namely the following corollary to theorem 4.

Corollary: If $b > a$, $0 \leq t, s, \leq 1$, then $[tb + (1-t)a]$ i $[sb + (1-s)a]$ if and only if $t = s$.

Theorem 5. If $b > a$, $0 < t < 1$ then $b > c > a$ where $c = tb + (1-t)a$

Proof. If either b i c or a i c then we would obtain that a i b by theorems 2 and 3 since b,a,c are collinear and distinct by corollary to theorem 4. So suppose that $a > c$. Thus $b > a > c$. Hence for some k, $0 < k < 1$ a i $[kb + (1-k)c]$ i $\{bk + (1-k)\ [tb + (1-t)a]\} = (1-t)(1-k)a + b(k + t - tk)$. The corollary to theorem 4 yields $(1-t)(1-k) = 1$, forcing both $t = 0$ and $k = 0$, contrary to hypothesis. Similarly we can show $b > c$.

Theorem 6. If a i $[tM + (1-t)m]$, b i $[sM + (1-s)m]$, $0 \leq t, s \leq 1$ then $a > b$ implies that $t > s$.

Proof. Suppose $a > b$ and $t \leq s$. By axiom 3 $t \neq s$ for otherwise a i b. So we may suppose that $t < s$. Now since $M > a > b$, (M i a implies $t = 1 \geq s$) a i $[rM + (1-r)b]$ i $[(rM + (1-r)\ sM + (1-s)m] = (r + s - rs)M + (1-r)(1-s)m$. That is, $[tM + (1-t)m]$ i $[(r + s - rs)M + (1-r)(1-s)m]$. Thus by theorem 4 $t = r + s - rs = s + r(1-s) \geq s$ since $1 - s \geq 0$. This contradicts $t < s$, and so the theorem is proved.

Theorem 7. If $0 \leq s < t \leq 1$, and a i $[tM + (1-t)m]$, b i $[sM + (1-s)m]$, then $a > b$.

Proof. Suppose that $b \geq a$. By theorem 4 a is not indifferent to b, so $b > a$; thus $M > b > a$ ($M > b$ by theorem 4, since $s \neq 1$, M i b is impossible). So by axiom 4, b i $[rM + (1-r)a]$ i $\{rM + (1-r)$

$[tM + (1-t)m]\}$, i.e. $[sM + (1-s)m]$ i $[(r + t - rt) M + (1-r)(1-t)m]$.

Thus $s = r - rt + t \geq t$, contradicting $t > s$.

Definition. If $a \in F$, a i $[tM + (1-t) m]$ then $u(a) = t$.

Definition. Function f from F to the reals is said to be linear if $f[ta + (1-t)b] = tf(a) + (1-t)f(b)$ for all $a, b \in F$, $0 \leq t \leq 1$.

Theorem 8. The u defined above defines a real-valued linear function from F into the reals with the property that $u(a) > u(b)$ if and only if $a > b$.

Proof. The linearity follows from a trivial computation and the second part of the theorem is merely theorems 6 and 7 combined.