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Optimum Transportation on Networks

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Summary: The present paper treats in a tentative manner optimum location subject to local availability limits on resources. Transportation of some or all primary, intermediate, or final commodities is admitted. At first a consumption program is supposed to be given, specifying the amount of final commodities to be covered at each place.

The model takes care of continuous, discrete and mixed location formations in one approach. For simplicity it is couched in terms of a 1-dimensional space economy;[1] but it is fairly easy to see how subsequent generalizations might be undertaken.

The first part of the paper does not rest explicitly on the activity analysis model. In the first section the analytical frame for the simultaneous treatment of the discrete and continuous pattern of location is introduced. Section 2 applies this to the case where local price dependent production functions exist. Section 3 deals with availability limits on only 1 resource, which is assumed to be transportable. In section 4 the general case of m resources and n products is analyzed without restrictive assumptions on their transportability. A simple principle

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1. That is, a linear graph, or network.

on the material index (A. Weber) is proved as a theorem. Finally section 5 investigates the relationships of the present approach to the activity analysis schema. A corresponding technology matrix is given. Thus the naive approach of the third section is based on more firm grounds, and conversely the applicability of activity analysis tools to location theory in the traditional sense is demonstrated.

1. In the theory of location it has been common to use schemas of continuous and discontinuous distributions of production, population, etc., side by side. For instance, isotimes (loci of equal price for a commodity) are usually conceived of as smooth curves; market and supply areas as containing an even distribution of buyers or sellers. In other instances, as when spatial monopoly and oligopoly are under consideration, the location of producers or markets is approached in terms of discrete points. A. Weber [2] assumed also raw material deposits and consumers' markets to be pointwise distributed. Attempts to integrate both methods and to approach the entire space economy in one sweep, are conspicuously few. Loesch's [4] frame of hexagons, designed to explain a hierarchy of economic regions, is a notable example. Another one, essentially stimulated by Loesch's proposals, is the 4-model schema of Miksch [5]. The Leontief-Isard [6] input-output schema on the other hand achieves a representation of the entire space economy by deliberately neglecting the finer structure and in particular the spatial pricing pattern. Since it does not purport to be a causal analysis of the locational phenomena, we are not concerned here with this interesting approach.

This section proposes a simple mathematical frame to deal, in the 1-dimensional case, with the discrete and continuous model simultaneously.

The typical picture is that of a network of routes, along which there is a continuous and/or discrete distribution of the excess production (local production minus consumption of commodity X). The routes may be curvilinear and arranged in whatever configuration. To identify points on this network let t be a current parameter such that every point P is associated with (at least) one parameter value t. $P=P(t)$. It will be useful to assign the routes R_m ($m=1, \ldots, \ell$) of the network to disjoint intervals of t on the real line, say

(1.0) $\quad 2m-1 \leq t \leq 2m$ for t of $P(t) \in R_m$.

Let t be unique for each point in the interior of a route. However, with each junction point P^j, $j = 1, \ldots, \ell$, of routes R_m, there is thus associated a set of indices or of routes on which P^j is located $\left\{m,\ P^j \in R_m\right\}$, and a set of parameter values t, $\left\{t,\ 2m-1 \leq t \leq 2m \ \& \ P^j \in R_m\right\}$, one value for each of the adjoining routes.[2] However P^j is specified by any one element of the set of t; therefore it is legitimate to define a set $\left\{m\right\}$ as a function of t by

$$(1.1) \qquad \left\{m(t)\right\} = \left\{m,\ P(t) \in R_m\right\} = \begin{cases} \left\{m,\ P^j \in R_m\right\} & \text{in the case of a junction point} \\ \left\{m,\ 2m-1 \leq t \leq 2m\right\} & \text{otherwise.} \end{cases}$$

In the following we consider the interior of one route R_m. At point t^r, $r=1, 2, \ldots, Z_m$) let the excess production function q have finite values which may depend on an as yet unspecified, parameter p: $\bar{q}^r=\bar{q}(t^r, p)$. At all other points the excess production function is assumed infinitesimal and an excess production density is defined there as follows. Let λ denote Lebesgue

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2. According to (1.0) t is chosen such as to assume integral values at each junction point.

measure, then on a stretch of length $d\lambda$ the excess production shall be given by $q^* = q^*(t, p)\, d\lambda$. Of course, q^* may vanish identically. If u is the flow of X, measured positive, if in the direction of increasing values of t, and if $t_1 < t_2$ are not any of the points t^r, then it is seen at once that

$$u(t_2)-u(t_1) = \int_{t_1}^{t_2} q^*(t,p)\,d\lambda + \sum_{t_1 < t^r < t_2} \bar{q}(t^r,p)$$

Let μ be a step function which jumps by 1 at each point of discreteness of q. Then we may write instead

$$u(t_2)-(t_1) = \int_{t_1}^{t_2} q^* d\lambda + \bar{q}\, d\mu.$$

Define $q=\bar{q}$ at points t^r and $q=q^*$ at all other points in the interior of R_m, and introduce a measure v with $dv = d\lambda + d\mu$. Then

(1.2) $$u(t_2)-u(t_1) = \int_{t_1}^{t_2} q\,dv.$$ [3]

(For the Lebesgue measure of the finite set of t^r vanishes). Now u is absolutely continuous /3,124/ with respect to v. Therefore its Radon-Nikodym derivative [, 133] exists.

(1.3) $$\frac{du}{dv} = q.$$ [4]

If a price p is associated with each position, then also the second equation of 293 is readily seen to hold

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3. Note that this is not a Stieltjes integral proper, since q and v are discontinuous at the same points.

4. This is the 1-dimensional generalized equivalent of the equation $\operatorname{div} \rho - q = 0$ of Cowles Commission Discussion Paper 293.

(1.4) $$\left|\frac{dp}{d\lambda}\right| \leq k \frac{dp}{d\lambda} = k \frac{u}{|u|} \qquad \text{if } u \neq 0$$

and

$$u=0 \text{ if } \left|\frac{dp}{d\lambda}\right| k.$$

A general proof of (1.3) will be given below.

So far we have considered relationships on only one route. How do these equations behave at junction points? Let the measure V_m on R_m be defined at the junction point P^j as follows:

$$dV_m = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \text{ if } t \begin{Bmatrix} \text{decreases} \\ \text{increases} \end{Bmatrix} \text{as } P^j \text{ is approached on } R_m.$$

Then the equation corresponding to (1.3) is

(1.5) $$\sum_{\{m(t)\}} \frac{du}{dV_m} - q = 0$$

where the sum runs over all indices of routes joining at P^j. This "continuity equation" in flows states that at each junction point inflows must equal outflows minus the excess production at the junction point itself. Because of (1.1), (1.5) is equivalent to (1.3) in the interior of routes, and thus valid everywhere. For (1.4) the corresponding generalization is

(1.6) $$\left|\frac{dp}{d\lambda}\right| \leq k, \frac{dp}{d\lambda} = k \frac{u}{|u|} \text{ if } u \neq 0 \qquad u=0, \text{ if } \left|\frac{dp}{d\lambda}\right| < k$$

and

$$p(t) = p(t_1) \text{ for } \{t, P(t) = P(t_1)\}$$

In order to prove (1.4), (1.6) more rigorously, we shall postulate here a general theorem of activity analysis on efficiency transportation. Namely, that prices can be attached to commodities at different positions, such that they do not permit a positive profit derivable from interlocal transportations, and entail a zero profit for all efficient shipments -- the price distribution must be a continuous function. For the only class of discontinuities conceivable in this context is that of jump discontinuities. But any jump would contradict the "law of indifference" (Jevons) at the particular point market.

The profit associated with the shipment of one unit of commodity from point t to t' is clearly $-p(t)+p(t') - \int_t^{t'} k\, d\lambda(t)$. Generally if the shipment program u is composed of a multitude of transactions, including possibly continuous ones, this expression becomes

$$(1.7)\qquad -\int_{\Gamma} p \frac{du}{dv}\, dv \;-\int_{\Gamma} k\,|u|\, d\lambda$$

where the integrals are taken over the entire network, denoted here as Γ. Because of the continuity of p, this integral is a Stieltjes integral. Let the entire graph be partitioned into two segments, Γ_1, Γ_2 the dividing points being points of continuity of u. Then $\int_{\Gamma} = \int_{\Gamma_1} + \int_{\Gamma_2}$ and on each of the segments, integration by parts may be carried out.

$$(1.8)\qquad -\int_{\Gamma_i} p \frac{du}{dv}\, dv = -\int_{\Gamma_i} pdu = -\, pu \Big|_{\{S_i^o\}}^{\{S_i'\}} + \int_{\Gamma_i} udp \qquad i=1,2$$

where the set (S_i^o) consists of those endpoints of the respective graph, where the parameter values t increase in the direction toward the graph, and the set (S') of those endpoints of the segments, at which t decreases toward the graph respectively. Adding the integrals over both segments makes the terms $-pu$ of (1.8) cancel. Thus

$$(1.9)\qquad -\int_{\Gamma} p \frac{du}{dv}\, dv = \int_{\Gamma} udp = \int u \frac{dp}{d}\, d\lambda,$$

since p is continuous and therefore absolutely continuous with respect to λ. Thus integral (1.7) becomes

(1.10) $$\int_{\Gamma} u\left(\frac{dp}{d} - k\frac{u}{|u|}\right)d\lambda$$

For this to be positive for all possible choices of u, a necessary condition on p is that

(1.11) $$\left|\frac{dp}{d\lambda}\right| \leq k, \text{ that } \frac{dp}{d\lambda} - k\frac{u}{|u|} = 0 \text{ if } u \neq 0$$

and that

$$u = 0 \text{ if } \left|\frac{dp}{d\lambda}\right| < k.$$ This is relation (1.6).

The two conditions (1.6) constitute the second principal equation on the flows in a network. This condition has intuitive appeal. It states that prices increase in the direction of shipments by the amount of transportation costs, and that the price differential can never exceed the local transportation cost.

2. The equation system (1.5), (1.6) plus a boundary condition contains the solution of the problem of efficient location on a single route, provided that the excess productions are given functions of prices. A boundary condition is necessary to link the "economy of the route" with the outer economy, if any interaction exists or to state explicitly that there is no such relationship (closed economic system). If sections on the route with identically vanishing flow do not occur, then one of the two differential equations may be reduced to an integral equation respectively.[5] Thus, if the price is known for t=0 by one of the boundary conditions, then

(2.1) $$p = p_o + \int_{t_o}^{t} k \,\text{sign}\left(\int_{t_o}^{t} q(p)dv\right) d\lambda$$

and

$$p_1 = p_o + \int_{t_o}^{t_1} k \,\text{sign}\left(\int_{t_o}^{t} q dv\right) d\lambda$$ (second boundary condition).

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5. The second differential equation becomes then trivial.

Alternatively [6]

(2.2) $$u = u_o + \int_{t_o}^{t} q\left(\int_{t_o}^{t} \frac{u}{|u|}\, d\lambda\right) dv$$

and

$$u_1 = u_o + \int_{t_o}^{t} q\left(\int_{t_o}^{t} \frac{u}{|u|}\, d\lambda\right) dv$$

These integral equations are solvable by a simple trial and error process. At any point, where u becomes zero, one of the alternatives $\dot{u} < 0$, $u > 0$ has to be tried in the next incipient interval. A final check comes then with the second boundary condition.

If instead of a single route a whole network is considered, then there are no such simple boundary conditions[7]. Let production and consumption be concentrated at the junction points. With suitable notations, the solutions becomes

(2.3) $$\sum_j u_{ij} = q_i(p_i)$$

(2.4) $$p_i - p_j = \frac{u_{ij}}{|u_{ij}|} k_{ij}, \text{ if } u_{ij} \neq 0$$

$$|p_i - p_j| \leq k_{ij} \quad \text{in general;}$$

where p_i is the price at junction i, and k_{ij}, u_{ij} are the transportation costs and flow vectors on route ij respectively.

This is essentially the equation system of the model by Koopmans [1] of efficient transportation. It can be seen that the routing system so described is a graph, possibly with inclusion of neutral circuits l.c. passim.

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6. The existence of several solutions is excluded if $\frac{dq}{dp} > 0$. This follows from the uniqueness proof, given elsewhere, [7] on solutions of boundary value problems of this type; it applies a forteriori to the 1-dimensional model.

7. However the above sketched integral equation method may be applied as well, provided that neighborhoods with $u \equiv 0$ are absent. Integration may be started at any point, assuming arbitrarily there a certain value of u. On each possible orbit, the flows must regain their original value after one circuit. This leads to corrections of the first assumption, until all flow systems match together. The domain of possible error is substantially broader here.

3. In this section the simplest case of resources with local availability limits is taken up. Suppose that there is only one primary commodity of this nature, and that any final commodities for which it is used has to be produced at the place of consumption. Thus we are only concerned with transportation of this primary commodity. Subsequently we relax this restriction on the technology.

Let $g=g(t)$ be the local availability limit of the resource, $q=q(t,p)$ be the net discharge (export) of it and $c=c(t,p)$ the local use for production purposes. Applying the well known conditions on the efficiency prices for primary commodities with availability limits, [1., 5.4, p. 82] which state that $p_{pri=} \geq 0$, $p_{pri>} = 0$[8] gives rise to

(3,1) $q(p)=0$ for $p=0$ almost everywhere[9]

$q(p)=g-c$ for $p>0$.

Equation (3.1) establishes a price-dependent export (or import resp.) program function, which is formally equivalent to the excess production function considered under section 2. This settles the problem.

We proceed to a slightly more general case. Let there be one final commodity produced from the primary one and capable of transportation.

A systematic treatment of this case is contained in the following section. Here we are only interested in the competition for transportation between the two goods. Let $\bar{k}$, $\bar{p}$ refer to the raw material, k, p to the final product. Let an input b of the primary commodity be needed to produce the unit of the final commodity.

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8. "Where $p_{pri=}$ stands for primary commodities whose availability limit is reached under the program and $p_{pri>}$ for those which are not fully used up."

9. More precisely: $q(p)=g-c$ on the closure of set $(p, p>0)$. For, including the limit points does not add to the transportation costs while it enlarges the domain of resources under access.

Then, assuming linearity,

(3.2) $p = a + b\bar{p}$ wherever local production of the final good takes place

and

$p \leq a + b\bar{p}$ otherwise.

For the primary and the final commodity it holds independently that

(1.3) $\left|\frac{dp}{d\lambda}\right| \leq k \quad \left|\frac{d\bar{p}}{d\lambda}\right| \leq \bar{k}$.

With (3.2) $\left|\frac{da}{d\lambda} + \bar{p}\,\frac{db}{d\lambda} + b\,\frac{d\bar{p}}{d\lambda}\right| \leq k$.

Suppose now that the production function is independent of the position (locality). Then $\left|b\,\frac{d\bar{p}}{d\lambda}\right| \leq k$. If shipment of the primary good takes place, then $\frac{d\bar{p}}{d\lambda} = \bar{k}$. It follows that then necessarily $b\bar{k} \leq k$. Let us suppose with A. Weber, that for the weight unit $k=\bar{k}$. Then $b \leq 1$, where b now represents "the proportion of the weight of localized material to the weight of the product" called the "material index" [2, 60]. This is the simplest instance of the "law of transport orientation."[10]

It follows incidentally that the existence of a gradient of production costs, $\frac{da}{d\lambda}$, $\frac{db}{d\lambda}$, interferes with the principle of transport orientation (or in modern language, the principle of minimizing transportation costs).

4. We proceed to the more general case of m final and n primary goods, a consumption program specifying the local demand of final goods, and availability limits restricting the local supply of primary goods. Since the interlocal flow relations have been developed with reference to price dependent net export functions q, it is convenient to attempt the construction of something like price dependent export functions from the efficiency conditions on local production.

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10. Clearly it is only within the frame of a 2-dimensional model that a general demonstration of the phenomena of transport orientation is possible.

Let there be a linear input output relationship with coefficients a_{ij}, $i = 1, \ldots, m$; $j = 1, \ldots, n$, denoting the input of the j^{th} primary commodity required to produce one unit of the final commodity i. Let g_i (≥ 0) be the consumption program and $\bar{g}_j$ (≥ 0) the locally available amount of primary commodities. In the case of raw materials these are the limits on the rate of depot exploitation. It should be remarked that the limits $\bar{g}_j$ refer only to the local resources not to supply from imports.

In the stationary economy which we consider

$$(4.1) \qquad q_i + g_i \geq 0 \qquad , i = 1, \ldots, m,$$

that is net imports can not exceed the consumption program. By definition

$$(4.2) \qquad \sum_i a_{ij} (q_i + g_i) \leq \bar{g}_j - \bar{q}_j , \qquad j = 1, \ldots, n$$

the total input of primary commodities can not exceed the local availability limits plus net imports. The efficiency principle applied to local production requires that

$$(4.3) \qquad \sum_j a_{ij} \bar{p}_j \leq p_i \qquad i = 1, \ldots, m,$$

that is that no production activity permits a positive profit and

$$(4.3a) \qquad q_i + g_i = 0 \qquad \text{if} \qquad \sum_i a_{ij} \bar{p}_j < p_i$$

that transformations involving loss do not take place. [1, passim.]

In Section 3 it was shown quite similarly that net exports of a commodity are zero at almost all points where the price of this commodity is zero.

$$(4.4) \qquad \begin{array}{llll} q_i = 0 & \text{a. e.} & \text{where } p_i = 0 & i = 1, \ldots, m \\ \bar{q}_j = 0 & \text{a.e.} & \text{where } \bar{p}_j = 0 & j = 1, \ldots, n. \end{array}$$

Conversely the availability limits are reached if the price of a primary commodity is positive.

$$(4.4a) \qquad \bar{q}_j = g_j - \sum_i a_{ij} (q_i + g_i) \qquad \text{if } \bar{p}_j > 0 \qquad j = 1, \ldots, n.$$

Equations (4.4), (4.4a) reduce the export programs of primary commodities to those of final commodities and prices of primary commodities. For final commodities we have conditions depending only on prices.

$$(4.5)\qquad \begin{array}{lll} q_i = 0 & \text{if } p_i = 0 & \text{(eqn. 4.4)} \\ q_i = -g_i & \text{if } 0 < p_i < \sum_j a_{ij}\bar{p}_j & \text{(eqn. 4.3a)} \\ q_i \geq -g_i & \text{if } \sum_j a_{ij}\bar{p}_j = p_i & \text{(eqn. 4.1)} \end{array}$$

$$i = 1, \ldots, m.$$

Each combination between an equation (4.4) on primary commodities and an equation (4.5) on final commodities is possible.

The system (4.4) (4.5) exhausts the conditions on efficient local allocation of resources and constitutes something like a price dependent production function.

Example: Let r final and m - r primary goods be nontransportable, $m - r < n$. Denote these by $i = 1, \ldots, r$, $j = 1, \ldots, m - r$ respectively. If all primary resources at a certain point should be utilized, their prices are positive and equation (4.4a) applies. Making use of the assumptions

$$q_i = 0 \qquad i = 1, \ldots, r$$
$$\bar{q}_j = 0 \qquad j = 1, \ldots, m - r$$

we have

$$(4.6)\qquad \sum_{i=r+1}^{m} a_{ij}(g_i + q_i) = \bar{g}_j - \sum_{i=1}^{r} a_{ij} g_i$$

for $j = 1, \ldots, m - r$, and

$$(4.7)\qquad \sum_{i=r+1}^{m} a_{ij}(g_i + q_i) + \sum_{i=1}^{r} a_{ij} - \bar{g}_j = \bar{q}_j$$

for $j = m - r + 1, \ldots, n$.

If the matrix (a_{ij}) has rank not smaller than $m - r$, then (4.6) possesses a unique solution q_i, $i = r + 1, \ldots, m$. The left hand of (4.7) is then known and thus $\bar{q}_j$, $j = m - r + 1, \ldots, n$, uniquely determined. This exemplifies that the system (4.4), (4.5) may determine a unique export program on primary and final commodities to any set of positive prices on resources.

The equations

$$(4.8) \qquad \frac{dp_i}{d\lambda} - k_i \frac{u_i}{|u_i|} = 0 \qquad i = 1, \ldots, m$$

$$(4.9) \qquad \frac{d\bar{p}_j}{d\lambda} - \bar{k}_j \frac{\bar{u}_j}{|\bar{u}_j|} = 0 \qquad j = 1, \ldots, n$$

$$(4.10) \qquad \sum_{\{\ell(t)\}} \frac{du_i}{dv_\ell} - q_i(t) = 0 \qquad i = 1, \ldots, m$$

$$(4.11) \qquad \sum_{\{\ell(t)\}} \frac{d\bar{u}_j}{dv_\ell} - \bar{q}_j(t) = 0 \qquad j = 1, \ldots, n$$

together with (4.4), (4.5) determine the flow systems u_i, $\bar{u}_j$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ uniquely up to "neutral circuits" [1, 248], provided the q_i, $\bar{q}_j$ are unique functions, by (4.4), (4.5), of p_i, $\bar{p}_j$. That this is not generally the case is easily seen. Suppose that in a whole neighborhood $t_1 < t < t_2$, $p_i = \sum_j a_{ij} \bar{p}_j$, a_{ij} = const., and $\frac{d\bar{p}_j}{d\lambda} = \bar{k}_j \frac{\bar{u}_j}{|\bar{u}_j|}$, $\bar{u}_j > 0$ and that $\frac{dp_i}{d\lambda} = k$ for some i. This means that on this route section, all primary goods are shipped in the same direction, and that at every point efficient production of each final commodity is possible. Then

$$\frac{dp_i}{d\lambda} = \sum_j a_{ij} \frac{d\bar{p}_j}{d\lambda} = \sum a_{ij} \bar{k}_j = k_i .$$

The coefficients a_{ij}, k_i, k_j are therefore not independent, and the system of unknowns remains underdetermined. This is what one would expect. It has to remain open to what extent shipments should be made in primary materials or in final goods, since the transportation costs amount to the same, and there is no difference in production cost with the place of production. Rules on the material index follow in the same way as under 3. If commodity i is transported rather than its primary components, then $1 \leq \sum_k a_{ik}$, the right handed expression being now called the material index. Conversely, suppose a primary commodity j is transported. Then for some final good i, $1 \geq \sum_k a_{ik}$; hence $a_{ij} \leq 1 - \sum_k' a_{ik}$ (where the prime indicates that j is to be omitted in the summation). The latter formula is derived under the assumption of a uniform direction of shipments (or no shipments) for all primary goods, whereas the first one holds in general.[11]

5. Throughout the preceding sections, implicit use has been made of theorems of the new welfare economics in the version of activity analysis. It is therefore natural to find a formulation of the present problem immediately in terms of the activity analysis schema.[12] In terms of the input coefficients a_{ij} used above the technology matrix of local production becomes

$$(5.1) \qquad \begin{bmatrix} -a_{11}^{t^r} & \cdots & -a_{1m}^{t^r} \\ -a_{n1}^{t^r} & \cdots & -a_{nm}^{t^r} \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = \begin{bmatrix} -B^{t^r} \\ I \end{bmatrix} = A^r$$

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11. Although this is not interesting any more one, can find a more general formula,

$$a_{ij} \leq 1 - \sum_{k \in j_2} a_{ik} + \sum_{k \in j_1}' a_{ik}; \qquad j \in j_1$$

$$a_{ij} \geq \sum_{k \in j_1} a_{ik} - \sum_{k \in j_2}' a_{ik} - 1; \quad j \in j_2$$

where j_1 is the set of primary goods, shipped in the direction of the final commodity, and j_2 the set of those shipped the other way.

12. This was pointed out first by Dr. Koopmans.

Since we have treated transportation inputs as an independent primary commodity (with no resource limitations) the technology matrix associated with transportation of primary or final goods from a point t_1 to a point t_2 has the following shape.

$$
\left[\begin{array}{c|c}
-I_{(n)} & \\
\hline
 & -I_{(m)} \\
\hline
I_{(n)} & \\
\hline
 & I_{(m)} \\
-c_1^{12} \ldots -c_n^{12} & -c_{n+1}^{12} \ldots -c_{n+m}^{12}
\end{array}\right] = \begin{pmatrix} -T \\ T \\ (c^{12})' \end{pmatrix}
$$

where $I_{(i)}$ stands for the unit matrix with i rows and columns. The vector $-c^{rs} = -(c_1^{rs}, c_2^{rs}, \ldots, c_{n+m}^{rs})$ expresses transportation inputs. $c_i^{rs} > 0$. With these matrices the composite technology matrix of the entire economy is easily constructed:

$$
A = \left[\begin{array}{cccc|c|c|c|c|c|c}
A_1 & & & & -T & -T & \ldots T & & & \\
\hline
 & A_2 & & & T & & -T & -T & -T & \\
\hline
 & & A_3 & & & T & & T & & \\
\hline
 & & & \ddots & & & & & T & \\
 & & & & & & & & & T \\
\hline
 & & & A_z & & & & & & \ldots -T \\
\hline
 & & & & -(c^{12})' & -(c^{13})' & -(c^{21})' & -(c^{23})' & -(c^{24})' & -(c^{z,z-1})'
\end{array}\right]
$$

However, does this matrix represent the same problem we have studied in Section 4? In other words, is the efficient set associated with this matrix identical with the optima determined in Section 4? Clearly not as long as the local consumption programs are fixed. But it is not difficult to see, that the efficient set to the present matrix is equal to the set of optima for all efficient consumption programs; an efficient consumption program being defined here as admissible [satisfying (4.2)] and providing for positive prices on all final goods.

The equivalence between the two follows readily from a comparison of the price vectors associated with either system under efficiency, and of the resource limitations observed. For convenience of language we introduce some notions first. An efficient set of activities in the total economy as represented by the technology matrix A will be called a totally efficient set. It is contrasted with a locally efficient set of production activities at one place within the limits imposed by local availability and the transportation program. It is further distinguished here from a set of efficient transportation activities for a given transportation program.

In accordance with activity analysis, the following notations are being used:

(5.4) $$x' = [(x^1)', (x^2)', \ldots, (x^z)'; (x^{12})', (x^{13})', \ldots, (x^{21})', \ldots, (x^{z,z-1})']$$
$$= (x_1^1, x_2^1, \ldots, x_m^1, x_1^2, x_2^2, \ldots, x_m^2, \ldots, x_1^z, \ldots, x_m^z;$$
$$x_1^{12}, \ldots, x_n^{12}, x_{n+1}^{12}, \ldots, x_{n+m}^{12}, x_1^{13}, \ldots, x_1^{z,z-1}, \ldots, x_{n+m}^{z,z-1})$$

(5.5) $$p' = [(p^1)', (p^2)', \ldots, (p^z)'; k] = [(p^1_{prim})', (p^1_{fin})',$$
$$(p^2_{prim})', (p^2_{fin})', \ldots, (p^z_{prim})', (p^z_{fin})', k]$$
$$= (p_1^1, p_2^1, \ldots, p_n^1, p_{n+1}^1, \ldots, p_{n+m}^1, p_1^2, p_2^2, \ldots, p_{n+m}^2, \ldots,$$
$$p_1^z, \ldots, p_{n+m}^z, k)$$

where superscripts denote the space coordinate t_r, $r = 1, \ldots, z$, and subscripts i the commodities. Since the primary commodities of the total model are in part restricted (the local resources) and in part unrestricted, but in that case desired in themselves (transportation performances), the price vectors associated with the efficient set has accordingly "mixed" properties [cf. 1, Theorem 4.7, p. 66 and Theorem 5.4.1, p. 82].

(5.6) $p^r_{fin} > 0, \; p^r_{pri=} \geq 0, \; p^r_{pri>} = 0 \qquad k > 0$

(5.7) $(p^r)' \; A^r = 0 \qquad (r = 1, \ldots, z)$ and

(5.8) $p^r_i - p^s_i \leq -kc^{rs}$ if $x^{rs}_i = 0$

$p^r_i - p^s_i = -kc^{rs}$ if $x^{rs}_i \neq 0$.

The limitations on locally available primary and final commodities come out as follows

(5.9) $y^r_{prim} + \hat{y}^r_{prim} \geq \eta_{prim}$, where $y^r = A^r x^r$

$y^r_{prim} = -B^r x^r$,

and where $\hat{y}^r$ denotes the vector of primary commodities exported from (or imported to, if negative) point t_r, obtained by row-wise summation over the right half of A:

(5.10) $\hat{y}^r = -\sum_s x^{rs} + \sum_s x^{sr}$

and

(5.11) $y^r_{fin} + \hat{y}^r_{fin} \geq 0$.

Now since $(\hat{y}^r)' = -[\bar{q}_1(t^r), q_1(t^r), \ldots, \bar{q}_n(t^r)]$ by definition of q, $\bar{q}$ (Section 4) the inequalities (5.9), (5.11) correspond exactly to the two inequalities (4.2). The price vectors do correspond if and only if $p^r_{fin} > 0$ for all points t_r. This outlines the proof of the equivalence of the preceding study with a case of activity analysis.[13]

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12. What we have done in Section 4 can be considered as breaking down the technology matrix A into handier components and studying the conditions induced in the corresponding smaller activity stystems.

A totally efficient set is thus the union of all locally efficient sets and an efficient set of transportation activities subject to the conditions 1) that the price vectors associated with the local activities and the price vector of the transportation program are proportional in corresponding components and 2) that the limitations on local resources match with the transportation program. In this form the statement is similar to Theorem 5.11 [1, p. 93] --exchange between economies.[14/]

"A necessary and sufficient condition that an efficient point y shall remain efficient after the addition to the technology of exchange activities at constant relative prices π, is that π be a price vector associated with y in the original technology."

The differences between the approach in Sections 3, 4 and on in terms of activity analysis proper, boil down to 3 points:

1. The first analysis included too few--only one--or too many--all conceivable but possibly inefficient--consumption programs. Two concessions have however to be made. If the local consumption program is price-dependent, then final commodities have always a positive price, provided that a price is at all defined. (The contrary to the latter may happen, if the minimum price necessary to induce local production or shipments of that commodity is still above the maximum consumers are willing to pay. Still there would be an efficient consumption program such that consumption of this commodity at the specified place were positive, this being essentially due to the assumption of unlimited transportation facilities). But then a price dependent program is efficient, and of all efficient sets the only interesting one in that case. The second specification is that to

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14. Nothing prevents us, incidentally, from considering the different localities as countries so that the model might apply to international trade.

each inefficient consumption program there exists an expanded local consumption program that renders all prices of final commodities positive, and is thus efficient.

2. The considerations in the first sections are especially suited to trace the geographical structure of prices and the pattern of interlocal flows. This approach also includes continuous spatial distributions.[15]

3. Within the framework of activity analysis it is more natural to consider transportation inputs as limited and moreover as not separated from other primary commodities.[16] This suggests a line of further investigation.

We do not enter here into the inadequacies in general, of linear models with respect to locational technology, since they do not yield a point of difference between the two set ups here pursued. It is recognized that such deviations of the models constitute an important problem that necessitates a separate investigation.

Conclusion: In a 1-dimensional economy, extended along a network of transportation routes, the optimum system of local production activities and interlocal flows can be described by a set of equations, (inequalities and differential equations), specifications of which had been known in the cases of a purely continuous and a purely discrete situation within a 2-dimensional economy. Alternatively, the optimum solutions (of the present discrete case) are given as the efficient set associated with a simply structured technology matrix of combined production and transportation activities.

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15. This is not necessarily an asset, since in reality there is hardly anything corresponding to it. But the continuous conception of population and industrial patterns, has always played a considerable part in location theory proper.

16. And also to introduce intermediate commodities, although the changes therefrom are not significant.

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