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A Comment on Marschak's Paper "A Simplification of the Axiomatics of 'Measurable Utility'", C.C.D.P. Economics 2012

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In his paper, "A Simplification of the Axiomatics of 'Measurable Utility'" C. C. D. P. Economics No. 2012, Marschak, on the basis of 4 simple and natural axioms tries to get a simple proof of the so-called continuity axiom. Hildreth objected to a certain point in the proof and though his objection is valid, one might still hope to prove the theorem using a different method of proof. Unfortunately, this is not the case, as we give an example where all of Marschak's axioms hold, but where the continuity axiom fails to hold.

Let us recall some of Marschak's concepts. He supposes that he has a set S, with elements A, B, C and an operation, denoted by $(AB)p$, a probability mixture of A with probability p and B with probability $1-p$, $0 \leq p \leq 1$, and $(AB)p \in S$. On this operation he has certain rules which we will not list. In addition he assumes a complete ordering $\leq$ exists on S. His axioms are (where $A < B$ means $A \leq B$ but not $B \leq A$)

1. $\leq$ defines a complete ordering on S.
2. If $A < B$, then $(AC)p < (BC)p$ for any $C \in S$, $0 \leq p \leq 1$

3. If $A = B$ (i.e. $A \leq B$, $B \leq A$) then $(AC)p = (BC)p$ for any $C \in S$

4. There is an A, B with $A < B$

What Marschak attempted to prove on the basis of these was

Continuity axiom: If $C < B < A$ then $B = (AC)p$

Let S be the unit square in the Euclidean two-space $S = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$

We define a complete ordering on S as follows:

$A = (a_1, a_2) < (b_1, b_2) = B$ if

(1) $a_1 < b_1$

and

(2) if $a_1 = b_1$, then $a_2 < b_2$

This is trivially a complete ordering for S.

We define $(AB)p = pA + (1-p)B$. Since Marschak's assumptions on $(AB)p$ are derived from such a definition, direct checking will show all the properties he desires on this operation hold.

We now check his axioms for S.

(1) $<$ constitutes a complete ordering on S (as shown above)

(2) Trivially if $A < B$, $C \in S$

$pA + (1-p)C < pB + (1-p)C$, $0 < p \leq 1$

(3) Likewise, if $A = B$ (i.e. $(a_1 a_2)$, (b_1, b_2) are such that $a_1 = a_2$, $b_1 = b_2$)

$pA + (1-p)C = pB + (1-p)C$.

(4) Some A, B, $A < B$ since $(0, 0) < (1, 1)$.

However the continuity axiom fails for consider $A = (1,1)$, $B = (0,1)$, $C = (0,0)$ then $C < B < A$, yet $B \neq pA + (1-p)C$ for then these three points in the plane would then be collinear, which they clearly are not.

Hildreth pointed out to me that this lexigraphic ordering also gives a counterexample to a conjecture of Goodman and Markowitz (C. C. D. P. Economics 2017, p. 8, second paragraph).

For if S is the set $\{(a_1,a_2) \mid a_1$ positive integers$\}$ using the lexigraphic ordering on the elements of S we see that the conditions 1-4 in C. C. D. P. Economics 2017 are satisfied. However no weights w_1, w_2 exist so that $(a_1,a_2) > (b_1,b_2)$ if and only if $w_1 a_1 + w_2 a_2 > w_1 b_1 + w_2 b_2$, for $(2,2) > (1,n)$ for every positive integer n, however $2w_1 + 2w_2 > w_1 + nw_2$ is only possible if $w_2 = 0$. Hence the ordering would be $(a_1,a_2) > (b_1,b_2)$ if and only if $a_1 > b_1$, which does not coincide with the original ordering.