COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 2016

The Stability of Systems with Nonnegative Coefficients

by John Chipman

I. An economy may be conceived of as partitioned into n sectors (such as countries, regions, classes, or industries), and as behaving according to the following equation:

(1) $dY(t) = dE + MdY(t-1)$

where $dY(t)$ and $dY(t-1)$ are column vectors of changes in sector receipts at times t and t-1 respectively, dE is a column vector of autonomous disbursements into these sectors, and M is a square matrix of which the typical element m_{ij} denotes sector j's marginal propensity to spend to sector i (or, in the Leontief model, industry j's inputs from industry i per unit output of j).[1]

Equation (1) gives rise, by virtue of initial conditions $dY(0) = 0$, $dY(1) = dE$, to a power series

(2) $dY(t) = [I + M + M^2 + \ldots + M^{t-1}]\, dE$

which converges to

(3) $dY(t) = (I-M)^{-1}\, dE$

if and only if the characteristic roots of M all lie within the unit circle of the complex plane. If this condition is satisfied, the system is said to be <u>stable</u>.

Elsewhere [1] I have discussed the necessary and sufficient conditions for stability. These conditions are very cumbersome, however, and since a priori

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1. For the derivation of this model, see Chipman [1].

economic considerations enable us to restrict the matrix M to sets of matrices with certain properties, it is useful to obtain simpler necessary and sufficient conditions under these restrictions.

An important property has been attributed to M by several economists, namely that of nonnegativeness. A matrix may be said to be nonnegative if all its elements are nonnegative. It has been shown by Metzler [8] and [9, p. 339] that a necessary and sufficient condition that the latent roots of a nonnegative matrix M lie within the unit circle is that the principal minors of I-M be all positive.[2] These are the well-known Hicks conditions for perfect stability [4, p. 315].

I shall now consider matrices which have further restrictions, and I shall prove a very simple necessary and sufficient stability condition for such matrices. The two conditions I shall consider separately are that (1) the diagonal elements of M are equal to zero, and (2) the column sums of M are less than or equal to unity.

II. A matrix is said to be hollow if its diagonal elements are all equal to zero. Hollow matrices have been considered by Leontief [7], Goodwin [2], and Waugh [12]. While this is an untenable assumption in the analysis of international, interregional and inter-class trade, it may often be safely assumed in the analysis of inter-industrial relationships.

I shall now demonstrate a fundamental property of nonnegative hollow matrices:

Theorem 1. Let M be a nonnegative hollow matrix. Then for any k-th order submatrix M_k symmetric about the principal diagonal of M, $(-1)^k \det M_k \leq 0$.

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2. See also Hawkins and Simon [3].

Proof: The theorem is easily verified for $k = 1$, 2, and 3. We shall show by induction that if it is true for any k, it is true for $k + 1$. We assume, then, that $(-1)^k \det M_k \leq 0$. We are to show that $(-1)^{k+1} \det M_{k+1} \leq 0$, or what is the same thing, $(-1)^k \det M_{k+1} \geq 0$.

Expand $\det M_{k+1}$ along the first row. The minor of the first (zero) element may be neglected. The minors of the remaining elements, multiplied by $(-1)^k$, are to be alternately nonnegative and nonpositive. The first column of all these minors has no zero. If, in the i-th of the k minors we consider, we permute this column to the right i-1 times, we obtain a hollow matrix except for the fact that its i-th row has no zero in the diagonal position. Call this a nearly hollow matrix. Now the i-th of these minors (before the normalization), multiplied by $(-1)^{k+1} (-1)^i$ is to be nonpositive. After normalization, this minor, now multiplied by $(-1)^{i-1} (-1)^{k+1} (-1)^i = (-1)^{k+2i} = (-1)^k$ (since an odd number of permutations changes the sign of the minor), must be nonpositive. Writing M_k' for a k-th order nearly-hollow matrix, we may then state that if $(-1)^k \det M_{k+1} \geq 0$ then we must have $(-1)^k \det M_k' \leq 0$. We shall suppose it to be true for a (k-1)th order nearly-hollow determinant. That is, we shall assume that $(-1)^{k-1} \det M_{k-1}' \leq 0$.

Now take a k-th order nearly-hollow matrix M_k'. Its i-th column has no zero element. Expand the determinant down this column. The minors so obtained, multiplied by $(-1)^{i+1} (-1)^k$, are to be alternately nonpositive and nonnegative. Hence the j-th minor, multiplied by $(-1)^j (-1)^{i+1} (-1)^k$, should be nonpositive. Therefore the i-th minor, multiplied by $(-1)^i (-1)^{i+1} (-1)^k = (-1)^{k-1}$, must be nonpositive. But the i-th minor is a (k-1)th order hollow determinant, and is indeed (when multiplied by $(-1)^{k-1}$) nonpositive.

The remaining k-1 minors of the i-th column of our k-th order nearly-hollow determinant can all be permuted into (k-1)th order nearly-hollow determinants.

It is easily seen that in the j-th minor, the column without a zero is in the j-th place for $j < i$ and in the $(j+1)$th place for $j > i$. In order for the j-th minor to be normalized into the nearly-hollow form, the column without a zero must go into the $(i-1)$th place for $j < i$ and into the i-th place for $j > i$. For $j < i$, the column without a zero must be permuted from the j-th place to the $(i-1)$th place, requiring $i-1-j$ permutations; for $j > i$ it must go from the $(j+1)$th place to the i-th place, requiring $i-j-1$ permutations. Thus in both cases, $i-j-1$ permutations are required. Now as we have seen, the j-th minor, multiplied by $(-1)^j (-1)^{i+1} (-1)^k$ is to be nonpositive. Hence the j-th minor, after normalization into nearly-hollow form, should, after multiplication by $(-1)^i (-1)^{i+1} (-1)^k (-1)^{i-j-1} = (-1)^{2i} (-1)^k = (-1)^k$, be nonnegative (since an odd number of permutation changes the sign of the determinant). That is to say the j-th minor (which is of order $k-1$), multiplied by $(-1)^{k-1}$, must be nonpositive. But this is indeed the case, by assumption. Hence the theorem is proved.

With the help of Theorem 1, the following important theorem may be derived:

<u>Theorem 2.</u> <u>Let M be a nonnegative hollow matrix. Then a necessary and sufficient condition that its latent roots all lie within the unit circle is that det $(I-M) > 0$.</u>

<u>Proof:</u> The characteristic equation of M is

$$(4)\quad \det (I\lambda - M) = \lambda^n - \sum_{k=1}^{n} \alpha_k \lambda^{n-k} = 0$$

where

$$-\alpha_k = (-1)^k \operatorname{tr}_k M.$$

By definition, $\operatorname{tr}_k M$ is the sum of the k-th order principal minors of M. Hence, by Theorem 1,

$$(5)\qquad \alpha_k \geqq 0 \qquad (k=1,\dots, n).$$

From (5) it follows, by a theorem of Smithies [10, p. 269] and Lange [6, p. 241] that the necessary and sufficient condition that the roots of the polynomial (4) be less than unity in absolute value is that $\sum_{k=1}^{n} \alpha_k < 1$. But from (4) we have

$$\det (I-M) = 1 - \sum_{k=1}^{n} \alpha_k > 0$$

as desired.

This striking theorem gives us simply one necessary and sufficient stability condition, namely the last of the Hicks conditions. It also sheds light on the theory of the consumption function with a distributed lag. Consider the difference equation

$$dY_s(t) = \alpha_1\, dY_s(t-1) + \alpha_2 dY_s(t-2) + \dots + \alpha_n\, dY_s(t-n) + dE_s$$

where s is the household (national income) sector of the economy, and the other n-1 sectors are industries. Such an equation has been used by Lange [6, p. 239], Hicks [5, p. 172], and others, with the following interpretation: The α_k are nonnegative components of the marginal propensity to consume, and $\sum_{k=1}^{n} \alpha_k$ is the marginal propensity to consume. We can now see that this interpretation is valid only if the interindustrial matrix M is nonnegative and hollow.

III. We shall now consider nonnegative matrices of a different type. Instead of hollowness, these matrices will now have the property that no column sum exceeds unity. It will be shown that the necessary and sufficient stability conditions are the same for these matrices as for hollow nonnegative matrices.

<u>Theorem 3.</u> <u>Let M be a nonnegative matrix such that none of its column sums exceeds unity. Then a necessary and sufficient condition that all the characteristic roots of M lie within the unit circle is that I-M be nonsingular.</u>

<u>Proof:</u> The necessity follows from the fact that if det (I-M) = 0, some characteristic root must be equal to unity. This has been shown elsewhere [1,p.365].

In order to prove the sufficiency, two properties of M are required. First, it follows from the condition that no column sum exceeds unity that the absolute value of every root must be less than or equal to unity. A proof of this is given by Solow in the first part of the proof of his first theorem [11, p. 16]. Secondly, it has been shown by Metzler [8, p. 287] that the root of a nonnegative[2a] matrix with largest absolute value must be positive and real. It follows from these two properties that if any root is not inside the unit circle, this root must be equal to unity. Consequently [1, p. 365] det (I-M) = 0, and the theorem is proved.

Now the following theorem has been proved by Solow [11, p. 15]:

If M is a nonnegative, indecomposable matrix none of whose column sums is greater than one, and at least one of whose column sums is less than one, then all the characteristic roots of M have modulus less than one.

An indecomposable matrix is defined as a matrix which, by permutation of rows and columns, cannot be partitioned into triangular form. A closed set is defined [11, p. 11] as "a collection of sectors each of which spends in no sector not in the collection." Hence M is indecomposable "if there is no closed set other than the set of all indices 1,..., n."

As a result of Solow's theorem and Theorem 3, the following theorem may be derived:

Theorem 4. Let M be a nonnegative matrix such that no column sum exceeds unity and at least one column sum is less than unity. Then if M is indecomposable, I-M is nonsingular; and if all the column sums of M but one are equal to unity, then M is indecomposable if and only if I-M is nonsingular.

Proof: In order to prove this, note that in Theorem 3 I might just as well have imposed the stronger condition that at least one column sum be less than unity, for the alternative is excluded by the nonsingularity of I-M. Then the first assertion follows from Solow's theorem (indecomposability implies stability) and Theorem 3 (stability implies nonsingularity of I-M). The second follows from Theorem 3 (nonsingularity of I-M implies stability) and the fact

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2a. This was shown for positive matrices, but Metzler's proof remains valid for nonnegative matrices.

that, if the latent roots all have modulus less than one, the only partition of M which excludes closed subsets all of whose column sums are equal to unity, is the matrix M itself. This theorem can also be shown in the following way. Suppose M is decomposable. If all its column sums are strictly less than one, obviously it is stable.[3] But if any one of its closed sets has a unit marginal propensity to spend, the matrix will be unstable. For if this is so, this closed set (each of whose sectors has a unit marginal propensity to spend) can be treated as a single sector. But then we have a single closed sector, which means (because of the nonnegativity of M) that the whole column in I-M corresponding to that sector vanishes. Hence I-M is singular, and therefore the system is unstable.

It is possible to state Theorem 3 in an equivalent form, which makes the necessary and sufficient stability conditions identical with those of Theorem 2. First we shall prove

<u>Theorem 5.</u> <u>Let M be a nonnegative matrix such that no column sum exceeds one and at least one[4] column sum is less than one. Then det (I-M) $\geqq$ 0.</u>

<u>Proof:</u> As we saw in the proof of Theorem 3, the hypothesis implies that the characteristic roots must all have moduli less than equal to one, and if any root has unit modulus, det (I-M) = 0. If the roots all have modulus less than one, I have shown elsewhere that det (I-M) > 0.

From this and Theorem 3 we have immediately:

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3. See Theorem 7 below.

4. The theorem is also true without this condition, but is somewhat weaker.

Theorem 6. Let M be a nonnegative matrix such that none of its column sums exceeds unity. Then a necessary and sufficient condition that the characteristic roots of M all lie within the unit circle is that det (I-M)>0.

The following rather obvious theorems may also be added:

Theorem 7. Let M be a nonnegative matrix whose column sums are all strictly less than unity. Then det (I-M)>0.

Proof: From Theorem 5, det (I-M) cannot be negative. Suppose it is zero. Then there exists a root equal to unity, which is impossible (Goodwin [1, p. 547], Metzler [9 ,p.340]

Theorem 8. Let M be a nonnegative matrix whose column sums are all equal to unity. Then det (I-M) = 0.

Proof: The column sums of I-M are all zero, hence its rows are linearly dependent.

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COWLES COMMISSION DISCUSSION PAPER No. 2016A

Supplement to Economics No. 2016

by John Chipman

June 18, 1951

1. Messrs. Gerard Debreu and Morton Slater, and also Professor Lloyd A. Metzler, have furnished counterexamples which establish the falsity of my Theorem 1 in Cowles Commission Discussion Paper Economics No. 2016 [1].

The following is the counterexample:

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 1 .$$

For, by Theorem 1, this is an even-order hollow determinant, and should therefore be nonpositive.

Similarly, if we have the matrix

$$\begin{bmatrix} 0 & \beta & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & \gamma & 0 \end{bmatrix}$$

where α, β, γ, δ are all positive, the determinant $\alpha\beta\gamma\delta$ is always positive. Yet the system clearly need not be stable, for if we pick $\alpha\beta \geqq 1$, $\gamma\delta \geqq 1$, and substitute into the characterisitc polynomial

$$(\lambda^2 - \alpha\beta) \ (\lambda^2 - \gamma\delta)$$

then none of the roots is within the unit circle. This contradicts Theorem 2, which was based on Theorem 1.

It should be pointed out that the presence of zeros among the off-diagonal elements of the matrix is inessential to the counterexample, since a determinant is a continuous function of the matrix elements. Hence if the zeros are replaced by small numbers, the counterexample will still hold.

The error in the proof of Theorem 1 occurred on page 3, the sixth line from the bottom. $(-1)^j$ should have been $(-1)^{j+1}$. This was pointed out to me by Edmond Malinvaud.

2. As a result of these counterexamples, we may draw a conclusion of considerable interest in dynamic economics. Consider the first-order matrix difference equation

(1) $$I\,dY(t) = M\,dY(t-1) + dE$$

where $dY(t)$ and dE are vectors of changes in receipts at time t and autonomous changes of disbursements in the various sectors of the economy. Let the h-th sector represent households (recipients of national income), and the remaining n-1 sectors represent industries. Then we may obtain from (1), as has been shown elsewhere [2, p. 362], the n-th order scalar difference equation

(2) $$dY_h(t) = \alpha_1\, dY_h(t-1) + \alpha_2\, dY_h(t-2) + \dots + \alpha_n\, dY_h(t-n) + d\Delta_h$$

where $d\Delta_h$ is the determinant of the matrix formed by substituting dE in the h-th column of I-M, and where the α_k are the negatives of the coefficients of the characteristic polynomial of M.

The above counterexample may be used to support the following contention: there is no economically-meaningful matrix which can give rise to a difference equation (2) in which the α_k's are all nonnegative. In other words, it does not appear possible to attach any economic meaning to the set of all matrices whose companion matrices have only nonnegative elements in their bottom row.

The above equation (2) has, as I pointed out previously [1, p. 5], been used by a number of writers with the interpretation that the α_k's are components of a marginal propensity to spend (or consume) distributed over time, according to a distributed Robertsonian lag. These authors (cf., for instance, Lange [6, p. 239], Hicks [5, p. 172], Solow [8; cf. *footnote* 4]) have assumed the α_k's to be nonnegative. It is contended here that if equation (2) is to be economically meaningful and realistic, the lags must represent production lags which are much more significant than any distributed Robertsonian lags in the spending functions of individuals. Hence (2) is derivable from an interindustrial matrix. But in this case, there is no justification for the assumption, so often made, that the coefficients of (2) should all be nonnegative.

3. The proof of my third theorem [1, p. 6] made use of a theorem by Metzler [6, pp. 287-288] stating that the root of a nonnegative matrix with largest modulus is positive and real. In his proof, Metzler did not allow for the possibility that there might be more than one distinct root (other than a pair of complex conjugates) of largest absolute value; that is, he assumed the matrix to be *primitive*. Hence Slater [7] has produced an alternative proof of Theorem 3.

The correct theorem has been proved by Frobenius [3].[1] He states (p. 456): If r is the largest positive root, or <u>maximal root</u>, [of a nonnegative matrix], then the modulus of any other root can never be greater than r, but may be equal to r. Any of the k roots which are absolutely

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1. This was pointed out to me by Martin Beckmann.

equal to r is simple, and their ratios, 1, $\frac{r'}{r}$, $\frac{r''}{r}$, ..., are the k roots of the equation $\rho^k = 1$.

A generalization of Metzler's proof is, also, readily obtained.[2] One has to allow for more terms in his equation (11) [6, p. 287] with modulus r_1. Then, if we assume there are s distinct roots or conjugate pairs with modulus r_1, the argument should proceed in the following way, paraphrasing Metzler's proof: Beyond a certain time interval, the sign of $y_k(t)$ will be the same as the sign of

$$\sum_{i=1}^{s} g_i(t) \cos(\theta_i t) + h_i(t) \sin(\theta_i t)$$

for at least some values of t. The latter expression, in turn, is dominated, for sufficiently large t, by the parts of the $g_i(t)$ and $h_i(t)$ with the highest power of t. Let ν be the greatest multiplicity of any of the roots or conjugate pairs with modulus r_1. Suppose $q \leqq s$ is the number of such roots or pairs. Then the leading expression will take the form

$$\sum_{i=1}^{q} t^{\nu-1} [A_i \cos(\theta_i t) + B_i \sin(\theta_i t)] \equiv \sum_{i=1}^{q} C_i t^{\nu-1} \cos(\theta_i t + \varphi_i).$$

Thus, if t becomes sufficiently large, $y_k(t)$ will have the same sign as $\sum_{i=1}^{q} C_i \cos(\theta_i t + \varphi_i)$ for at least some values of t. If none of these θ_i is an integral multiple of 2π (including zero), then $\sum_{i=1}^{q} C_i \cos(\theta_i t + \varphi_i)$ will assume both positive and negative values for different integral values of t, and $y_k(t)$ therefore becomes negative for certain values of t. Since this is clearly impossible when all the y_k's are initially positive, it follows that one of the θ_i's, the amplitude of one of the roots with largest

2. I am indebted to Professor Metzler for indicating this generalization.

modulus, must be an integral multiple of 2π, and this root must be a positive real root.

Hence if a nonnegative matrix has more than one root with largest absolute value, one of them must be positive and real. This is sufficient both for Metzler's own proof of the necessity and sufficiency of the Hicks conditions for the stability of a market system in which all commodities are gross substitutes [6, p. 285], and for the proof of my Theorem 3 [1, p. 6].

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