Optimal Aggregation of Inventories under Certainty

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The following note illustrates the fact that economic "aggregation problems" are problems in maximization.

It has been shown in RAND P-189 that, under conditions of certainty, the best inventory policy for a single commodity is to reorder it at regular intervals, of duration

$$\Theta = \sqrt{K/(c-b_1x)} \text{ weeks,}$$

where K is the cost of handling an order of any size, 2c is the cost of carrying one unit of commodity over one week, x is the quantity to be delivered per week (the demand flow), and b_1 is the price reduction coefficient in the assumed linear supply function (price $= b_o - b_1x$). With pipeline time assumed to be zero, the amount ordered at the beginning of each interval is $x\Theta$.

Denote by Ω the set of all commodities delivered by the organization. Denote by π a partition of Ω. That is, the elements $\alpha_1^{(\pi)}, \alpha_2^{(\pi)}, \ldots$ of π are pairwise disjoint subsets of Ω, whose sum is exactly Ω. A partition π will be called a method of inventory aggregation, and its elements $\alpha_1^{(\pi)}, \alpha_2^{(\pi)}, \ldots$ will be called aggregates, if all commodities belonging to a given $\alpha_h^{(\pi)}$ are always ordered simultaneously.

Let Θ_α be the optimal period length for the aggregate $\alpha \subseteq \Omega$; that is, the cost per week, assignable to α, will be smallest if a joint order for all commodities in α is issued every θ_α weeks. Let this cost be C_α and suppose α is an element of the partition π. Then the total cost per week, over all aggregates in π is

$$C^{(\pi)} = \sum_{\alpha \in \pi} C_\alpha .$$

Our aggregation problem consists in choosing a partition π that will minimize $C^{(\pi)}$.

Let x_i be the demand flow of the commodity $i \in \alpha$. Then its quantity ordered is $y_i = x_i \theta_\alpha$. If we measure demand in "dollarworths", $x_i = 1$, $y_i = \theta_\alpha$. Using a notation analogous to that used for the case of a single commodity, we can write the cost assignable to the aggregate α over the period θ_α, as a sum of three components:

(1) Cost of handling an order, K_α;

(2) Cost of carrying the stock over the period θ_α,

$$\sum_{i \in \alpha} 2c_i \cdot \theta_\alpha \cdot \frac{y_i}{2} = \theta_\alpha^2 \sum_{i \in \alpha} c_i ,$$

where $2c_i$ = cost of carrying one unit of i over one week;

(3) Payment to seller, a function of all $y_i (i \in \alpha)$ which we shall approximate by

$$\sum_{i \in \alpha} b_i y_i + \sum_{i,j \in \alpha} b_{ij} y_i y_j = \theta_\alpha \sum b_i + \theta_\alpha^2 \sum_{i,j} b_{ij};$$

the second order coefficients b_{ij} allow for the price reduction on bulk purchases. Adding the three components, and dividing by θ_α, we obtain the cost per week

$$C_\alpha = K_\alpha / \theta_\alpha + \Sigma b_i + \theta_\alpha L_\alpha , \text{ where}$$

$$L_\alpha = \Sigma c_i + \Sigma\Sigma b_{ij}, \tag{2.15}$$

the summations being over all i, j in α. Since C_α is a minimum with respect to the length of the period, we have $\partial C_\alpha / \partial \theta_\alpha = 0$, $\partial^2 C_\alpha / \partial \theta_\alpha^2 > 0$,

and hence

(2.16) $$\theta_\alpha = \sqrt{K_\alpha / L_\alpha}\,;\ C_\alpha = \sum_{i \in \alpha} b_i + 2\sqrt{K_\alpha L_\alpha}\,;\sqrt{L_\alpha} > 0.$$

The total cost

(2.17) $$C^{(\pi)} = \sum_{\alpha \in \pi} C_\alpha = \sum_{i \in \Omega} b_i + 2D^{(\pi)},$$ where

(2.18) $$D^{(\pi)} = \sum_{\alpha \in \pi} \sqrt{K_\alpha L_\alpha}.$$

Since $\sum_{i \in \Omega} b_i$ does not depend on π, our problem becomes that of computing $D^{(\pi)}$ for each possible partition π, and choose that π that results in the smallest $D^{(\pi)}$. While in general the number of partitions to be compared is formidable, it is made more manageable in practice by considerations of previous experience. Even so, there remains the problem of fast computation of $D^{(\pi)}$ for selected partitions π.

An important simple case will illustrate the result. Consider a set of N commodities with only two partitions to be compared, π=I and π=II: with π=I each commodity is ordered separately; with π=II all commodities constitute jointly one aggregate.

$$\mathrm{I} = \left\{\alpha_1^{(\mathrm{I})}, \alpha_2^{(\mathrm{I})}, \ldots, \alpha_N^{(\mathrm{I})}\right\} = \left\{(1),\ (2), \ldots, (N)\right\},$$

$$\mathrm{II} = \left\{\alpha^{(\mathrm{II})}\right\} = \left\{(1,\ 2, \ldots, N)\right\}.$$

Assume the cost of handling an order for any single commodity to be the same: $K_\alpha(\mathrm{I}) = k$ (say), $i = 1, \ldots, N$. Further, write

$$K_\alpha(\mathrm{II}) = K \geq k.$$

By (2.15), (2.16), we have, for the case of partition I,

$$\sqrt{L_\alpha(\mathrm{I})} = \sqrt{c_i + b_{ii}} = d_i \text{ (say)};\ d_i > 0;\ i = 1, \ldots, N.$$

For the case of partition II, assume for simplicity $b_{ij} = 0$, $i \neq j$ (i.e., neglect "complementarity"). Then

$$L_{\alpha}(II) = \Sigma(c_i + b_{ii}) = \Sigma d_i^2.$$

(Σ will denote, from now on, summation from 1 through N). Hence by (2.18)

$$D^{(I)} = \sqrt{k\,\Sigma d_i}\,;\quad D^{(II)} = \sqrt{K}\cdot\sqrt{\Sigma d_i^2}\,;$$

$$(2.19)\qquad \left(\frac{D^{(I)}}{D^{(II)}}\right)^2 = \frac{k}{K}\cdot\frac{\left(\Sigma\, d_i\right)^2}{\Sigma\, d_i^2} > \frac{k}{K}\,,$$

since $(\Sigma d_i)^2 - \Sigma d_i^2 = \Sigma\Sigma d_i d_j > 0$. Moreover

$$(\Sigma d_i)^2/\Sigma d_i^2 = N/(1+V^2),$$

where $V = \dfrac{\text{standard deviation of the d's}}{\text{mean of the d's}}$, so that $(1+V^2)$ indicates the dissimilarity between the N commodities with respect to their storage costs and bulk price reductions. It follows from (2.18) that the separate ordering of all commodities is preferable to their simultaneous ordering if and only if the cost of handling a joint order multiplied by a "dissimilarity index" is more than N times the cost of handling the order for each commodity separately. In particular, if all d's are equal, $V=0$ and aggregation is advantageous whenever $K < Nk$.

In an extreme case, the cost of handling an order does not depend on its content, $K=k$; then, by (2.19), aggregation is always advantageous.