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Efficient Allocation of Resources and Capital Accumulation

By Edmond Malinvaud

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1. Introduction

The fundamental theorem of the new welfare economics states that an efficient allocation of resources is associated with a set of prices and with some rules concerning the behavior of the economic units. In accordance to these rules the firms should maximize their net profit as computed with the given prices. In this note it is shown that the introduction of the capitalistic process leads to a similar theorem in which a scalar interest rate is introduced: The firms should still maximize their net profit defined as the discounted value of all their present and future incomes. For computing the income P_t during a given time period, the formula should be:

$$P_t = \text{Value of Net Production} + \text{Capital Gains} - \text{Interest of Capital}$$

This result will probably appear quite reasonable to most economists. However, it seems that only one proof of it has ever been given[1] and this was built on an oversimplified model. Moreover the inclusion of capital gains in the definition of profit was not usually considered as a necessary requirement

1. M. Allais - Economie et Intérêt (Paris 1947) - Annexe III.

of welfare economics.

Except for a somewhat restrictive convexity assumption, the proof given here is quite general. Any kind of capitalistic process can be considered and no restriction is imposed concerning the smoothness of the economic expansion. The definition of efficiency is roughly the following: An "economic history" is said to be nonefficient if it is possible to increase during one time period the consumption of some commodity without decreasing either the consumption of any other commodity during the same time period or the consumption of any commodity in any other time period.

The model used here is a generalization of the von Neumann model; it has its formal simplicity but is not subject to the same restrictions (absence of consumption, linearity, no technological changes).

In Economics 2007 (Supplement) the theorem has been established for a stationary state in an economy without any technological change. But these assumptions are not necessary and in fact hide some interesting aspects of the theorem. Besides this, the model defined in Economics 2007 is impaired by some technical imperfections and the concept of "admissibility" defined there is not so direct as the concept of "efficiency" used here. It seems therefore that the present proof is an improvement over the former one.

2. <u>Notation</u>

Vectors are denoted by ordinary letters (a, b, p.....), scalars by Greek letters (α, β .....) and sets by thick capitals ($\mathbb{T}$, $\mathbb{U}$, $\mathbb{V}$).

The inequality $u \leqq v$ applied to vectors u and v, means that no component of u is greater than the corresponding component of v.

$u \leq v$ means: $u \leqq v$ and $u \neq v$.

$\mathbb{R}_\ell$ is the ℓ-dimensional Euclidean space. ℓ is the number of commodities in the model. An element in $\mathbb{R}_\ell$ is a vector whose components represent quantities of the ℓ commodities.

$\mathbb{R}_{2\ell}$ is the 2ℓ-dimensional Euclidean space. An element in $\mathbb{R}_{2\ell}$ is written as a pair (a,b) in which a and b are vectors in $\mathbb{R}_\ell$. Hence an element in $\mathbb{R}_{2\ell}$ represents two different bunches of ℓ commodities. Sets in $\mathbb{R}_{2\ell}$ are characterized by a star ($\mathbb{T}^* \subset \mathbb{R}_{2\ell}$).

The ℓh-dimensional and $(2h-1)\ \ell$-dimensional Euclidean spaces will also be used later in the proof ($\mathbb{R}_{\ell h}$ and $\mathbb{R}_{(2h-1)\ell}$).

The addition of sets is defined in the following way: $\mathbb{V} = \mathbb{U}_1 + \mathbb{U}_2$ means: any $v \in \mathbb{V}$ may be written as $v = u_1 + u_2$ where $u_1 \in \mathbb{U}_1$ and $u_2 \in \mathbb{U}_2$.

3. "Economic history"

In this section we are looking for a characterization of each possible future economic evolution. As we are concerned here with the productive sector we will not take into consideration the distribution of final goods among consumers. With respect to time the model is essentially discrete in the following sense: We suppose that it is possible to find a time period θ (elementary time period) and to divide the future into successive periods of length θ such that the consumption takes place only at the end of each period. This assumption does not seem to be restrictive because it is possible to choose θ small enough to have a satisfactory representation of the economic reality.

It is also convenient to suppose that the time unit is chosen such as to make $\theta = 1$. Each period is characterized by a value of $t(t=1,2.....h...)$.

Inside the time period t, the economic activity can be described as follows: At the beginning the existing quantity of goods is represented by a vector $a_t \in \mathbb{R}_\ell$: which describes the capital of the period and the stocks which are left over from the preceding period. The productive activity consists in a transformation of these goods (agricultural or industrial production, transportation and so forth). The result of this activity is a new vector $b_t \in \mathbb{R}_\ell$ which represents the total quantity of commodities after production. At this stage the consumption takes place; it can be represented by a vector $x_t \in \mathbb{R}_\ell$.

It is clear that the commodities which will be transferred to the next time period can be represented by: $b_t - x_t$. Hence the following holds:

$$a_{t+1} = b_t - x_t$$

From the point of view of total production, the future will be completely known if a_t and b_t are known for all values of t (t=1, 2......).

Hence it is reasonable to state the following:

<u>Definition</u> An "economic history" is a sequence of vectors $(a_t, b_t) \in \mathbb{R}_{2\ell}$ with t=1, 2...... To represent it, we write $\{a_t, b_t\}$; i.e., $\{a_t, b_t\} = (a_1, b_1), (a_2, b_2) \ldots\ldots\ldots (a_t, b_t) \ldots\ldots\ldots\ldots$

4. <u>Possible economic histories</u>

It is clear that not all economic histories can be realized; some of them would require an amount of natural resources or a technical knowledge exceeding the actual possibilities. For a formal representation of this fact, we shall say:

Given the quantities of commodities at the beginning of the period t^o, (i.e., given a_{t^o}) b_{t^o} is restricted to a set $\mathcal{B}_{t^o, a_{t^o}}$,

$$b_{t^o} \in \mathcal{B}_{t^o, a_{t^o}} \subset \mathbb{R}_\ell$$

The essential assumption is that $\mathcal{B}_{t^o, a_{t^o}}$ depends on t^o and a_{t^o} <u>only</u>, it is independent of the values taken by a_t and b_t for $t < t^o$. This hypothesis is indeed essential because it makes possible the proof we are looking for.

At first, it may seem somewhat unfounded. However, when one thinks about it, it appears quite reasonable for the following reason: concerning each kind of natural resources or each part of our technological knowledge, either we consider that the future cannot be modified by us, in which case our assumption is justified, or we think that we can develop some activity so as to improve the future, in which case the corresponding activity should be included in the productive sector and its product in the bunch of commodities. (If not, how could it be made the object of any economic choice?) Hence, if the definition of commodities is broad enough, our hypothesis is clearly a good picture of the reality.

Instead of saying that b_{t^o} is limited to a set $\mathbb{B}_{t^o, a_{t^o}}$ dependent on a_{t^o}, we can say that (a_{t^o}, b_{t^o}) is limited to a set $\mathbb{T}^*_{t^o} \subset \mathbb{R}_{2\ell}$. $\mathbb{T}^*_{t^o}$ depends on t^o only. The sequence $\left\{\mathbb{T}^*_t\right\}$ for all positive values of t (t=1, 2......) describes completely the limitations imposed on economic histories:

$\left\{a_t, b_t\right\}$ is a possible economic history if and only if:

$$(a_t, b_t) \in \mathbb{T}^*_t \text{ for all } t = 1, 2\ldots\ldots\ldots\ldots$$

Notice here that if $(a^o_t, b^o_t) \in \mathbb{T}^*_t$ and $b^1_t \leqq b^o_t$ then $(a^o_t, b^1_t) \in \mathbb{T}^*_t$. Indeed, if it is possible from a^o_t to get b^o_t, it is also possible to get any bunch of commodities which is smaller in every respect.

We can also say: $b^o_t \in \mathbb{B}_{t, a^o_t}$ and $b^1_t \leqq b^o_t$ imply: $b^1_t \in \mathbb{B}_{t, a^o_t}$.

The "<u>production units</u>" 2/ can now be defined as corresponding to a decomposition of the sets $\mathbb{T}^*_t$. Suppose $\mathbb{T}^*_t$ can be written as:

$$\mathbb{T}^*_t = \sum_{j=1}^{n_t} \mathbb{T}^*_{t,j}$$

We shall say that during the period t, there are n_t effective production units

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2. In order to understand the proof and the meaning of the theorem, it is not necessary to consider the decomposition of $\mathbb{T}_t$. Hence it is probably advisable to skip over the next page in a first lecture.

each one of them being associated with a set $\Pi^*_{t,j}$ such that its activity $(a_{t,j}, b_{t,j})$ is subject to the constraint:

$$(a_{t,j}, b_{t,j}) \in \Pi^*_{t,j} \subset \mathbb{R}_{2\ell}$$

By definition of addition for sets the following relations hold:

$$a_t = \sum_{j=1}^{n_t} a_{t,j} \qquad b_t = \sum_{j=1}^{n_t} b_{t,j}$$

More generally, there are N production-units (N can be infinite); each one is characterized by an index k (k=1, 2......N) and is associated with a sequence $\left\{\Pi^*_{t,k_t}\right\}$ of sets in $\mathbb{R}_{2\ell}$ (t = 1, 2.........).

A production-unit k is said to be effective during the period t if Π^*_{t,k_t} is not the empty set. During the period t there are n_t effective production-units (n_t is necessarily finite).

Obviously the definition makes sense only if: $k^o \neq k^1$ implies $k^o_t \neq k^1_t$ for all t.

It should be noticed here that it is not implied that each production-unit is specialized in some given type of activity.

A convexity axiom is necessary for the proof and will be introduced here. As a rough first approximation it is probably admissible, though it would not be difficult to find in the economic reality instances which disprove it.

Axiom. The sets $\Pi^*_{t,j}$ are convex.

This means:

If $(a^o_{t,j}, b^o_{t,j}) \in \Pi^*_{t,j}$ and $(a^1_{t,j}, b^1_{t,j}) \in \Pi^*_{t,j}$

Then $\left(\alpha a^o_{t,j} + (1-\alpha)\, a^1_{t,j},\ \alpha b^o_{t,j} + (1-\alpha)\, b^1_{t,j}\right) \in \Pi^*_{t,j}$ where α is a scalar between o and 1.

From the convexity of each $\Pi^*_{t,j}$, the convexity of Π^*_t follows.

5. "Efficient" economic histories

We want to study the properties of economic histories which are efficient in the following sense: Given an efficient economic history $\left\{a_t^o, b_t^o\right\}$, it is not possible to find an economic history $\left\{a_t^1, b_t^1\right\}$ which would be better in every respect; by which we mean that $\left\{a_t^1, b_t^1\right\}$ would allow a consumption x_t^1 such that:

$$x_t^1 \geqq x_t^o \text{ for all } t$$
$$x_t^1 \gneqq x_t^o \text{ for at least one } t$$

But this is not quite sufficient because we need to require that $\left\{a_t^1, b_t^1\right\}$ starts with the same bunch of commodities as $\left\{a_t^o, b_t^o\right\}$ or:

$$a_1^1 = a_1^o$$

Indeed, if we allow any a_1 at the beginning of the first period, it is always possible to make greater the consumption of the first period by increasing a_1. But this is meaningless because the quantity of goods at the time of our economic decision is given to us.

For the purpose of our demonstration we need even a more restrictive definition for the following reason: We are looking for infinite efficient economic histories, but an infinite time is conceived only as the result of an infinite extension of some economic horizon. Hence we will study the properties of an efficient economic history limited to a given horizon and see what happens to these properties when the horizon goes to infinity.

Now, if we compare two economic histories limited to a horizon h, we can only say that $\left\{a_t^1, b_t^1\right\}$ is better than $\left\{a_t^o, b_t^o\right\}$ if it allows for a disposable quantity of commodities after the last period, at least equal to that given by $\left\{a_t^o, b_t^o\right\}$. Hence we must require:

$$a_{h+1}^1 \geqq a_{h+1}^o$$

We are now able to give the two following formal definitions of efficiency:

Definition 1. The (infinite) economic history $\left\{a_t^o, b_t^o\right\}$ is h-efficient if there is no possible economic history $\left\{a_t^1, b_t^1\right\}$ with the following properties:

i) $a_1^1 = a_1^o$

ii) $a_{h+1}^1 \geqq a_{h+1}^o$

iii) $b_t^1 - a_{t+1}^1 \geqq b_t^o - a_{t+1}^o$ for all $t = 1, 2, \ldots \ldots h$

iv) $b_t^1 - a_{t+1}^1 \geq b_t^o - a_{t+1}^o$ for at least one $1 \leqq t \leqq h$

Definition 2. The economic history $\left\{a_t^o, b_t^o\right\}$ is said to be efficient if it is h-efficient for all $h > 0$.

Section 7 is devoted to the study of necessary and sufficient conditions for h-efficiency and is indeed the important part in the proof. But we need first to define one more set.

6. Definition of Π_h

Our purpose in this section is to represent an economic history by a single point in some space. This appears to be convenient for the proof of the h-efficiency theorem.

An economic history is a sequence $\left\{a_t, b_t\right\}$ with $t = 1, 2, \ldots\ldots$ Consider the following subsequence:

$$\left\{b_1, a_2, b_2, \ldots\ldots\ldots, a_t, b_t \ldots\ldots\ldots\ldots a_h, b_h\right\}$$

and write as a shorthand denomination $(a,b)_h$ for this subsequence.

Each a_t and b_t being vectors in $\mathbb{R}_\ell$, $(a,b)_h$ can be represented by a point in $\mathcal{R}_{(2h-1)\ell}$.

Define now the set $\mathcal{T}_h \subset \mathcal{R}_{(2h-1)\ell}$, by:

$$(a,b)_h \in \mathcal{T}_h \text{ if } \begin{cases} (a_1^o, b_1) \in \mathcal{T}_1^* \\ (a_t, b_t) \in \mathcal{T}_1^* \text{ for } t = 2,3\ldots\ldots h \end{cases}$$

where a_1^o is the total quantity of commodities at the present time.

Clearly, there is a one to one correspondence between the points of $\mathcal{T}_h$ and the possible economic histories which start by a_1^o and are limited to the horizon h.

The two following observations are trivial:

i) $\mathcal{T}_{h+1}$ is related to $\mathcal{T}_h$ by: $\mathcal{T}_{h+1} = \mathcal{T}_h \times \mathcal{T}_{h+1}^*$

ii) $\mathcal{T}_h$ <u>is convex</u>. Indeed,

If $(a^o, b^o)_h \in \mathcal{T}_h$ and $(a^1, b^1)_h \in \mathcal{T}_h$

Then $\alpha(a^o, b^o) + (1-\alpha)(a^1, b^1) \in \mathcal{T}_h$ for $o \leq \alpha \leq 1$

because:

$\alpha(a_t^o, b_t^o) + (1-\alpha)(a_t^1, b_t^1) \in \mathcal{T}_t^*$ for all t=1, 2......h by the convexity of $\mathcal{T}_t^*$.

In the same way, corresponding to the production-unit k, we can define $\mathcal{T}_{h,k}$ from $\mathcal{T}_{t,k_t}^*$ t=1, 2.....h. $\mathcal{T}_{h,k}$ is convex and there is only a finite number of $\mathcal{T}_{h,k}$ which are not empty. The following is also true:

$$\mathcal{T}_h = \sum_{k=1}^{N} \mathcal{T}_{h,k}$$

because $\mathcal{T}_h$ and $\mathcal{T}_{h,k}$ are cartesian products of sets $\mathcal{T}_t^*$ and $\mathcal{T}_{t,j}^*$ which satisfy:

$$\mathcal{T}_t^* = \sum_{j=1}^{n_t} \mathcal{T}_{t,j}^*$$

7. <u>Condition for h-efficiency</u>

The following lemma is the object of this section:

<u>Lemma 1</u> - $\left\{a_t^o, b_t^o\right\}$ is h-efficient if and only if there is a sequence of h nonnegative vectors $p_1 \ldots p_t \ldots p_h$ in $\mathcal{R}_\ell$ such that:

$$\underset{(a,b)_h \in \mathcal{T}_h}{\text{Max}} \left[\sum_{t=1}^{h-1} p_t(b_t - a_{t+1}) + p_h b_h\right] = \sum_{t=1}^{h-1} p_t(b_t^o - a_{t+1}^o) + p_h b_h^o$$

Define in $\mathcal{R}_{(2h-1)\ell}$ the (displaced) convex polyhedral cone $\mathcal{V}_h$ by:

$$(a,b)_h \in \mathcal{V}_h \quad \text{if:} \begin{cases} b_h \geqq b_h^o \\ \text{and } b_t - a_{t+1} \geqq b_t^o - a_{t+1}^o \\ \qquad\qquad\qquad \text{for all } t=1, 2, \ldots, h-1 \end{cases}$$

Now, $\left\{a_t^o, b_t^o\right\}$ is h-efficient if and only if there is no point $(a^1, b^1)_h \in \mathcal{T}_h$ which is interior point of $\mathcal{V}_h$. (This is easily seen from the definition of h-efficiency).

$\mathcal{T}_h$ and $\mathcal{V}_h$ are convex, hence their intersection is convex. It is contained in the boundary of $\mathcal{V}_h$ and therefore at most of dimension $(2h-1)\ell-1$. There is in $\mathcal{R}_{(2h-1)\ell}$ a hyperplane of dimension $(2h-1)\ell-1$ which contains this intersection. This hyperplane separates $\mathcal{T}_h$ and $\mathcal{V}_h$. Conversely if there is such a separating plane $\left\{a_t^o, b_t^o\right\}$ is h-efficient.

Hence, $\left\{a_t^o, b_t^o\right\}$ is h-efficient if and only if there is a vector:

$u = \left\{p_1, q_2, p_2 \ldots q_t, p_t \ldots q_h, p_h\right\} \in \mathcal{R}_{(2h-1)\ell}$ such that:

i) u is normal to a support plane to $\mathcal{V}_h$.

ii) u is directed toward the interior of $\mathcal{V}_h$.

iii) u is directed toward the exterior of $\mathcal{T}_h$.

iv) u is normal to a support plane of $\mathcal{T}_h$.

We will now show that i), ii) and iii) hold for a normal to $\mathcal{T}_h$ if and only if:

$$q_{t+1} = -p_t \qquad \text{for } t = 1, 2, \ldots, h-1$$

$$\text{and} \quad p_t \geqq 0.$$

This will complete the proof, because iv) is equivalent to saying that $\sum_{t=1}^{h} p_t b_t + \sum_{t=2}^{h} q_t a_t$ reaches an extremum at $(a^o, b^o)_h$ and iii) says that this is in fact a maximum.

(1) If i), ii), and iii) hold, then $q_{t+1} = -p_t$ and $p_t \geqq 0$.

It is clear that if u is normal to a support plane of $\mathbb{V}_h$, it is normal to the maximal linear space contained in $\mathbb{V}_h$. This space is defined by:

$$\begin{cases} b_t - a_{t+1} = b_t^o - a_{t+1}^o & \text{for } t = 1, 2 \ldots\ldots h-1 \\ b_h = b_h^o \end{cases}$$

A normal to it is a linear combination of the normals to each defining linear space. The normals to $b_t - a_{t+1} = b_t^o - a_{t+1}^o$ are given by: $\{0, 0 \ldots\ldots 0, v_t, -v_t, 0 \ldots\ldots 0\}$

where v_t is any vector in $\mathbb{R}_\ell$.

The normals to $b_h = b_h^o$ are given by: $\{0, 0 \ldots\ldots 0 \ldots\ldots 0, 0, v_h\}$

where v_h is any vector in $\mathbb{R}_\ell$.

Hence u can be written as:

$$\{p_1, -p_1, p_2 \ldots\ldots -p_{t-1}, p_t \ldots\ldots -p_{h-1}, p_h\}$$

which is exactly the condition $q_{t+1} = -p_t$.

By the same kind of argument, it is easy to see that u is directed toward the inside of $\mathbb{V}_h$ if and only if it is a positive linear combination of normals to each defining plane, each of these normals being directed toward the inside of:

$$b_t - a_{t+1} \geqq b_t^o - a_{t+1}^o \quad (\text{or } b_h \geqq b_h^o)$$

This is equivalent to saying that $v_t \geqq 0$.

Hence by positive linear combination $p_t \geqq 0$.

(2) If $q_{t+1} = -p_t$ and $p_t \geqq 0$, then i), ii) and iii) hold.

ii) is a consequence of the preceding argument.

i) can be proved in the following way:

u is normal to the maximal linear subspace of $\mathcal{V}_h$. u is contained in the orthogonal complement $\mathcal{W}$ to this maximal linear subspace. The intersection of $\mathcal{W}$ and $\mathcal{V}_h$ is an orthant in $\mathcal{W}$. As u is directed toward the interior of $\mathcal{V}_h$, it is normal to a support plane to $\mathcal{V}_h \cap \mathcal{W}$ in $\mathcal{W}$. The hyperplane spanned by this support plane and the maximal linear subspace of $\mathcal{V}_h$ is a support plane of $\mathcal{V}_h$ in $\mathcal{R}_{(2h-1)\ell}$.

Now, u is normal to $\mathcal{T}_h$ and $p_t \geqq 0$. We know that the set obtained from $(a^o, b^o)_h$ in $\mathcal{T}_h$ by variation of b_t^o (without changing any other component) is $\mathcal{B}_{t,\, a_t^o}$ which contains all $b_t \leqq b_t^o$. Also p_t is normal to a support plane of $\mathcal{B}_{t, a_t^o}$. $p_t \geqq 0$ implies that p_t is directed toward the outside of $\mathcal{B}_{t,\, a_t^o}$, u is therefore directed toward the outside of $\mathcal{T}_h$ and iii) holds.

8. <u>Condition for efficiency</u>

The following lemma is the object of this section:

<u>Lemma 2</u> - $\left\{a_t^o, b_t^o\right\}$ is an efficient economic history if and only if there is a sequence of nonnegative vectors in $\mathcal{R}_\ell$: $p_1, p_2 \ldots\ldots p_t \ldots\ldots$ such that,

<u>for all h:</u>

$$\operatorname*{Max}_{(a,b)_h \in \mathcal{T}_h} \left[\sum_{t=1}^{h-1} p_t (b_t - a_{t+1}) + p_b b_h \right] = \sum_{t=1}^{h-1} p_t (b_t^o - a_{t+1}^o) + p_h b_h^o .$$

Lemma 2 follows from Lemma 1, using the following trivial remark:

If $\left\{a_t^o, b_t^o\right\}$ is h-efficient with the vectors $\left\{p_1 \ldots p_t \ldots p_h\right\}$, it is also (h-1)-efficient with the vectors $\left\{p_1 \ldots\ldots p_t \ldots\ldots p_{h-1}\right\}$.

This is clear because if $\mathcal{T}_h$ contains $(a^o, b^o)_h$ and admits the normal $\left\{p_1 q_2 p_2 \ldots\ldots q_t p_t \ldots q_h p_h\right\}$ satisfying $q_{t+1} = -p_t$; then $\mathcal{T}_{h-1}$ contains $(a^o, b^o)_{h-1}$ and admits the normal $\left\{p_1 q_2 p_2 \ldots\ldots q_t p_t \ldots\ldots q_{h-1} p_{h-1}\right\}$ satisfying $q_{t+1} = -p_t$.

Now, if there is a sequence $\left\{p_t\right\}$ satisfying the requirement of Lemma 2, $\left\{a_t^o, b_t^o\right\}$ is h-efficient for all h, hence is efficient.

Conversely we shall show that if $\{a_t^o, b_t^o\}$ is h-efficient for all h, there is a sequence $\{p_t\}$ satisfying the requirement of Lemma 2.

Select a fixed h. The set of all vectors which satisfy the requirement of Lemma 1 for this h is a closed cone in $\mathbb{R}_{\ell h}$ whose intersection with the unit ℓh-sphere is a closed set $\mathbb{P}_o$.

Similarly the set of all vectors which satisfy the requirement of Lemma 1 for (h+n)-efficiency is a closed cone, whose intersection by the space $\mathbb{R}_{\ell h}$ of its h first components is a closed cone. Call $\mathbb{P}_n$ the intersection of this last cone with the unit ℓh-sphere. $\mathbb{P}_n$ is closed.

By the remark made earlier we know that:

$$\mathbb{P}_o \supset \mathbb{P}_1 \supset \ldots\ldots\ldots \supset \mathbb{P}_{n-1} \supset \mathbb{P}_n \supset \ldots\ldots\ldots$$

The efficiency condition says that no one of the $\mathbb{P}_n$ is empty. Hence:

$$\bigcap_{n=0}^{\infty} \mathbb{P}_n \neq 0 \quad \text{because the unit } \ell h\text{-sphere is compact.}$$

As this is true for all h, there is at least one sequence $\{p_t\}$ satisfying the conditions of Lemma 2.

9. Economic interpretation of the condition

Our purpose is now to see what means the maximisation of the quantity :

$$p = \sum_{t=1}^{h-1} p_t \, (b_t - a_{t+1}) + p_h \, b_h$$

Obviously the vector p_t could be interpreted as a price vector, but this would not be so good because the prices are usually subject to some rule of normalization. For instance, the price of some commodity, called numeraire, is to be equal to 1. The fulfilment of this condition would not be possible in general with our sequence $\{p_t\}$. However from $\{p_t\}$ we can construct the two following sequences:

$\{\bar{p}_t\}$ sequence of normalized price vectors

$\{\beta_t\}$ sequence of scalar positive coefficients

with $p_t = \beta_t \, \bar{p}_t$ for all t.

It is even possible to put $\beta_1 = 1$, because if $\{p_t\}$ satisfies the requirements of Lemma 2, $\{\alpha p_t\}$ satisfies them as well.

Without changing the result of the maximisation, we can add $-\bar{p}_1 a_1^o$ to P which becomes:

$$P = \sum_{t=1}^{h-1} \beta_t \bar{p}_t (b_t - a_{t+1}) + \beta_h \bar{p}_h b_h - \bar{p}_1 a_1^o$$

or $$P = \bar{p}_1 (b_1 - a_1^o) + \sum_{t=2}^{h} \beta_t \left[\bar{p}_t b_t - \frac{\beta_{t-1}}{\beta_t} \bar{p}_{t-1} a_t \right]$$

Write now: $$1 + \rho_t = \frac{\beta_{t-1}}{\beta_t}$$

We can put: $$P = \bar{p}_1 (b_1 - a_1^o) + \sum_{t=2}^{h} \beta_t \left[\bar{p}_t (b_t - a_t) + (\bar{p}_t - \bar{p}_{t-1}) a_t - \rho_t \bar{p}_{t-1} a_t \right]$$

or: $$P = \bar{p}_1 (b_1 - a_1^o) + \sum_{t=2}^{h} \beta_t P_t$$

with $$P_t = \bar{p}_t (b_t - a_t) + (\bar{p}_t - \bar{p}_{t-1}) a_t - \rho_t \bar{p}_{t-1} a_t$$

With reference to the period t, the economic interpretation of the preceding variables are the following:

- ρ_t is the rate of interest
- $\bar{p}_{t-1} a_t$ the value of capital at the beginning of the period
- $\bar{p}_t (b_t - a_t)$ the value of the net production
- $(\bar{p}_t - \bar{p}_{t-1}) a_t$ the capital gains (or losses, if negative)
- $\rho_t \bar{p}_{t-1} a_t$ the interest of the value of capital
- P_t is the net profit of the period
- P the value of the net production of the first period plus the discounted value of the net profit of the following periods.

The computation of the discount coefficients is made according to the usual formula:

$$\beta_t = \frac{\beta_{t-1}}{1 + \rho_t}$$

10. Rules of decision for the firms

Lemmas 1 and 2 give rules of decision for the economy as a whole and do not tell how the decentralization of economic choices should be realized. However, we know that, for each h : $\Pi_h = \sum_{k=1}^{N} \Pi_{h,k}$

and that, even if N is infinite, there is only for each h a finite number of $\Pi_{h,k}$ which are not empty.

Hence if to $\left\{a_t^o, b_t^o\right\}$ corresponds a point on the boundary of Π_h with normal u; then to $\left\{a_{t,k}^o, b_{t,k}^o\right\}$ corresponds a point on the boundary of $\Pi_{h,k}$ with normal u; this is true for all k. And conversely.

As Lemmas 1 and 2 give properties of the normal vector u, they can be generalised easily to take into account the decomposition of Π_h. This leads to the final theorem:

<u>Theorem.</u> An economic history $\left\{a_t^o, b_t^o\right\}$ is efficient if and only if there is a sequence of normalised price vectors:

$$\bar{p}_1, \bar{p}_2 \dots \bar{p}_t \dots \text{ in } \mathbb{R}_\ell \quad (\text{with } \bar{p}_t \geq 0)$$

and a sequence of interest rates

$$1, \rho_2 \dots \rho_t \dots \qquad (\text{with } \rho_t > -1)$$

such that, subject to the constraint: $(a_k, b_k)_h \in \Pi_{h,k}$, each production-unit k, maximizes:

$$P_k = \bar{p}_1 \,(b_{1,k} - a_{1,k}^o) + \sum_{t=2}^{h} \beta_t \; P_{t,k}$$

for all h.

In this last formula, β_t is determined from ρ_t by:

$$\beta_t = \frac{\beta_{t-1}}{1 + \rho_t}$$

$P_{t,k}$ is computed as:

$$P_{t,k} = \bar{p}_t (b_{t,k} - a_{t,k}) + (\bar{p}_t - \bar{p}_{t-1}) a_{t,k} - \rho_t \bar{p}_{t-1} a_{t,k} .$$