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Concluding Remarks on the Continuous Transportation Model

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This paper concludes the exposition of the continuous model of transportation and location, introduced in Cowles Commission Discussion Papers 293 and 2006. No systematic completeness is intended, since the continuous approach is tentative and limited by its nature. Notations and basic relations are those of CCDP 2006.[1] They will not be repeated here.

Since the following considerations are of a rather technical nature and have, for that reason, been condensed vigorously, it might be useful to the hurried reader, to have the main features shortly annoted here.

The purpose of CCDP 293 was to demonstrate that in a model of location based on the assumption of a continuous spatial distribution of production the optimum system is obtainable in terms of a system of differential equations. These equations are further discussed in the first section of the present paper. It is shown that their number can be reduced substantially (1.1, 1.2, 1.4), and that an equation system for the spatial distribution of either the prices or the local production in excess of consumption is obtained. (The equation system (1), (5) of CCDP 293 was in terms of both trans-

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1. On p. 2, under 1.2, read: λ can...be identified with the negative local price. This inversion of the sign of λ results from that introduced on p. 1 of CCDP 2006.

portation vectors and prices.) It follows incidentally that the conditions for linearity of these equations imply that the integrand of the variation problem be of the second degree in the vector functions, which is characteristic for models employed in physics. The second section attempts the generalization for the cases of several commodities and price-dependency of the production program of the "principle" discussed at length in CCDP 2006, Section 2. This principle asserts the equivalence of minimizing the transportation cost-integral and of maximizing the weighed (negative) sum of the locational potentials, both subject to certain side-conditions, of course. This would allow us to formulate the problem of optimum location entirely in terms of the spatial price-distribution.

In the third section the same principle is constructed for the discrete case by lining out a possible correspondence between the continuous and the discrete model. Since the interest of this paper is in refining and articulating the continuous conception of transportation and location rather than in establishing a common basis for both the continuous and discrete case, the possibilities of a merger by use of, say, more general notions of integration are not touched upon here.

1. The purpose of this section is to prove that the equation system

(1) $$k \frac{v}{|v|} + \text{grad}\,\lambda = 0$$

(2) $$\text{div}\, v - q = 0$$

which characterizes the field of transportation vectors in a closed area under consideration can always be reduced to a single differential-equation. In the general case the result is a second order differential equation in λ.

1.1. Note first, that (1) may be written

(1a) $$f_1(x_1, x_2; v_1, v_2) + \frac{\partial \lambda}{\partial x_1} = 0$$

(1b) $$f_2(x_1, x_2; v_1, v_2) + \frac{\partial \lambda}{\partial x_2} = 0.$$

This constitutes an implicit functional representation of v_1, v_2, that may be solved into:

$$v_1 = g(x_1, x_2; \frac{\partial \lambda}{\partial x_1}, \frac{\partial \lambda}{\partial x_2})$$

$$v_2 = h(x_1, x_2; \frac{\partial \lambda}{\partial x_1}, \frac{\partial \lambda}{\partial x_2}) ,$$ provided, that the Jacobian

$$\begin{vmatrix} \frac{\partial f_1}{\partial v_1}, & \frac{\partial f_1}{\partial v_2} \\ \frac{\partial f_2}{\partial v_1}, & \frac{\partial f_2}{\partial v_2} \end{vmatrix} \not\equiv 0.$$

Introducing this in (2) we arrive at the desired equation:

(3) $$0 = \frac{dg}{dx_1}(x_1, x_2; \frac{\partial \lambda}{\partial x_1}, \frac{\partial \lambda}{\partial x_2}) + \frac{dh}{dx_2}(x_1, x_2; \frac{\partial \lambda}{\partial x_1}, \frac{\partial \lambda}{\partial x_2}) - q(x_1, x_2; \lambda).$$

By spelling it out one may get the equivalent form:

(3a) $$0 = a \frac{\partial^2 \lambda}{\partial x_1^2} + b \frac{\partial^2 \lambda}{\partial x_1 \partial x_2} + c \frac{\partial^2 \lambda}{\partial x_2^2} + d - q$$

where a, b, c, d are functions of x_1, x_2, $\frac{\partial \lambda}{\partial x_1}$, $\frac{\partial \lambda}{\partial x_2}$, and q is a function of x_1, x_2, λ.

1.2. If the Jacobian vanishes identically in v_1 it follows from well-known theorems[2] that there exists a functional relationship between f_1 and f_2: $F(x, f_1, f_2) \equiv 0$. Substituting $\frac{\partial \lambda}{\partial x_1}$, $\frac{\partial \lambda}{\partial x_2}$ from (1a), (1b) in this we get $F(x, \frac{\partial \lambda}{\partial x_1}, \frac{\partial \lambda}{\partial x_2}) = 0.$

Since it can be supposed that $\frac{\partial q}{\partial \lambda} \neq 0$, which means that the local excess production is price-dependent, we may resolve (2) by

$\lambda = p(x, \text{div } v)$ and introduce it in F. Thus

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2. Cf., for instance, C. A. Stewart, Advanced Calculus, London, 1940, p. 178-179.

(4) $F(x, p_{x_1}(x, \text{div } v), p_{x_2}(x, \text{div } v)) = 0$

which is a first order differential equation in div v. In any case the system of differential equations (1), (2) is shown to be reducible.

1.3. In order for (3) to be a linear differential equation in λ, it is necessary that (1a), (1b) be linear in v_1, v_2, which implies that $k\,|v|$ is of the second degree in v_1, v_2.[3] It is easily seen that in the simplest case of $k = k_o(x).(v_1^2 + v_2^2)$ a self-adjoint differential equation of the elliptic type results, which is

(5) $$\frac{d}{dx_1}\left(\frac{-1}{2k_o}\frac{\partial \lambda}{\partial x_1}\right) + \frac{d}{dx_2}\left(\frac{-1}{2k_o}\frac{\partial \lambda}{\partial x_2}\right) - q\,\lambda - q_o = 0$$

where it is assumed that q be linear in λ; $q = q_o + \lambda q_1$ (q_o and q_1 being functions of x). The common methods of potential theory, solving boundary value problems by means of integral equations or the Green function, etc., are applicable in this case. This need not imply that computational difficulties are overcome in principle.

1.4. For extending our considerations to the case of several commodities, call $k_i|v_i| = t_i = t_i\,(x_1, v_1, \ldots, v_n)$ where i denotes the i-th commodity ($i = 1, \ldots, n$). The equation system to start with is

(1c) $$\frac{\partial t_i}{\partial v_{i1}} + \frac{\partial \lambda_i}{\partial x_1} = 0 \qquad i = 1, \ldots, n$$

(1d) $$\frac{\partial t_i}{\partial v_{i2}} + \frac{\partial \lambda_i}{\partial x_2} = 0$$

(1e) $$\frac{\partial v_{i1}}{\partial x_1} + \frac{\partial v_{i2}}{\partial x_2} - q_i = 0$$

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3. Thus the quadratic character of the integrand function $k|v|$ is not convenient just because it is characteristic to physical problems, but for genuine mathematical reasons; it renders the resulting differential equations linear, that is as simple as possible.

Call $\frac{\partial t_i}{\partial v_{i1}} = f_{i1}; \frac{\partial t_i}{\partial v_{i2}} = f_{i2}$.

The system (1c), (1d) of 2n equations in v_{i1}, v_{i2} can be made explicit in v_{i1}, v_{i2}, provided that

$$\begin{vmatrix} \frac{\partial f_{11}}{\partial v_{11}}, & \frac{\partial f_{12}}{\partial v_{11}}, & \frac{\partial f_{21}}{\partial v_{11}}, \dots & \frac{\partial f_{n2}}{\partial v_{11}} \\ \dots & & & \\ \frac{\partial f_{11}}{\partial v_{n2}} & \dots & & \frac{\partial f_{n2}}{\partial v_{n2}} \end{vmatrix} \neq 0$$

With $\frac{\partial v_{i1}}{\partial x_1} + \frac{\partial v_{i2}}{\partial x_2} - q_i = 0$ it follows

$$(6) \quad \frac{\partial g_{i1}}{\partial x_1}\left(x, \frac{\partial \lambda_1}{\partial x_1}, \frac{\partial \lambda_1}{\partial x_2}, \frac{\partial \lambda_2}{\partial x_1}, \dots, \frac{\partial \lambda_n}{\partial x_2}\right) + \frac{\partial g_{i2}}{\partial x_2}\left(x, \frac{\partial \lambda_1}{\partial x_1}, \dots, \frac{\partial \lambda_n}{\partial x_n}\right) - q_i(x, \lambda, \dots, \lambda_n) = 0$$

where $v_{i1} = g_{i1}(x, \frac{\partial \lambda_1}{\partial x_1}, \dots, \frac{\partial \lambda_n}{\partial x_2})$; $\qquad v_{i2} = g_{i2}(x, \dots, \frac{\partial \lambda_n}{\partial x_2})$

are the explicit representations of v_{i1}, v_{i2} from (1c), (1d). The original system of 3n equations containing v_{i1}, v_{i2} and λ_i is thus reduced to a system of n differential equations in λ_i ($i = 1, \dots, n$) alone.

In order to obtain a system of n <u>linear</u> equations in λ_i it is again clear, that (1c), (1d) must be linear in v_{i1}, v_{i2}, which implies that $t_i \equiv k_i|v_i|$ is of the second degree in v_j. ($j = 1, \dots, n$). The resulting system may be written:

$$(7) \sum_{K=1}^{n} a_{iK} \frac{\partial^2 \lambda_K}{\partial x_1^2} + b_{iK} \frac{\partial^2 \lambda_K}{\partial x_1 \partial x_2} + c_{iK} \frac{\partial^2 \lambda_K}{\partial x_2^2} + d_{iK} \frac{\partial \lambda_K}{\partial x_1} + e_{iK} \frac{\partial \lambda_K}{\partial x_2} + f_{iK} \lambda_K + g_{iK} = 0 \qquad i = 1, \dots, n$$

with coefficients dependent only on x. In the simplest case

$$k_i|v_i| = k_i(x)\,(v_{i1}^2 + v_{i2}^2)$$

(7) becomes

$$\frac{d}{dx_1}\left(-\frac{1}{2k_i} \frac{\partial \lambda_i}{\partial x_1}\right) + \frac{d}{dx_2}\left(-\frac{1}{2k_i} \frac{\partial \lambda_i}{\partial x_2}\right) - q_i(\lambda) = 0 \qquad i = 1, \dots, n.$$

With the further simplications k_i = const., and $q_i(\lambda) = q_{iK}\lambda_K + q_{io}$[4],

and notations $a_{iK} = -2k_i q_{iK}$

$b_i = -2k_i q_{io}$

we have eventually:

(8) $\Delta \lambda_i + \sum_K a_{iK} \lambda_K + b_i = 0 \qquad i = 1,\ldots,n$

with coefficients depending on x alone. This may also be written in vector form.

(8a) $\Delta \lambda + A\lambda + b = 0$

where λ, b are n-dimensional vectors, and A is a n-row square matrix. Although this constitutes the simplest case of an n-commodity model that can be set-up without significant loss of generality, the mathematics of its final solution is anything but simple.

2. In this section we attempt to extend the maximum principle, put forward in Section 1 of CCDP 2006, to the case of several commodities. The first step is to show that for any given production program the optimum system of transport routes is characterized by a maximum of the weighed potential sum $\sum_i \iint \lambda_i q_i \, dx_1 \, dx_2$, subject to certain side conditions, which essentially secure the character of a gradient function of λ. Thereafter the principle will be extended to a transportation program, that is not fixed in advance but price-conditioned. As indicated in Cowles Commission Discussion Paper 293, Section 2, this reflects a spatial equilibrium under competitive conditions, and an optimum constellation of not only the transportation system but the locational pattern as well.

2.1. As a preliminary stage consider once more the case of a single commodity the transport program of which shall be price-conditioned, however.

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4. This means the assumption of linear demand and supply functions with respect to all commodities.

In the same fashion as in CCDP 2006, equation (4a) we have

$$\delta \iint \left\{ \lambda q(\lambda) + u\left(\frac{y}{|y|} k + \operatorname{grad} \lambda\right) \right\} dx_1\, dx_2 = 0$$

which yields $q + \lambda \frac{\partial q}{\partial \lambda} - \operatorname{div} u = 0$.

If we now make the assumption that the excess supply function q be homogeneous of the first degree in λ, $q(x, \lambda) = q_o(x) \cdot \lambda$ and put $u = 2u^*$, then $2q - \operatorname{div} 2u^* = 0$ which is identical with (6) of CCDP 2006. The further proceeding of the proof is the same as in CCDP 2006, Section 1. The assumption of homogeneity of q with respect to λ is both sufficient and necessary for our assertion. The only difference is in the interpretation of the Lagrange parameter u, which represents the flow vector with double length now.

2.2. We proceed to several commodities in a fixed production program $q_i(x_1, x_2)$ $i = 1, \ldots, n$. The objective is to prove that

$$\max \sum_i \iint \lambda_i q_i\, dx_1\, dx_2 \quad = \quad \min_v \sum_i \iint k_i |v_i|\, dx_1\, dx_2$$

$$(9) \text{ subject to } \begin{cases} k_i \frac{y_i}{|y_i|} + \operatorname{grad} \lambda_i = 0 \\ \oint (\lambda_i u_i)_n\, ds = 0 \text{ for any} \\ u_i \text{ with } \operatorname{div} u_i - q_i = 0. \end{cases} \qquad \text{subject to } \begin{cases} \operatorname{div} v_i - q_i = 0 \\ v_i = 0 \text{ on } B. \end{cases}$$

The proof in Section 1 of CCDP 2006 can almost literally be applied. It makes no difference if transportation cost of empty carrier movements are taken into account, that is:

$$q_n = -\sum_{i+1}^{n-1} t_i q_i$$

(cf., page 7 of CCDP 2006) since we have full freedom in the choice of the q_i's ($i = 1, \ldots, n$).

2.3. The most interesting case is, however, that of the program being price-conditioned. $q_i = q_i(x, \lambda_1, \ldots, \lambda_n)$. Since the λ_i represent prices, we must assume some homogeneity relations between the q and λ. If we hold that the $q_i(x, \lambda_1, \dots, \lambda_n)$ be homogeneous of zero degree in λ, then it follows easily that the "potential sum" $\sum_i \lambda_i q_i$ is homogeneous of the first degree in λ. For:

(10) $$\sum_j \lambda_j \frac{\partial}{\partial \lambda_j}\left(\sum_i \lambda_i q_i\right) = \sum_j \lambda_j q_j + \sum_i \sum_j \lambda_i \lambda_j \frac{\partial q_i}{\partial \lambda_j} = \sum_j \lambda_j q_j.$$

A maximisation of $\iint \sum_i \lambda_i q_i \, dx_1 \, dx_2$ with respect to the system of λ_i is therefore meaningless, unless further restrictions are imposed on the absolute magnitudes of the λ.

For the particular part that transportation costs are to play in our model it is suggestive, to fix the level of the λ_i $(i = 1, \dots, n)$ by requiring that the total of absolute transportation costs, homogeneous of the first degree in λ, attain a given constant level.

(11) $$\iint \sum_i k_i \, |v_i| \, dx_1 \, dx_2 - c = 0.$$

Still another condition is necessary. It comes out, if one tentatively requires $\iint \sum_i \lambda q_i \, dx_1 \, dx_2$ to be homogeneous of zero degree with respect to the function λ_i. It is said "tentatively," because the notion of homogeneity is not generally defined for functionals (functions of functions).

$$\sum_j \lambda_j \frac{\iint \sum_i \lambda_i q_i \, dx_1 \, dx_2}{\delta \lambda_j} = \sum_j \lambda_j \iint \left(q_j + \sum_i \lambda_i \frac{q_i}{j}\right) dx_1 \, dx_2 = 0.$$

Since $\iint q_j \, dx_1 \, dx_2 = 0$ (equation (9) of CCDP 293), we are left with

$$\sum_j \lambda_j \iint \sum_i \lambda_i \frac{\partial q_i}{\partial \lambda_j} \, dx_1 \, dx_2 = 0.$$

Sufficient for this condition is

(12) $$\sum_i \lambda_i \frac{dq_i}{d \lambda_j} = 0 \qquad (j = 1, \dots, n).$$

Since this condition will play a crucial part in the following argumentation and it will prove to be necessary for the solving equation div u - q = 0 to hold, an ad hoc justification is clearly unsatisfactory. There is however an economic interpretation available. Consider the change in the trade budget, or total of transactions, $\sum_i \lambda_i q_i$ of a given place with the outside world, that arises with the change in a single price λ. The effect $\frac{d}{d\lambda_j}(\sum_i \lambda_i q_i)\, d\lambda_j = (q_j + \sum_i \frac{dq_i}{d\lambda_j} \cdot \lambda_i)\, d\lambda_j$ can be separated into the direct one $q_j\, d\lambda_j$, giving the change in immediate returns from the j-th commodity, and into the indirect repercussions contained in $\sum_i \lambda_i \frac{dq_i}{d\lambda_j} d\lambda_j$ $(j = 1,\ldots,n)$. Our condition (12) imposes that these repercussions should cancel out. This, it must be stressed, should not only hold approximately, but strictly, at least in a sufficiently small neighborhood of the optimal situation.

Including restriction (11) our maximum problem can be set up in terms of Lagrange parameters, as the variation with respect to λ_j, $(j = 1,\ldots,n)$, of

$$(13) \quad \iint \left\{ \sum_i q_i \lambda_i + \sum_i u_i \left(\frac{y_i}{|y_i|} k_i + \operatorname{grad} \lambda_i\right) + \mu \left(\sum_i k_i |y_i| - c\right) \right\} dx_1\, dx_2 .$$

Here μ is a number[5] while the parameter system of u_i's represents functions. The Euler equations are

$$(14) \quad q_j + \sum_i \lambda_i \frac{dq_i}{d\lambda_j} + \sum_i u_i \frac{y_i}{|y_i|} \cdot \frac{\partial k_i}{\partial \lambda_j} + \mu \sum_i \frac{dk_i}{d\lambda_j} \cdot |y_i| - \operatorname{div} u_j = 0, \quad j = 1,\ldots,n$$

Here the second term vanishes by virtue of assumption (12). By substituting q_i in (13) through (14) it becomes

$$\iint \left\{ \sum_i \lambda_i \left(- \sum_j \lambda_j \frac{dq_j}{d\lambda_i} - \sum_j u_j \frac{y_j}{|y_j|} \cdot \frac{dk_j}{d\lambda_i} - \mu \sum_j \frac{dk_j}{d\lambda_i} |y_j| + \operatorname{div} u_i \right) + u_i \left(\frac{y_i}{|y_i|} k_i + \operatorname{grad} \lambda_i\right) + \mu (k_i |y_i| - c) \right\} dx_1\, dx_2 .$$

Because of the zero degree homogeneity of q and k all terms vanish which contain $\lambda_i \frac{dq_j}{d\lambda_i}$ or $\lambda_i \frac{dk_j}{d\lambda_i}$. There remains

$$(15) \quad \iint \left\{ \sum_i \left(\lambda_i \operatorname{div} u_i + u_i \left(\frac{y_i}{|y_i|} k_i + \operatorname{grad} \lambda_i\right)\right) + \mu \left(\sum_i k_i |y_i| - c\right) \right\} dx_1\, dx_2$$

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5. Because μ stands for an isoperimetric problem. Cf., Courant-Hilbert, Methoden der Mathematischnen Physics, Berlin, 1931, Vol. 1, Chapter III.

The first term can be transformed by partial integration, in the fashion of CCDP 2006, equation (4b), into

$$\iint \sum_i u_i \frac{y_i}{|y_i|} k_i \, dx_1 \, dx_2$$

and a boundary integral which is assumed to vanish (9). Eventually:

(16) $$\iint \left\{ \sum_i u_i \frac{y_i}{|y_i|} k_i + \mu \left(\sum_i k_i |y_i| - c \right) \right\} dx_1 \, dx_2.$$

Now the second term in (16) vanishes at a maximum, for the side condition (1) must be satisfied. This gives one equation in the $|y_i|$, $i = 1, \ldots, n$. With $\iint \sum_i u_i \frac{y_i}{|y_i|} k_i \, dx_1 \, dx_2$ left, the argument remains the same as in CCDP 2006 (p. 4 after equation (4b)) that the y_i should point in the direction of u_i.

(17) $$y_i = \alpha_i u_i \qquad \alpha_i > 0, \text{ scalar}.$$

Thus we have for (14)

(18) $$q_i + \sum_j |u_j| \frac{dk_j}{d\lambda_i} + \mu \sum_j \alpha_j \frac{dk_j}{d\lambda_i} |u_j| - \operatorname{div} u_i = 0 \qquad (i = 1, \ldots, n).$$

Our other equations are

(9a) $$\frac{u_i}{|u_i|} k_i + \operatorname{grad} \lambda_i = 0$$

(11a) $$\iint \sum_i k_i \alpha_i |u_i| \, dx_1 \, dx_2 = c$$

(9b) $$\oint_B (u_i \lambda)_n \, ds = 0 \qquad (i = 1, \ldots, n).$$

These are 3n + 1 equations. (The boundary conditions which are accessory to the differential equations in u_i, do not count.) We are faced with 4n + 1 unknowns (functions and a number) u_i, λ_i, α_i and μ. Thus we may impose n more conditions. These will be the postulate, that the particular form of dependency of the k_i on the prices λ in general shall not enter. This is done by $\mu \cdot \alpha_i(x_1, x_2) = -1$, $(i = 1, \ldots, n)$.

Equation (14) then assumes

(14a) $\qquad q_i - \operatorname{div} u_i = 0.$

From (9) and (14a) it is then seen that u_i is identical with the flow vector v_i, hence, that λ_i assumes the value of the original minimum problem. Thus

$$\max_{\lambda} \iint \sum_i \lambda_i q_i \, dx_1 \, dx_2 = \min_{v} \iint \sum_i k_i |v_i| \, dx_1 \, dx_2$$

$$\text{(19) subject to} \begin{cases} k_i \dfrac{y_i}{|y_i|} + \operatorname{grad} \lambda_i = 0 \\ \oint (u_i \lambda_i)_n \, ds = 0 \text{ for all} \\ u_i \text{ with } \operatorname{div} u_i = q_i \\ \iint \sum_i k_i y_i \, dx_1 \, dx_2 = c \\ \text{the particular form of } k_i(\lambda) \text{ is irrelevant.} \end{cases} \quad \text{subject to} \begin{cases} \operatorname{div} v - q = 0 \\ v = 0 \text{ on } B \end{cases}$$

The left side is seemingly a much more complicated version of the problem. It should however be noticed that the number of unknown functions λ is only half that of flow functions (these being vector-components), and that the boundary conditions $v_i = 0$ are the strongest conceivable ones. This equivalence is however dependent on a rather strong assumption, that is, $\sum_j \lambda_j \frac{dq_i}{d \lambda_i} = 0$ $(i = 1, \ldots, n)$. It is seen from that that optimum conditions are basically different on the level of the whole economy from what they are if only one commodity is under consideration.

3. It is not difficult to "translate" most of the considerations from the continuous into the discrete model. A different question is however the power of such results. We shall outline the procedure for the λ-max-principle, applied to one single commodity in a given production program.

3.1. The minimum problem for $\iint k|v| \, dx_1 \, dx_2$ subject to $q - \operatorname{div} v = 0$ is equivalent to minimizing

(20) $\qquad \sum_{i,j=1}^{n} \frac{1}{2} k_{ij} v_{ij}$ subject to $q_i - \sum_j v_{ij} = 0,$

where $i,j = 1, \ldots, n$ denote the n discrete points (ports, etc.), between which transportation is to be carried out. k_{ij} is the transportation cost on the discrete route from i to j (in this sequence), v_{ij} the amount of commodity being shipped directly from i to j.

By definition $v_{ij} = - v_{ji}$; and by convention put

$$\left.\begin{array}{l} k_{ij} = - k_{ji} \\ k_{ij} > 0 \text{ if } v_{ij} > 0 \end{array}\right\} \tag{21}$$

that is: measure transport cost positive only in the direction of positive flow. The factor $\frac{1}{2}$ in (20) is necessary, because the double sum covers every route twice (once in each direction). By application of Lagrange-parameters

$$K = \sum_{i,j} \frac{1}{2} k_{ij} v_{ij} + \lambda_i (q_i - \sum_j v_{ij}) \tag{20a}$$

$$\frac{d}{dv_{ij}} K = \frac{1}{2} (k_{ij} - k_{ji}) - \lambda_i + \lambda_j = 0 \text{ or}$$

$$k_{ij} = \lambda_j - \lambda_i, \text{ which corresponds to } k \frac{v}{|v|} + \text{grad}\,\lambda = 0 \tag{22}$$

and defines a potential function λ. It must be observed that (22) applies only to such pairs, i,j for which v_{ij} is nonzero. By (22) it is expressed then, that the nonvanishing v_{ij} have to be distributed so as to be consistent with the imputation to each point of potentials λ_i defined by the requirement: differences in potential are equal to the transportation cost in positive flow. (The absolute value is undetermined.) It is well-known (and obvious by visual evidence), that this constraint is equivalent to the condition that the structure of flows constitutes either a "tree" or a connected graph containing only such orbits as are neutral. (These latter may be deformed into a tree without change in the total of transportation cost. Henceforth we shall speak simply of trees.) While this condition is necessary to secure a transportation cost minimum, it is by no means sufficient[6], so that it

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6. That is because the inequalities $k_{ij} + \lambda_j - \lambda_i \geq 0$ for all i, j, which in addition are necessary to determine the resolving tree uniquely, cannot be obtained on the grounds of Lagrangian differentiation alone.

leaves the solution open. It restricts the solution however to a finite set.

3.2. With (22), (21a) we obtain

$$K = \sum_{i,j} \frac{1}{2} (\lambda_i - \lambda_j)\, v_{ij} + \sum_i \lambda_i q_i - \sum_{i,j} \lambda_i v_{ij} = \tag{23}$$

$$= \sum_{i,j} \frac{1}{2} (\lambda_i + \lambda_i)\, v_{ij} + \sum_i \lambda_i q_i - \sum_{i,j} \lambda_i v_{ij} = \sum_i \lambda_i q_i$$

which corresponds to $\int k|v|\, dx_1\, dx_2 \quad = \quad \int \lambda\, q\, dx_1\, dx_2$

subject to $k \frac{v}{|v|} + \text{grad}\, \lambda = 0$ subject to $q = \text{div}\, v$.

Following Section 1, CCDP 2006, one may raise the problem of maximizing $\sum_i \lambda_i q_i$ subject to a definition of λ by

$$\lambda_j - \lambda_i + k_{ij} = 0 \tag{22a}$$

where the signs of k_{ij} have to be consistent with the possible directions of flow on the graph so selected.

With Lagrange parameters u_{ij} this can be set-up as to minimize

$$\sum_{i,j} \lambda_i q_i + \sum_{i<j} u_{ij} (\lambda_j - \lambda_i + k_{ij}) = \tilde{K}. \tag{24}$$

The restriction $i < j$ in the side conditions is natural, since there are only n-1 such conditions.

$$\frac{d}{dx_i} \tilde{K} = 0 = q_i - \sum_{i<j} u_{ij} + \sum_{j<i} u_{ji}. \tag{25}$$

By convention put $u_{ij} = - u_{ji}$ then

$$q_i - \sum_j u_{ij} = 0 \tag{25a}$$

$$\tilde{K} = \sum_{i,j} \lambda_i u_{ij} + \sum_{i<j} u_{ij} (\lambda_j - \lambda_i + k_{ij})$$

$$= \sum_{i,j} \lambda_i u_{ij} - \sum_{i,j} u_{ij} \lambda_i + \sum_{i<j} k_{ij} u_{ij} = \frac{1}{2} \sum_{i,j} k_{ij} u_{ij}. \tag{26}$$

(26) shows that the Lagrange parameters u_{ij} may themselves be interpreted as flows consistent with the production program q_i. This system of flows u_{ij} constitutes a tree on account of the provisions in (22a). Since $\frac{1}{2} \sum_{i,j} k_{ij} u_{ij}$

of (26) is maximised by choosing the tree that underlies the definition of k_{ij} so as to have positive flows wherever $u_{ij} > 0$, it follows: Of all trees defining a λ-potential, which are possible among n given points, those trees will maximize the potential sum $\sum_i \lambda_i q_i$, that are consistent with the production program q_i. But since for any such tree $\sum_i \lambda_i q_i = \sum_{i,j} k_{ij} v_{ij} =$ the total of transportation cost, it follows that that one is optimal which minimzes $\sum_i \lambda_i q_i$ relative to trees, that is, provides for

$$(27) \qquad \underset{\text{trees}}{\text{Min}} \quad \underset{\text{flows}}{\text{Max}} \quad \sum_i \lambda_i q_i \, ; \qquad \lambda \text{ subject to } \lambda_j - \lambda_i + k_{ji} = 0.$$

This example may suffice to indicate the kind and limitations of connections between the discrete and the continuous conception of transportation that can be expected. While, in some cases, the continuous set-up may be employed as a preliminary stage in transportation and location problems, which are by nature discrete in character, there can be no question as to its intrinsic failure to reflect characteristic features of a discontinuous reality.