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A Simplification of the Axiomatics of "Measurable Utility"[1]

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Sec. 1. von Neumann and Morgenstern [1] have proved, in essence, that certain axioms defining rational choice imply for each individual the existence of a utility function (unique up to linear transformations) that has the following property: if two decisions are associated with different mathematical expectations of utility, then the decision associated with the higher mathematical expectation of utility is chosen. The proof given by von Neumann and Morgenstern occupies 13 pages not counting interpretative sections; and the presentation has led to serious misunderstandings. In [2] the present writer supplied a modified (and, I believe, intuitively more appealing or "transparent") set of axioms defining rational choice, and a geometrical proof that those axioms imply the von Neumann-Morgenstern theorem. The present paper attempts to simplify the matter still further by making some lemmas or theorems of [2] into axioms, and conversely. The new axioms (especially II.1) seem to appeal to intuition[2] at least as strongly as those used previously, yet the proof is considerably simpler.

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1. With acknowledgments to L. J. Savage (Sec. 4) and E. Malinvaud (Sec. 9). Sec. 6 was criticized by Hildreth. Further comments are invited.

2. By saying that a proposed set of axioms defining rational choice appeals to the intuition of a given person (e.g., the reader) we mean the following conjecture: if that person were challenged with making a choice, and given sufficient opportunity for deliberation, he would reject decisions inconsistent with the proposed axioms. This is quite analogous to the intuitive

A comparison between this presentation and that of [1] and [2] will be given in Sections 4 and 9.

Sec. 2. Let [A] be the set of all possible distributions of a certain random vector x, and denote elements of this set by the first letters (A,B,...) of the capital alphabet, with or without subscripts. By ABp we shall mean a distribution obtained by assigning probability p to A and probability 1-p to B. Thus ABp is an element of [A]. The following identities are valid: (unless otherwise stated, $0 \leq p \leq 1$, here and throughout the paper).

(2.1) $A \equiv AB1 \equiv BA0$

(2.2) $AAp \equiv A$

(2.3) $BA(1-p) \equiv ABp$

(2.4) $(BAp)Aq \equiv BA(pq)$; $B(BAp)q \equiv BA[q+(1-q)p]$

(2.5) $(BAp)(BAq)r \equiv BA[rp+(1-r)q]$

Sec. 3. We interpret each component of x as a (continuous or discrete) quantity of a certain commodity (in the most general sense) at a certain future time. We call [A] the set of all prospects. The following axioms are proposed:

Ax I. [A] is completely ordered by a relation $\leq$, (read: "not preferred to").

<u>Definitions</u>: ($A \leq B$ and not $B \leq A$) means ($A < B$) means ($B > A$). (read "B preferred to A").

($A \leq B$ and $B \geq$) means ($A = B$). (read "A and B are indifferent").

To indicate two identical prospects we shall always write $A \equiv B$.

Obviously ($A \equiv B$) implies ($A = B$) but not conversely.

(Footnote 2 continued)
appeal of axioms of arithmetic.

It is a remarkable fact that a proposition may be implied in a set of intuitively appealing propositions and yet be not itself intuitively appealing: there would be otherwise no need for mathematical textbooks. This answers the question "What is to be gained by not directly requiring rationality to imply maximization of mathematical expectations of utility"? (This question was asked by William J. Baumol in a stimulating correspondence in the course of which the need for simplified axiomatics and proofs became clear to me).

Ax II.1. $A < B,\ 0 < p < 1 \Rightarrow ACp < BCp$

Ax II.2. $A = B,\ 0 < p < 1 \Rightarrow ACp = BCp$

Ax III. There exist A, B such that $A < B$.

Sec. 4. Comparison with paper [2].

Ax I is identical with Postulate I of [2].

Ax II.1 says that if A is preferred to B, then the "mixture" of A with C is preferred to the "mixture" (with the same odds) of B with C. One can also translate II.1 into the language of a payoff matrix:

	States of Nature:	
	N_1	N_2
Strategies	Payoff:	
S_1	A	C
S_2	B	C

(N_1 and N_2 have, respectively, probabilities p and 1-p). If $A < B$, strategy S_1 is "inadmissible" and is rejected. This interpretation is probably in line with an oral remark by Savage.

Note that Ax II.1 excludes the "love of danger" ([2], Sec. VIII).

Ax II.2 is identical with Postulate IV of [2].

Ax III corresponds to but is weaker than Postulate III' (existence of four non-indifferent prospects) or III* (existence of two non-indifferent prospects, each involving positive probabilities only) of paper [2]. It is therefore weaker than Postulate III (existence of indifference surfaces) of that paper.

In addition to weakening Postulate III* (or III') the present paper disposes of Postulate II (continuity): this becomes a theorem (Th 6).

Sec. 5. The following theorems are derived from Axioms I, II.1, II.2 and the identities of Sec. 2.

Th 1. $A < B$, $0 < p \leq 1 \Longrightarrow ACp < BCp$, by II.1 and (2.1).

Th 2. $A = B$, $0 \leq p \leq 1 \Longrightarrow ACp = BCp$, by II.2 and (2.1).

Th 3. $A = B \Longrightarrow BAp = A$.

Proof: Replace C by A in Th 2 and use (2.2).

Th 4. $A < B$, $0 < p < 1 \Longrightarrow A < BAp < B$.

Proof: Replace C by A in Ax II.1 and use (2.2); then replace C by B and p by 1-p and use (2.2), (2.3).

Note to Th 3, 4: One might consider them as axioms, and weaken II.1, II.2 by the restriction $C \neq A$, $C \neq B$.

Th 5'. Let $A < B$ and $p > 0$, $q < 1$; then $p < q \Longleftrightarrow BAp < BAq$.

Proof: <u>Sufficiency</u>: Let $p < q$. By Th 4 $A < BAq$; and since $0 < p/q < 1$ we have, by Th 4, $A < (BAq)\, A\, \frac{p}{q} < BAq$. But, by (2.4) the middle term $= BAp$. Hence $BAp < BAq$.

<u>Necessity</u>: Let $BAp < BAq$; this is inconsistent with $p \geq q$ because $A < B$, $p = q$ would imply $BAp = BAq$; and $A < B$, $p > q$ would imply (as just proved) $BAq < BAp$.

Th 5. If $A < B$; then $p < q \Longleftrightarrow BAp < BAq$. This extends Th 5 to the cases $p = 0$ or $q = 1$, using (2.1) and Th 4.

Sec. 6. <u>Continuity theorem:</u>

Th 6. If $A_0 < A_1$ and $A_0 \leq B \leq A_1$ then there exists one and only one $p (0 \leq p \leq 1)$ such that $B = A_1 A_0 p$.

Proof: <u>Existence</u>: If $B = A_0$ or A_1 put $p = 0$ or 1. If $A_0 < B < A_1$ let p_0 be the smallest p such that $A_1 A_0 p < B$; and let p_1 be the largest p such that $B < A_1 A_0 p$. Since p is continuous, there exist p' and p" such that $p_0 < p' < p_1$

and $p_0 < p'' < p$. Then $B \leqslant A_1A_0p'$ and also $A_1A_0p' \leqslant B$; hence $A_1A_0p' = B$.

Similarly $A_1A_0p'' = B$.

Uniqueness: By Th 5, $p' \gtrless p''$ would imply $A_1A_0p' \gtrless A_1A_0p''$ and hence $B \gtrless B$. Therefore $p' = p''$.

Sec. 7. Theorem on Moral Expectations. We can give this name to theorem 7 because the expression in (7.2),

$$p \cdot v(B_1) + (1-q) \cdot v(B_0)$$

has been sometimes called the "moral expectation" of the prospect B_1B_0q, where $v(B_1)$, $v(B_0)$ are the "utilities" of B_1, B_0, respectively.

Th 7. There exists a numerical function $v(A)$ on set [A] such that for any B_0, B_1 and for $C \equiv B_1B_0q$,

(7.1) $v(B_0) < v(B_1) \Longleftrightarrow B_0 < B_1$;

(7.2) $C = q \cdot v(B_1) + (1-q) \cdot v(B_0)$.

Proof: A variable prospect B must fall into either of the following classes:

[J]: $A_0 \leqslant B \leqslant A_1$;

[K]: $B > A_1$; and

[L]: $B < A_0$.

By Th 6, there exists, for a given B, a unique p defined by one of the following relations:

$B = A_1A_0p$ $\quad (0 \leqslant p \leqslant 1)$ if B is in [J],

$A_1 = B\,A_1p$ $\quad (0 < p < 1)$ if B is in [K],

$A_0 = A_1Bp$ $\quad (0 < p < 1)$ if B is in [L].

Now define $v(B)$ as follows:

If B is in $[J]$, $v(B)=p\cdot 1+(1-p)\cdot 0$ hence $v(B)=p$; $0\leqslant v(B)\leqslant 1$.

If B is in $[K]$, $1=pv(B)+(1-p)\cdot 0$ hence $v(B)=1/p$; $1<v(B)<\infty$.

If B is in $[L]$, $0=p\cdot 1+(1-p)v(B)$ hence $v(B)=-p/(1-p)$; $-\infty<v(B)<0$.

Putting $p=1$ or 0 in the case $[J]$, we obtain $u(A_1)=1$, $u(A_0)=0$.

The necessity part of (7.1) follows from the definition of the function v: for, if $B_0<B_1$ and they are in different classes, then $v(B_0)$ and $v(B_1)$ are in different intervals such that $v(B_0)<v(B_1)$; and if B_0 and B_1 are in the same interval, then $v(B_0)<v(B_1)$ follows from $B_0<B_1$ by applying Th 5. Repeating the argument for the cases $B_1<B_0$ and $B_1=B_0$, one shows that (7.1) is also sufficient.

To prove (7.2) note that by (2.2) it is an identity for $B_0=B_1$. If $B_0\neq B_1$ assume, without loss of generality, $B_0<B_1$; hence $B_0\leqslant C\leqslant B_1$ by Th 4. First consider the case when both B_0 and B_1 are in class $[J]$. Then, by Th 6 and 5,

$$(7.3)\quad B_0=A_1A_0p_0,\ B=A_1A_0p_1,\quad 0\leqslant p_0<p_1\leqslant 1;$$

so that, by definition $v(B_0)=p_0$, $v(B_1)=p_1$. Then by (2.5)

$$C\equiv(A_1A_0p_0)(A_1A_0p_1)q\equiv A_1A_0[qp_0+(1-q)p_1],$$

and hence, since C is in class $[J]$ (because B_0 and B_1 are), the definition of v yields

$$v(C)=qp_0+(1-q)p_1=qv(B_1)+(1-q)v(B_0).$$

This would complete the proof of Th 7, if there existed a "worst" and a "best" prospect -- i.e., two prospects $\underline{A}$ and $\bar{A}$ such that, for no D, $D<\underline{A}$ or $D>\bar{A}$ were true; we would then pick $A_0\equiv\underline{A}$ and $A_1\equiv\bar{A}$ and the case (7.3) would apply. If, however, the "worst" and "best" prospect do not exist, we have to consider further cases.

Consider the case when B_0 and B_1 are in $[K]$. Then

$$(7.4)\quad A_1=B_0A_0p_0=B_1A_0p_1,\ p_0>0,\ p_1>0.$$

Since $A_0 < A_1 < C$, let $A_1 = CA_0r$ (by Th 6), $r > 0$. Then, B_0, B_1 and C all being in [K],

$$v(C) = 1/r, \ v(B_0) = 1/p_0, \ v(B_1) = 1/p_1.$$

We have to prove (7.2) which has become

$$(7.4.a) \quad \frac{1}{r} = \frac{q}{p_1} + \frac{1-q}{p_0}.$$

Let

$$(7.4.b) \quad B_0 = B_1A_0s, \ 0 < s < 1.$$

Then by (7.4) and (2.4)

$$(7.4.c) \quad A_1 = (B_1A_0s)A_0p_0 = B_1A_0p_0s = B_1A_0p_1; \text{ hence } s = p_1/p_0.$$

By (7.4.b), (2.4),

$$C = B_1B_0q = B_1(B_1A_0s)q = B_1A_0[q+(1-q)s] = B_1A_0t,$$

where

$$t = q+(1-q)s = \frac{q}{p_1} + \frac{1-q}{p_0} \cdot p_1,$$

since $s = p_1/p_0$. Now since we had put $A_1 = CA_0r$, we have

$$A_1 = (B_1A_0t)A_0r = B_1A_0(rt) \text{ by (2.4); so that, by (7.4.c)}$$

$$rt = p_1, \ \frac{1}{r} = \frac{t}{p_1},$$

thus proving (7.4.a). Hence the theorem is proved, also for the case when B_0 and B_1 are both in [K].

The remaining cases are: B_0 and B_1 both in [L]; B_0 in [J], B_1 in [K] or [L]; B_0 in [J], B_1 in [K]. The proofs are similar, always using the identities (2.4), (2.5). Perhaps some unifying yet not less elementary proof is possible.

Sec. 8. Theorem on Measurable Utilities

Th 8. If, for any B, $u(B) = m + nv(B)$, $n > 0$, then conditions (7.1), (7.2) are satisfied, with $u(B)$ replacing $v(B)$. Conversely, if $u(B)$ satisfies those conditions, $u(B)$ must be a linear increasing function of $v(B)$.

Proof: Sufficiency: substitute $m+nv(B)$ for $v(B)$ in (7.1), (7.2).

Necessity: If (7.2) is satisfied by both functions u and v, then q, 1-q, and 1 can be regarded as the variables of the following homogeneous linear system:

$$u(B_1)\cdot q+u(B_0)\cdot(1-q)-u(C)\cdot 1=0$$

$$v(B_1)\cdot q+v(B_0)\cdot(1-q)-v(C)\cdot 1=0$$

$$1\cdot q + 1\cdot 1-q - 1.1=0$$

Hence the coefficients are linearly dependent. Therefore, there exist m, n such that $u(A)=m+nv(A)$, $(A\equiv B_1, B_0, C)$. To satisfy (7.1), m must be positive.

Sec. 9. Comparison with v. Neumann-Morgenstern

As stated in paper [2], p. 136, the class U of "entities" u,v,w,..., considered in v. Neumann-Morgenstern's axiomatics, [1], p. 26, has to be interpreted as the class of all indifference sets of prospects. In our notation, an indifference set $[A_0]$ is a subset of [A], such that for any two elements A_0' and A_0'' in $[A_0]$, $A'=A''$. The relation $[A_0]<[A_1]$ ($u<v$ in v. N.-M. notation) can be interpreted in our terminology as follows:

For any A_0' in $[A_0]$ and A_1' in $[A_1]$, $A_0'<A_1'$. Further, for any number $0<p<1$ ($0<\alpha<1$ in the v. N.-M. notation), the following operation is said to be set up:

(9.1) $p\cdot[A_0]+(1-p)\cdot[A_1]=[A_2]$;

it was pointed out by E. Malinvaud to the present writer that in (9.1) von Neumann-Morgenstern not only defined an operation (on the left hand side) but also, by using the equation sign, introduced an axiom — viz., that the result of the operation is itself an indifference set. It is, in fact, stated by von Neumann and Morgenstern that the "entity" w — corresponding in v. N.-M. notation to our [A] in (9.1) — is an element of U.

On p. 26 of [1], the operation in our (9.1) is interpreted as constructing a lottery with two outcomes, the odds being p:1-p. If, in addition, one would interpret the elements of $[A_0]$, $[A_1]$,... as (generally uncertain) prospects, i.e., probability distributions, then "equation" (9.1) becomes immediately our Axiom II.2; also, the complicated looking Axiom (3.B:C) of von Neumann-Morgenstern becomes a theorem, and need not be postulated separately.

The remaining part of v. N.-M. axiomatics corresponds to our Ax. I (complete ordering) and Th. 6 (continuity).