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The Utility of Wealth

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1.1. Friedman and Savage[1] have explained the existence of insurance and lotteries by the following joint hypothesis:

(1) Each individual (or consumer unit) acts as if he: a) ascribed (real) numbers (called utility) to every level of wealth;[2] and b) acts in the face of known odds so as to maximize expected utility.

(2) The utility function is as illustrated in Figure 1. We may assume it to be a continuous curve with at least first and second derivatives.[3] Let U be utility; W, be wealth. Below some point A $\frac{\partial^2 U}{\partial W^2} < 0$; between A and B, $\frac{\partial^2 U}{\partial W^2} > 0$; above B, $\frac{\partial^2 U}{\partial W^2} < 0$.

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1. M. Friedman and L. J. Savage, "The Utility Analysis of Choices Involving Risk," Journal of Political Economy, Vol. 56, August 1948, pp. 279-304.

2. We wish to avoid delicate questions of whether the relevant utility function is the "utility of money" or the "utility of income." We will assume that income is discounted by some interest rate and we will speak of the "utility of wealth."

3. The existence of derivatives is not essential to the hypothesis. What is essential is that the curve be convex below A and above B; concave between A and B. Our arguments would be essentially unaffected if we made these more general assumptions.

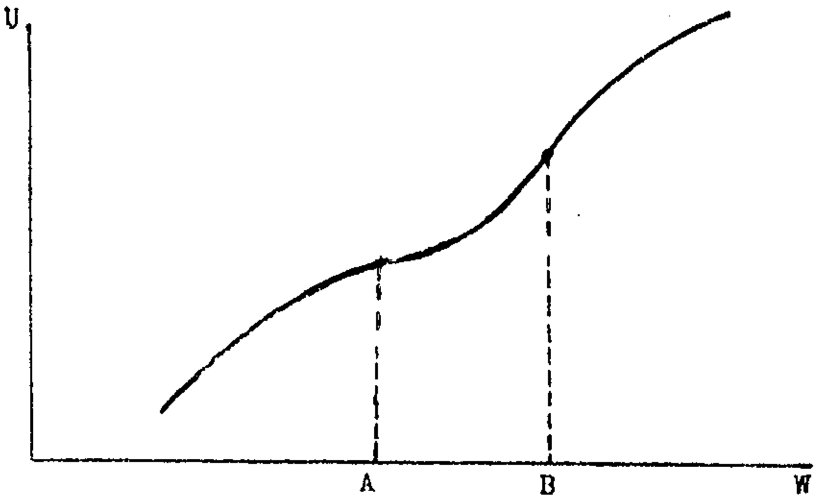

Figure 1

To tell geometrically whether or not an individual would prefer W_o with certainty or a "fair"[4] chance of rising to W_1 or falling to W_2, draw a line from the point $(W_1, U(W_1))$ to the point $(W_2, U(W_2))$. If this line passes above the point $(W_o, U(W_o))$ then the expected utility of the fair bet is greater than $U(W_o)$; the bet is preferred to having W_o with certainty. The opposite is true if the line $\overline{(W_1, U(W_1)), (W_2\ U(W_2))}$ passes below the point $(W_o, U(W_o))$. In Figure 2, W_o is preferred to a fair chance of rising to W_1 or falling to W_2. The chance of rising to W_1' or falling to W_2'' is preferred to having W_o' with certainty. The first example may be thought of as an insurance situation. A person with wealth W_1 would prefer to be sure of W_o

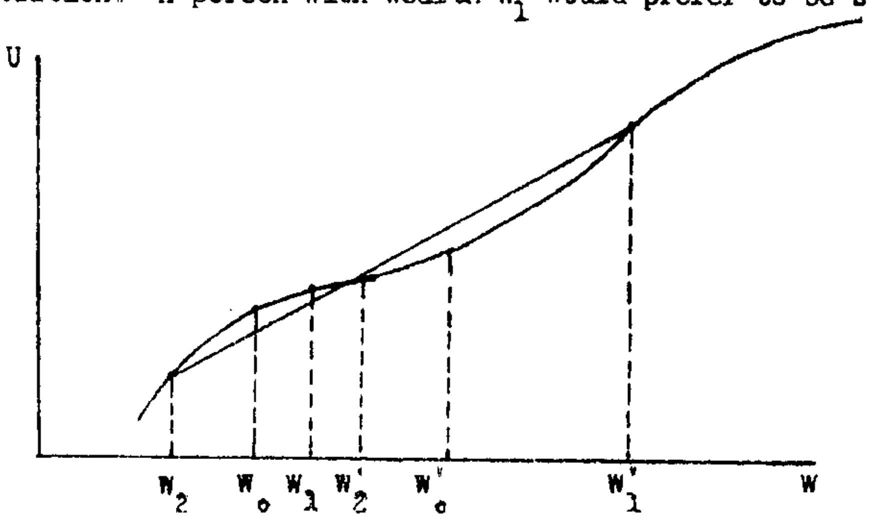

Figure 2

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4. A fair bet is defined to be one with expected gain or loss of wealth equal to zero. In particular if   is the probability of W_1 and $(1 -\ \ )$ is that of W_2 then $\ W_1 + (1 -\ \ )\ W_2 = W_o$.

than to take a chance of falling to W_2. The second example may be thought of as a lottery situation. The person with wealth W_0 pays $(W_0 - W_1)$ for a lottery ticket in the hope of winning $(W_2 - W_1)$. Even if the insurance and the lottery were slightly "unfair,"[5/] the insurance would have been taken and the lottery ticket bought.

Thus the Friedman-Savage hypotheses explains both the buying of insurance and the buying of lottery tickets.

1.2. In this section we will argue that the Friedman-Savage (F-S) hypothesis contradicts common observation in important respects. In the following section we will present a hypothesis which explains what the F-S hypothesis explains; avoids the contradictions with common observation to which the F-S hypothesis is subject; and explains still other phenomena concerning behavior under uncertainty.

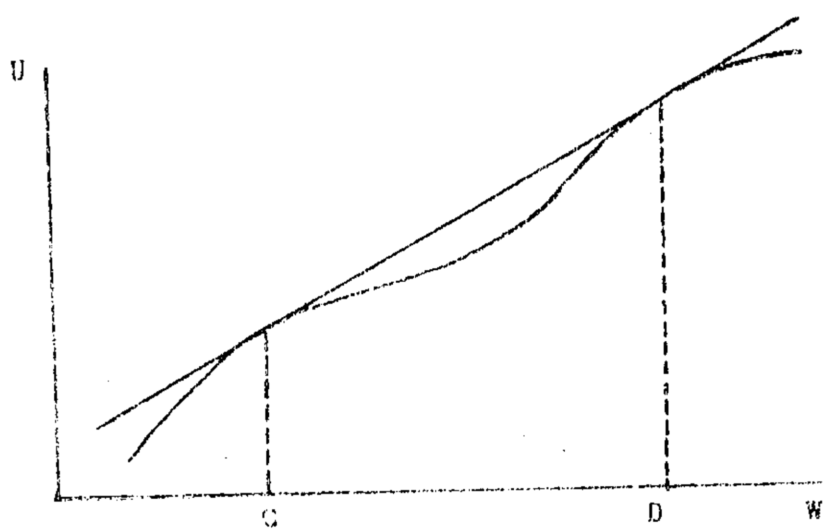

Figure 3.

In Figure 3 a line ℓ has been drawn tangent to the curve at two points.[6/]

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5. I.e., Even if $\alpha W_1 + (1 - \alpha) W_2 > W_0$, $\alpha' W_1' + (1 - \alpha') W_2' < W_0'$. For limits on the amount of unfairness which an individual would accept. See Friedman and Savage, op. cit., p. 291.

6. If such a "double tangent" exists it is unique. If it does not exist the hypothesis is subject to the St. Petersberg paradox. We will assume therefore that the unique double tangent exists.

A person with wealth less than C is persumably "poor;" a person with wealth greater than D is presumably well to do. Friedman and Savage go so far as to interpret these as different social classes. The amount (D - C) is the size of the optimal lottery prize (i.e., the size of prize which it is most profitable for lottery managers to offer). Those poorer than C will never take a fair bet. Those richer than D will never take a fair bet. Those with wealth between C and D will take some fair bets.

We will now look more closely at the hypothesized behavior of persons with various levels of wealth. We will see that for some points on the W axis the F-S hypothesis implies behavior which is not only not observed, but would be considered shocking if it were. At other points on the curve the hypothesis implies less absurd, but still peculiar, behavior. At only one region of the curve does the F-S hypothesis imply behavior which is commonly observed. This in itself may suggest how the analyses should be modified.

Consider two men with wealth equal to C + 1/2 D (i.e., two men who are midway between C and D). There is nothing which these men would prefer, in the way of a fair bet, than one in which the looser fell to C and the winner rose to D. The amount bet would be $\frac{C - D}{2}$ - half the size of the optimal lottery prize. At the flip of a coin the loosor would become poor; the winner, rich. Not only would such a fair bet be acceptable to them, none would please them more.

But we do not observe persons of middle income taking large symmetric bets. We expect people to be repelled by such bets. If such a bet were made it would be certainly considered unusual, irrational, and perhaps immoral.

Consider a person with wealth slightly less than D. This person is "almost rich." The bet which this person would like most, according to the F-S hypothesis, is one which if won would raise him to D, if lost would lower him to C. He would be willing to take a small chance of a large loss for a large chance of a small gain. He would not insure against a loss of wealth to C. On the contrary he would be anxious to be a one man insurance company. He would even be willing to extend insurance at an expected loss to himself.

Again such behavior is not observed. No moderately well to do individual would accept a small insurance premium for assuming insurance risks. The insurance business is done by companies (or persons) of such great wealth that they can diversify to the point of almost eliminating risk (consider the stability of our insurance companies). The one man insurance company is as absurd as the large symmetric bets spoken of above.

According to the F-S hypothesis a person with wealth less than C or more than D will never take any fair bet (and a-fortiori, never an unfair bet). This seems peculiar, since even poor people buy sweepstakes tickets, play the horses and participate in other forms of gambling. Rich people play roulette and the stock market. We might defend the F-S hypothesis at this point by ascribing the above behavior to the "fun of participation" or to inside information. But people gamble even when there can be no inside information; and as to the joy of participation, if people like participation but don't like taking chances, why don't they always play with stage money? It would be desirable (at least according to Occam's Razor) if an alternative utility analysis could help explan chance taking among the rich and the poor, as well as avoid the less defensible implications of the F-S hypothesis.

Another level of wealth which is of interest is that which corresponds to the first infection point on the F-S curve. We will find that the implications of the F-S hypothesis are quite plausible for this level of wealth. We will not discuss these implications at this point; for, the analysis is essentially the same as that of the modified hypothesis to be presented below.

2.1. We will introduce our hypothesis by means of a set of questions and answers. The writer has asked these questions informally of many people and has typically received the answers indicated. But these "surveys" have been too unsystematic to serve as evidence; we present these questions and typical answers only as a heuristic introduction. After our hypothesis is introduced we will compare its ability and that of the F-S hypothesis to explain well established phenomena. (The hypothesis as a whole is presented on page 9.)

Suppose a stranger offered to give you either 10¢ or else one chance in ten of getting $1 (and nine chances in ten of getting nothing). The situation was quite impersonal; and you knew the odds were as stated. Which would you prefer,

(1) 10¢ with certainty, or one chance in ten of getting $1?

Similarly which would you prefer (why not circle your choice?):

(2) $1 with certainty or one chance in ten of getting $10?

(3) $10 " " or " " " " " " $100?

(4) $100 " " or " " " " " " $1000?

(5) $1000 " " or " " " " " " $10,000?

(6) $1,000,000 with certainty or one chance in ten of getting $10,000,000?

Suppose that you owed the stranger 10¢, would you prefer to pay the

(7) 10¢ or take one chance in ten of owing $1?

Similarly would you prefer to owe

(8) $1 or take one chance in ten of owing $10?

(9) $10 or " " " " " " " $100?

(10) $100 or " " " " " " " $1000?

(11) $1,000,000 or take one chance in ten of owing $10,000,000.

The typical answers (of my middle income acquaintances) to these questions are as follows: most prefer to take a chance on $1 rather than get 10¢ for sure; take a chance on $10 rather than get $1 for sure. Preferences begin to differ on the choice between $10 for sure or one chance in ten of $100. Those who prefer the $10 for sure in situation (3) also prefer $100 for sure in situation (4); while some who would take a chance in situation (3), prefer the $100 for sure in situation (4). By situation (6) everyone prefers the $1,000,000 for sure rather than one chance in ten of $10,000,000.

All this may be explained by assuming that the utility function for levels of wealth above present wealth is first concave and then convex:

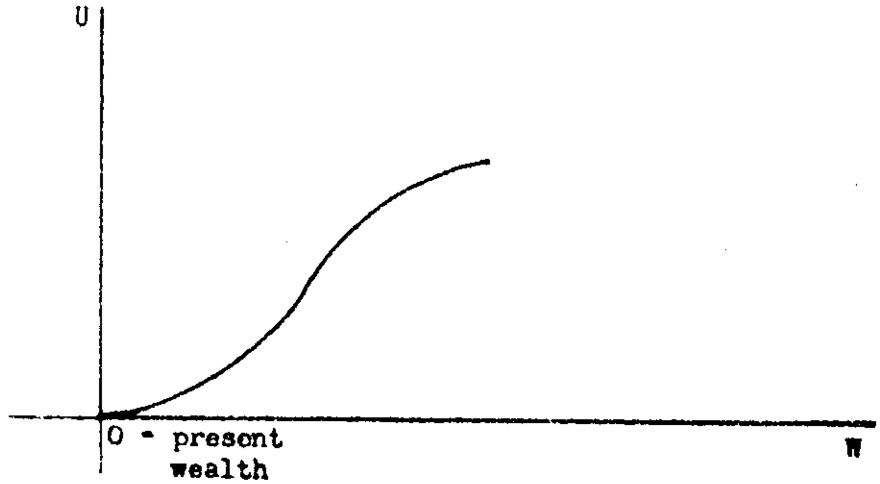

Figure 4

Let us continue our heuristic introduction. People have generally indicated a preference for owing 10¢ for sure, rather than one chance in ten of owing $1; owing $1 for sure rather than taking one chance in ten of owing $10; $10 for sure rather than one in ten of $100. There comes a point, however, where the individual is willing to take a chance. In situation (11) for example, the individual generally will prefer one chance in ten of owing $10,000,000 rather than owing $1,000,000 for sure. All this may be explained by assuming that the utility function going from present wealth downward is first convex and then concave. Thus we have a curve as in Figure 5, with three inflection points. The middle inflection point is at present wealth. The function is concave immediately above present wealth; convex, immediately below.

How would choices in situations (1) - (11) differ if the chooser were rather rich? Our guess is that he would take a chance on getting the $10 rather than take $1 for sure; take a chance on $100 rather than take $10 for sure; perhaps take a chance on $1,000 rather than take $100 for sure. But the point would come when he would become cautious. E.g., he would prefer $1,000,000,000 rather than

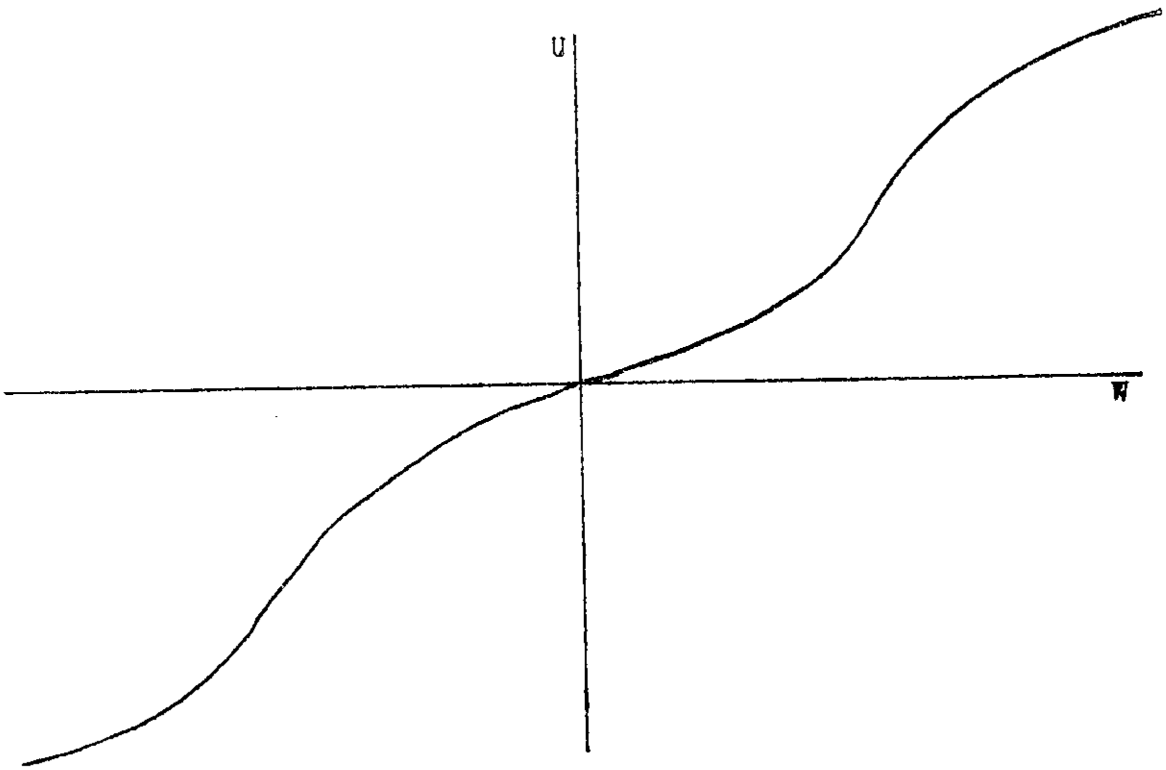

Figure 5

one chance in ten of \$10,000,000,000. In other words he would act essentially the same, in situations (1) - (6), as someone with more moderate wealth, except that his third inflection point would be further from the origin. Similarly we hypothesize that in situations (7) - (11) he would act as if his first inflection point also was further from the origin.

Conversely, if the chooser were rather poor we would expect him to act as if his first and third inflection points were closer to the origin.

Generally people avoid symmetric bets. This suggests that the curve falls faster to the left of the origin than it rises to the right of the origin. (I.e., $U(X) > |U(-X)|$ $X > 0$).

So far we have assumed that the second inflection corresponds to present wealth. There are reasons for believing that this is not always the case. For example, suppose that our hypothetical stranger, rather than offering <u>to</u> give you \$X or a chance of \$Y, had instead first given you the \$X and then had offered you a fair bet which if lost would cost you - \$X and if won would net you \$(Y - X). These two situations are essentially the same, and it is plausible to expect the chooser to act in the same manner in both situations. But this will not always be the implication of our hypotheses if we insist that the second inflection point always corresponds to present wealth. We can resolve this dilemma by assuming a) that the second inflection point corresponds to "customary wealth;" b) that, in cases of recent windfall gains or losses, present wealth may differ from customary wealth. (In such cases we will let the second inflection point remain at the origin of our graph.) We will return later to the concept of "customary wealth." Unless we specify otherwise, we will assume that there have been no recent windfall gains or losses and therefore customary wealth equals present wealth.

To summarize our hypothesis: the utility function has three inflection points. The middle inflection point is located at the "customary" level of wealth. Except in cases of recent windfall gains and losses, customary wealth equals present wealth. The first inflection point is below, the third inflection point is above customary wealth. The distance between the inflection points is a non-decreasing function of wealth. The curve is monotonically increasing; it is first concave, then convex, then concave and finally convex. We may also assume that $|U(-X)| > U(X)$, $X > 0$ (where $X = 0$ is defined to be at the second inflection point). A curve which is consistent with our specifications is given in Figure 5. 2.2. An examination of Figure 5 will show that our hypothesis is consistent with the existence of both "fair" (or slightly "unfair") insurance and "fair" (or slightly "unfair") lotteries. The same individual will buy insurance and lottery tickets. He will take large chances of a small loss for a small chance for a large gain.

Our hypothesis implies that his behavior will be essentially the same whether he is poor or rich—except the meaning of "large" and "small" will be different. In particular we have no levels of wealth where people prefer large symmetric bets to any other fair bet; or desire to become one man insurance companies, even at an expected loss.

Thus we see that our hypothesis is consistent with both insurance and lotteries, as was the F-S hypothesis. We also see that our hypothesis avoids the contradictions with common observations to which the F-S hypothesis was subject.

2.3. We will now apply our hypothesis to other phenomena. We will only consider situations wherein there are objective odds. This is because we are concerned with a hypothesis about the utility function and do not want to get involved in questions concerning subjective probability beliefs. It may be hoped, however, that a utility function which is successful in explaining behavior in the face of known odds (risk) will also prove useful in the explanation of behavior under uncertainty.

It is a common observation that, in card games, dice games and such, people play more conservatively when loosing moderately, more liberally when winning moderately.[7] This phenomena can be explained in two different ways. These two explanations apply to somewhat different situations.

A bet which a person makes during a series of games ("plays" in the von Neumann sense) can not be explained without reference to the gains and losses which have occurred before and the possibilities open afterward. What is important is the outcome for the whole series of games: the winnings or loosings for "the evening" as a whole. Suppose "the evening" consists of a series of independent games (say matching pennies); suppose that the probability (frequency) distribution of wins and losses for a particular game is symmetric about zero.

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7. For anyone who wishes evidence of this fact we refer to Mosteller and Nogee, "An Experimental Measurement of Utility, Table 10. Participants in the experiment were asked to write instructions as to how their money should be bet by others. The instructions consisted of indicating what bets should be accepted when offered, and "further (written) instructions." The "further instructions" are revealing; for example, "A-II--Play till you drop to 75¢ then stop!!"; "B-V--If you get low, play only very good odds;" "C-I--If you are ahead you may play the four 4's for as low as $3;" "C-III--If player finds that he is winning, he shall go ahead and bet at his discretion;" "C-IV--If his winnings exceed $2.50 he may play any and every hand as he so desires but if his amount should drop below 60¢ he should use discretion in regard to the odds and hands that come up." No one gave instructions to play more liberally when loosing than when winning.

Suppose that at each particular game, the player has a choice of betting liberally or conservatively (i.e., he can influence the dispersion of the wins and losses). If he bet with equal liberality at each game, regardless of previous wins or losses, then the frequency distribution of final wins and losses (for the evening as a whole) would be symmetric. The effect of playing conservatively when loosing, liberally when winning, is to make the frequency distribution of final outcomes skewed to the right. Such skewness is implied as desirable (in a large neighborhood of customary income) by our utility function. In sum, our utility function implies the desirability of some positive skewness of the final outcome frequency distribution; which in turn implies the desirability of playing conservatively when loosing moderately, and playing liberally when winning moderately.

This implication holds true whatever be the level of customary wealth of the individual. In the F-S analysis, a person with wealth equal to D in Figure 3 would play liberally when loosing, conservatively when winning so as to attain negative skewness of the frequency distribution. This, I would say, is another one of those peculiar implications which flow from the F-S analysis.

Now let us consider the effect of wins or losses on the liberality of betting when we do not have the stratigic considerations which were central in the previous discussion. For example, suppose that the "evening" is over. The question arises as to whether or not the game should be continued into the morning (i.e., whether or not a new series of games should be initiated). There is also a question of whether or not the stakes should be higher or lower. We abstract from tiredness or loss of interest in the game.

How does the evening's wins or losses effect the individual's preferences on these questions? Since his gain or loss is a "windfall" the individual is moved from the middle inflection point (presumably by the amount of the gain or loss).

A person who broke even would, by hypothesis, have the same preferences as at the beginning of the evening.

A person who had won moderately would (by definition of "moderate") be between the second and third inflection point. The moderate winner would wish to continue the game and increase the stakes.

A person who had won very much would (by the definition of "very much") be to the right of the third inflection point. He would wish to play for lower stakes or to not play at all. In the vernacular, the heavy winner would have made his "killing" and would wish to "quit while winning."

The moderate looser, between the first and second inflection points, would wish to play for lower stakes or not to play at all.

A person who lost extremely heavily (to the left of the first inflection point) would wish to continue the game.(somewhat in desperation).

We see above the use of the distinction between customary and present wealth.

We have assumed that asymmetric bets are undesirable. This assumption could be dropped or relaxed without much change in the rest of our hypothesis; but I believe this assumption is correct and should be kept. Symmetric bets are avoided when moderate or large amounts are at stake. Sometimes small symmetric bets are observed. How can these be explained? We offer three explanations, one or more of which may apply in any particular situation. First, we saw previously that a symmetric bet may be part of a strategy leading to a positively skewed probability distribution of final outcome for the evening as a whole . Second, for very small symmetric bets the loss in utility from the bet is negligible and is compensated for by the "fun of participation." Third, (this reason supplements the second) there is an inflection point at $W = 0$, therefore the utility function is almost linear in the neighborhood of $W = 0$, therefore there is little loss of utility from small symmetric bets.

3.1. Above we used the concept of "fun of participation." If we admit this as we must--as one of determinants of behavior under uncertainty, then we must contend with the following hypothesis: The utility function is everywhere convex; all (fair) chance taking is due to the "fun of participation." This "classical" hypothesis is simpler than ours and is probably rather popular. If it explained observable behavior as well as our hypothesis, this classical hypothesis would be preferable to ours.

Before examining the hypothesis we must formulate it more exactly. It seems to say that the utility of a gamble is the expected utility of the outcomes plus the utility of playing the game (the latter utility is independent of the outcome of the game). This can be presented graphically as in Figure 6.

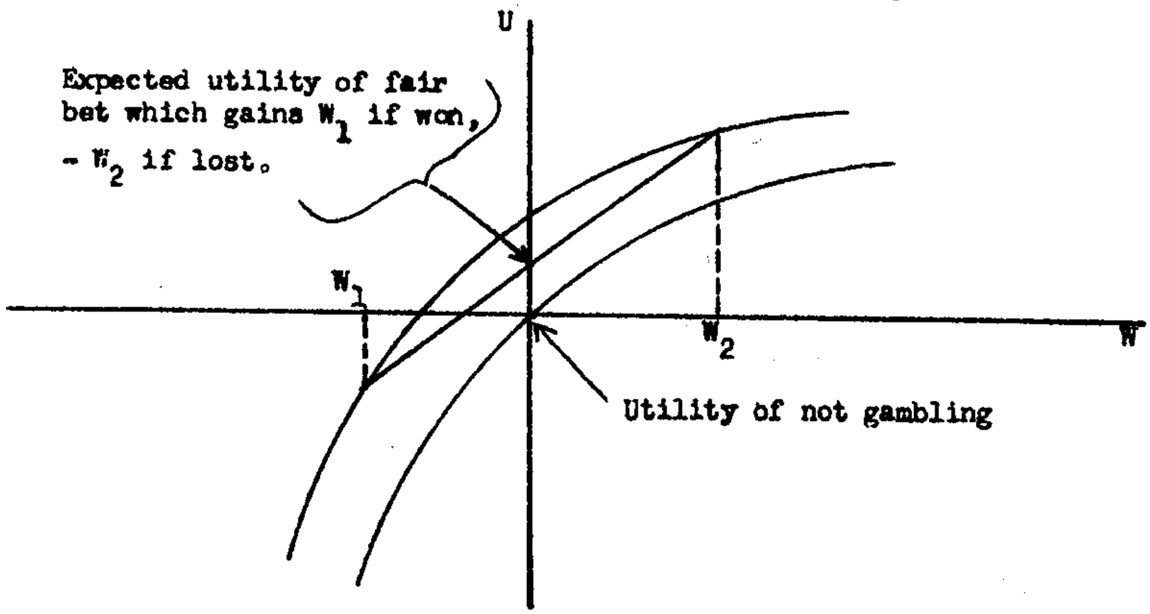

Figure 6.

One implication of this hypothesis is that for given (fair) odds, the smaller the amount bet the higher the expected utility. In particular, when millionaires play poker together they play for pennies; and no man will buy more than one lottery ticket. This contradicts observation.[8/]

One might hypothesize that the utility of the game, to be added to the utility of the outcomes, is a function of the possible loss ($- W_2$) or the difference between gain and loss ($W_1 - W_2$). Neither of these hypothesis explains why people

8. The statement that millionaires "ought" to play for pennies is irrelevant. We seek a hypothesis to explain behavior, not a moral principle by which to judge behavior.

prefer small chances of large gains with large chances of small losses, rather than vice versa. Nor do they explain why people play more conservatively when loosing than when winning.

In short the classical hypothesis may be consistent with the existence of chance taking, but it does not explain the particular chances which are taken. To explain such choices, while maintaining simple hypotheses concerning "fun of participation," we must postulate a utility function as in Figure 5.

4.1. In review, we have seen that the F-S hypothesis has implications which contradict common observation. We have presented a hypothesis which is consistent with insurance and lotteries, as was the F-S hypothesis, avoids the contradictions with observation to which the F-S hypothesis was subject, and explains other phenomena concerning behavior under uncertainty. A commonly accepted classical hypothesis was also considered and found lacking.

Our goal in these considerations is not to explain gambling, but to test hypotheses concerning behavior in the face of known odds. It is hoped that the knowledge gained will prove useful in the understanding of economic behavior under uncertainty.