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Quantitative Description of the Effect of Technological Change on Production Potential

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Quantitative Description of the Effect of Technological Change on Production Potential[1]

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1 Introduction

The main purpose of this paper is to study the effect of technological change on production potential and to attempt to give a simple quantitative description of this effect.

The significant words in this description of our purpose are susceptible of many definitions; the interrelations between the concepts they introduce are complex. For these two reasons at least, our first task will be to set up a mathematical model of the economic system on whose unambiguous definitions a reasoning can be built, and whose compact symbolism permits the handling of its interrelations.

As some readers may wish to avoid the burden of getting acquainted with this model in its full generality, several sections (specifically 2-7-8-10) have been divided into two parts. In such a section the first part deals only with the most simple model of the type considered: it will contain only two commodities, and two individuals, variables will be continuous, curves smooth, etc.. The second part, beginning with the words More generally, actually deals with the model we wish to present. It is possible to neglect

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[1] Based on a Cowles Commission Discussion Paper, Economics No. 299 (hectographed) December 1950. My acknowledgments are due to R. Solow of Massachusetts Institute of Technology, Cowles Commission Staff members and their guests, particularly C. Hildreth, L. Hurwicz, T. C. Koopmans, H. Markowitz, J. Marschak, H. Simon, J. Leveugle, E. Malinvaud for their critical comments.

these second parts and still to grasp the main ideas of this paper; however it should be kept in mind in this case that the first parts of the sections mentioned constitute an intentionally oversimplified illustration of a more general theory.

The model presented here is by no means final; the main unsolved conceptual difficulties are emphasized. It only suggests a point of view from which technological change may be looked at. If this model is found conducive to a better overall picture, when used with the other points of view, many improvements can be brought to the adaptable frame we present.

2 Description of the Model

We assume that in the economic system only two commodities, characterized by the subscript $h = 1$ or 2, are produced and consumed.

We assume also that there are only two consumption-units characterized by the subscript $i = 1$ or 2.

It will therefore be possible to have two dimensional figures illustrating the reasoning; on the other hand, as soon as there is more than one commodity and more than one consumption-unit most of the significant features of the general model are present.

The concept of commodity is, of course, not defined: the mathematical model is capable of various interpretations and in such an interpretation the content given to any concept is only required to have the formal properties of this concept in the model. A commodity can thus be a good or a service, direct or indirect, playing a role in any production or consumption process.

Similarly a consumption-unit can be a consumer, a household unit, a governmental agency,... In an economy provided with a central planning board incarnating a social welfare function there is only one consumption-unit. The only things that matter for us are that

1) the activity of the i-th consumption-unit, is completely represented by the quantities of commodities it "consumes" x_{1i}, x_{2i} (the quantity of a commodity produced, a certain type of labor for example, would be represented by a negative number) i.e., by a point of the commodity plane (Fig. 1).

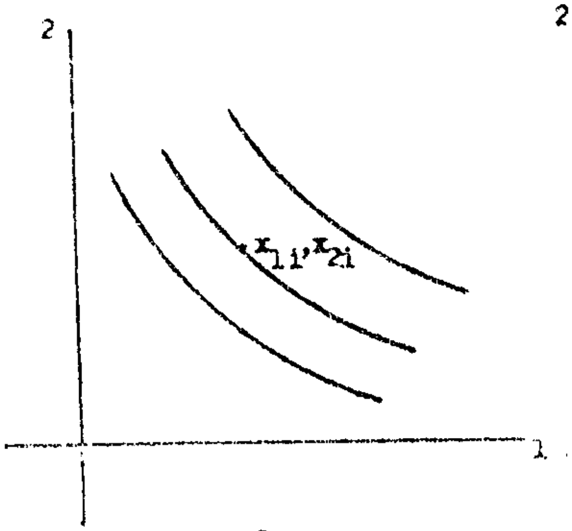

Fig. 1

2) there is a preference ordering of its "consumption" points and therefore a family of indifference curves.

A numerical function $s_i(x_{1i}, x_{2i})$ can be constructed such that it increases when, and only when, one goes from a point to another one which is preferred. The value of s_i indicates to what extent the needs of the i-th consumption-unit are satisfied; we can call it its <u>satisfaction</u> or, if it is an ordinary consumer, its <u>standard of living</u>. Any monotonic increasing function of s_i provides an equally satisfactory expression of the standard of living. The point whose coordinates are s_1 and s_2 describes completely the standard of living of the economic system (Fig. 2). It is the aim of this system to make the various s_i as large as possible.

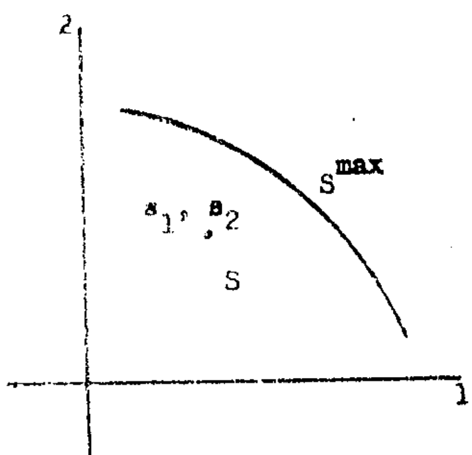

Fig. 2

The production activity of the system is represented by the <u>net</u> quantities y_1, y_2 of the two commodities consumed by the whole production sector (the quantity of a commodity produced would be negative). It can be represented also by the point of the commodity plane whose coordinates are y_1 y_2 (Fig. 3). Constraints which we describe broadly by the expression "technological knowledge" determine whether such a "production" point is actually achievable or not.

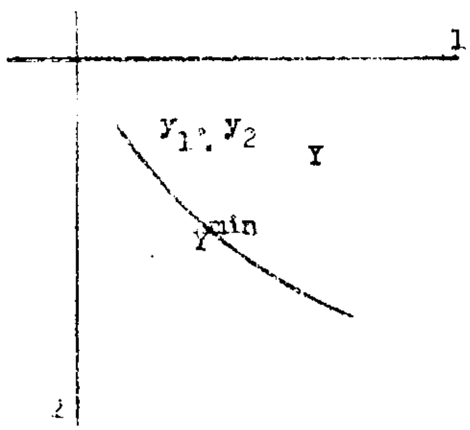

Fig. 3

Those constraints are defined independently of the limitation of physical resources (which will be dealt with later), and of conditions in the consumption sector. One wants to know, for example, in a crude two-commodity model, if it is possible to produce 600 million tons of steel out of 500 million tons of a certain type of ore.

All possible points form the region Y of technological possibilities whose boundary is a curve Y^{min} the locus of the efficient points in production. It is impossible to decrease any coordinate (i.e., to decrease any input or to increase any output) of a point of Y^{min} without increasing another coordinate (on Fig. 3 the first commodity is the input, the second one is the output.)

The quantity of the h-th commodity consumed by all consumption-units and by the production sector is $z_h = x_{h1} + x_{h2} + y_h$; it can come only from the utilizable physical resources z_1^o, z_2^o and therefore $z_h \leq z_h^o$,[2] h = 1, 2.

[2] A certain type of labor can be treated in two different ways:
if one wishes to emphasize the fact that the quantity of this labor produced by consumption-units is variable, one includes it in the individual consumptions (x_{1i} or x_{2i}); one of the significant aspects of the increase of the standard of living associated with technological progress is, in fact, the reduction of the working week.
otherwise one treats it as a physical resource whose quantity is a datum

z_1, z_2 can conveniently be called the utilized physical resources.

More generally in the vector spaces which we consider we use the following notations for inequalities among vectors:

$u \geqq v$ if for every pair of components $u_\iota \geqq v_\iota$

$u > v$ if for every pair of components $u_\iota > v_\iota$

$u \geq v$ if $u \geqq v$ and $u \neq v$.

A function $w(u)$ is said to be increasing (resp. nondecreasing) if "$u^2 \geq u^1$" implies "$w(u^2) \geq w(u^1)$ [resp. $w(u^2) \geqq w(u^1)$]."

According to the usual terminology, a vector v is a maximal (resp. minimal) element of a set U if 1) $v \in U$ (v belongs to U), 2) there is no u such that $u \in U$ and $v \leq u$ (resp. $v \geq u$). The set of maximal (resp. minimal) elements of U is denoted by U^{max} (resp. U^{min}).(3)

3 Points of rigor will not be discussed here; for instance, the sets we deal with will always be tacitly assumed to be closed.

The vector sum of a finite number of sets U_ι, $V = \sum_\iota U_\iota$ is the set of $v = \sum_\iota u_\iota$, $u_\iota \in U_\iota$.

With the help of these concepts the economic system can be described in all the generality that we intend to give to it.

A commodity is characterized by a subscript h which defines its physical nature, its location (we can assume that in both cases there is but a finite number of them), and the time interval it refers to (all successive time intervals are supposed of equal length). The quantity of the h-th commodity can either be varied continuously or is restricted to be an integral multiple of a given indivisible unit.

A consumption-unit is characterized by a subscript i; its activity is represented by a consumption-vector x_i of the commodity space R. We assume that there is a preference ordering of those vectors. One can therefore construct equivalence classes (an equivalence class may happen to contain only one vector) and define a nondecreasing numerical function $s_i(x_i)$, constant in, and only in, each class of equivalence; this index of the satisfaction of the needs of the i-th consumption-unit is defined but for a monotonically increasing function. The m satisfaction levels s_i are considered as the components of a vector $\check{s}$ of the satisfaction space $\tilde{R}$.

The production activity of the system is represented by the <u>total input-vector</u> $y \in R$ whose components are inputs (<u>net</u> quantities of commodities consumed by the <u>whole</u> production sector) or negatives of outputs (defined in a symmetrical way). Constraints such as the limitation of technological knowledge determine the set Y of possible y. The set of efficient vectors in production is Y^{min}.

A family of sets Y_j is a decomposition of Y if $Y = \sum_j Y_j$; in other words for every $y \in Y$ we can write $y = \sum_j y_j$, $y_j \in Y_j$.[4] The <u>input-vector</u> y_j characterizes the activity of the j-th <u>production-unit</u> whose set of possibilities is Y_j. The concept of <u>production-unit</u> can be interpreted alternatively as that of industry, firm, plant,... . This formulation allows for production and consumption of intermediate commodities, even in a circular way, with as many intermediate steps as one wants; it allows, of course, for discontinuities of variables, or if they are continuous, for nonsmooth efficiency surfaces Y_j^{min}, for the existence of fixed ratios between some variables, etc... . If $y \in Y^{min}$ it is necessary but not sufficient that $y_j \in Y_j^{min}$ for all j.

$x = \sum_i x_i$ is the <u>total consumption-vector</u> and $z = x + y$ is the total <u>net</u> consumption of the whole economy, we call it the <u>utilized-physical resources-vector</u> as opposed to z^o the <u>utilizable-physical resources-vector</u>. One has necessarily $z \leqq z^o$.

4 This decomposition is not meant to be unique. If $0 \in Y_j$ for every j, $Y_j \subset Y$ (Y_j is contained in Y) for every j.

3 Discussion of the Model

It would probably seem unsatisfactory to have an economic model whose whole activity would be concentrated at a single time-point. The alternative is to consider this activity as extending over a sequence of equal time-intervals 1,2,...,t,... . The interpretation of the subscripts that we have introduced can easily include a characterization of the time-interval; several difficulties however arise.

We have first to make a choice between certainty and uncertainty: we take the first one, which is much easier to handle, but we do not thereby underestimate the importance of the second one in the problem we are concerned with.

At the end of the sequence of time intervals under consideration, certain quantities of the various commodities will exist. Shall we require that each one of these quantities has a minimum value given a priori? This would be rather unsatisfactory for they are in fact the result of optimizing processes. The alternative however is to allow the sequence to extend to infinity and sizable difficulties arise here.

Next we have to choose between two ways of looking at the problem of change:

-- we can center our attention on preference orderings of the consumption-units, at the initial time point, and the plans that they make at this time. The purpose of our study would then be to find out the effects of a change of the set Y of possible input-vectors, and/or of the utilizable-physical-resources-vector s^o. We call this a structural change.

-- or we can study step by step the changes occurring from one time-interval to the next one when the structure does not change. In such a study we should take into account the relations existing between preference orderings at two successive time points and between the sets of production possibilities at two successive time points. This can be described as a study of the dynamic evolution of the system. Here again the first problem is easier and it is the one we take up.

To sum up, the nature of our subject suggests the study of the dynamic evolution of a system whose activity extends to infinity under uncertainty. The problem of structural change of a system whose activity extends over a finite period of time under certainty has been taken up for the sake of analytic convenience.

The set Y can be interpreted in many different ways. Its definitions range from an abstract one, such as the set of possibilities which can be imagined on the basis of some ideal pooling of all existing knowledge, to concrete ones taking into account the innumerable obstacles which prevent it; it can take into account research activity. The existence of various interpretations is natural for any model aiming at a certain degree of generality.

4 Technological Change

Technological change has sometimes been defined as a change in the quantities consumed and produced by Industry y_1, y_2 (more generally in y) but this definition is much too wide since it can result from a change in the preference orderings, in the set of technological possibilities Y, in physical resources z_1^o z_2^o (more generally in z^o) or in economic organization (income distribution, tax system, etc...). We will define it as a change of Y. Such a change can clearly come from invention as well as from diffusion of technological knowledge.

The changes of Y are considered as exogenous. Their complex and little known interrelations with changes of the other structural data are not investigated in spite of their importance.

5 Measurement

Our purpose is to associate one or a few numbers with the replacement of Y^o by Y^1 that the system undergoes. The exact definition of such a number results from the use to which it will be put. If problems of prediction or action, for example, are completely specified the optimal properties of this number are unambiguously known and its definition follows.

However, in the absence of specific problems of this kind a quantitative description may be sought. It is a natural idea and a general procedure to try to represent a situation too complex to be grasped at one glance by one or a few descriptive numbers. Intuitive considerations will, in this case, lead to the definition of the measure of the technological change which occurred. One of the most important requirements will be made explicit at the end of the next section.

6 Change of Technological Knowledge

The change of technological knowledge is represented in this model by the replacement of Y^0 by Y^1. The difficulties raised by the quantitative description of the change of technological knowledge are discussed in more detail in other papers (see in particular Carl Kaysen, Topic 1); the present mathematical formulation calls for a measure to be associated with those sets in the commodity space; our inability to suggest such a measure which would not be artificial supports the negative conclusion that they reached.

The change from Y^0 to Y^1 will therefore be measured in connection with other characteristics of the economic system. However the more of them we include in this measurement process the less representative the result is of the change to be measured. One of the most important of our requirements will be that these additional characteristics are as few as possible.

7 Production Potential

The constraints imposed on the system we have described in Section 2 are

1) that the combination of inputs and outputs represented by the point of coordinates y_1, y_2 belongs to the region of technological possibilities Y (Fig. 3).

2) that the utilized quantity of each commodity is at most equal to the utilizable quantity: $z_1 \leq z_1^o$, $z_2 \leq z_2^o$.

These constraints determine whether a certain standard of living for the system represented by a point of coordinates s_1, s_2 (Fig. 2) is attainable or not, in other words whether the first individual can achieve the standard of living s_1, while simultaneously the second one achieves the standard of living s_2. All the achievable points form a region S (Fig. 2) of the satisfaction plane. This region seems to be, in the most general sense, the production potential of the economic system. Let Y^o be the region of production possibilities before the technological change and S^o the corresponding production potential; Y^1, the region which replaces Y^o and which contains it; S^1, the corresponding production potential. Since Y^1 contains Y^o, S^1 contains S^o and we are looking for a quantitative description of the change of production potential represented by the replacement of S^o by S^1. A solution will be suggested at the end of the next section.

More generally we can say that the constraints

$$y \in Y^o, \quad z \leq s^o$$

determine the set S^o of attainable s. S^o is the production potential of the system. If Y^o is replaced by $Y^1 \supset Y^o$, S^o is replaced by $S^1 \supset S^o$.

8 Quantitative Description of the Effect of Technological Change

Let us choose two numerical values for the standards of living of the two consumers: s_1^o for the first, s_2^o for the second. We assume that the point (s_1^o, s_2^o) of the satisfaction plane belongs to the borderline $S^{o\,max}$ (Fig. 2) of the region S^o in such a way that there is no achievable point both of whose coordinates are at least as great as s_1^o, s_2^o, one of them being actually greater. The situation (s_1^o, s_2^o) is therefore optimal in the Pareto sense.

Let us consider an arbitrary point (z_1, z_2) of the commodity plane, and determine whether by utilizing those quantities of physical resources and any one of the production processes described by a point of the new region of technological possibilities Y^1 it is possible for the two consumers to attain standards of living at least equal to s_1^o and s_2^o. All the points (z_1, z_2) for which it is possible to do this form a region Z^1 of the commodity plane whose border is a curve Z^{1min}. (Fig. 4.)

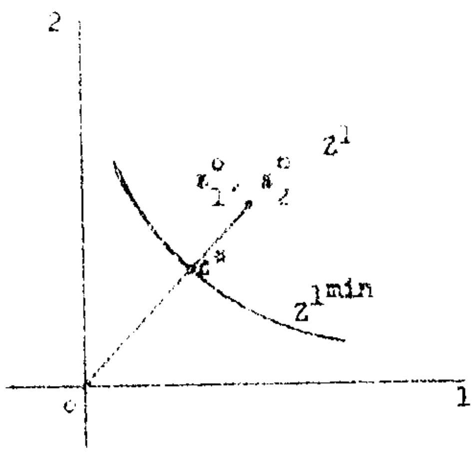

Fig. 4

Any point of Z^{1min} characterizes a complex of physical resources which allows one to achieve the standards of living s_1^o, s_2^o by using the new production possibilities Y^1 and such that it is impossible to decrease one of its coordinates without increasing at least another one.

Before the technological change the curve $Z^{o min}$ went through (z_1^o, z_2^o) (since the point (s_1^o, s_2^o) was on $S^{o max}$). The effect of the technological change can therefore be described on Fig. 4 by the relative positions of the curve Z^{1min} and the point (z_1^o, z_2^o) which is above. The problem of representation of this situation (whose complexity increases with the number of commodities) by a number is considered in more detail in the second half of this section; it leads to the introduction of the point z^* of Z^{1min} which results from (z_1^o, z_2^o) by a proportional reduction of both coordinates in a ratio which we call $1 - \rho$. The gain originating from the technological change (Y^o replaced by Y^1) in connection with the physical resources (z_1^o, z_2^o) and the standard of living (s_1^o, s_2^o) would thus be described by the maximum saving of physical resources which it is possible to achieve while still preserving the given standards of living.

This gain can alternatively be described by the ratio $1 - \rho$, the quantities of commodities $(1 - \rho)\, z_1^o$, $(1 - \rho)\, z_2^o$ or the value $(1 - \rho)(p_1 z_1^o + p_2 z_2^o)$.

where p_1, p_2 are the prices for the commodities.

These measures depend on the particular point (s_1^o, s_2^o) chosen on the curve $S^{o\max}$. If we allow the point (s_1^o, s_2^o) to vary on this curve, the maximum and the minimum of ρ seem to describe best the effect of the technological change on the production potential. These two values give bounds for the resource saving which can be expected; they depend only on the new technological possibility region Y^1, the preference orderings, and the actually available resources (z_1^o, z_2^o). They seem to satisfy best the requirement that we have set up at the end of 6.

<u>More generally</u>, let $s^o \in S^{o\max}$ and Z^1 be the set of z defined by

$$y \in Y^1 \qquad s \geq s^o.$$

To describe in a simple manner the relative positions of z^o and $Z^{1\min}$ we can try to define the distance from z^o to $Z^{1\min}$ and for this purpose use as weights the intrinsic price vectors p associated with the various points of $Z^{1\min}$. The distance mentioned would thus be $\underset{z \in Z^{1\min}}{\mathrm{Min}} \frac{p\cdot(z^o - z)}{p\cdot z}$ where z is an arbitrary point varying in $Z^{1\min}$, p one of the intrinsic price-vectors associated with z. This amounts to finding $\underset{z \in Z^{1\min}}{\mathrm{Max}} \frac{p\cdot z}{p\cdot z^o}$. Taking into account convexity properties of Z^1 it is easy to show that this is equal to ρ defined by

$$z^* = \rho\, z^o \quad , \qquad z^* \in Z^{1\min}$$

(z^* is collinear with z^o and belongs to $Z^{1\min}$).

A more complete explanation is given by the relations

$$\rho = \underset{p \in P'}{\mathrm{Max}}\ \underset{z \in Z^1}{\mathrm{Min}}\ \frac{p\cdot z}{p\cdot z^o} = \underset{z \in Z^1}{\mathrm{Min}}\ \underset{p \in P'}{\mathrm{Max}}\ \frac{p\cdot z}{p\cdot z^o}$$

where P' is the closed positive orthant, origin excluded (set of vectors $p \geq 0$).(5)

5 The detailed proofs of these properties are given in two Cowles Commission Discussion Papers: Economics 297 and Economics 2004.

The gain originating from the technological change is thus described by

$1 - \rho$, or $z^o(1 - \rho)$, or $p^o . z^o(1 - \rho)$.

$\underset{s^o \in S^{o max}}{\text{Max}} \rho$ and $\underset{s^o \in S^{o max}}{\text{Min}} \rho$ describe the effect of the technological change on production potential.

If $s^o \notin S^{o max}$, $z^o \notin Z^{o min}$, and the distance from z^o to $Z^{o min}$ can be described by a number ρ^o defined as above; after the technological change $Z^{o min}$ has been replaced by $Z^{1 min}$, ρ^o by ρ^1, and the effect of technological change is described by $\rho^o - \rho^1$, or $z^o(\rho^o - \rho^1)$, or $p^o . z^o(\rho^o - \rho^1)$.

9 Other Definitions

If s_1^1, s_2^1 (more generally z^1) are the available physical resources in the new structure (denoted by the superscript 1 in contrast with the old structure denoted by o), the gain attributable to technological change could alternatively be defined as the minimum proportional expansion of the new resources which permits the achievement of the new standards of living s_1^1, s_2^1 (more generally s^1) with the old technology Y^o.

A definition of this type allows for changes in the array of commodities, population, tastes and resources. It gives for the technological gain an expression directly comparable with the gain from a change of physical resources.

Among the definitions of the gain from technological change currently used or suggested we quote only two typical ones:

1) The variation of real national income which written in our notations would be

$$p_1(x_1^1 - x_1^o) + p_2(x_2^1 - x_2^o) \text{ [or } p.(x^1 - x^o)].$$

2) The variation of the ratio of the values of global output and global input (let us call y^- the vector whose components are corresponding components of y when they are negative, zero when they are positive, y^+ the vector defined in a symmetrical way; the ratio mentioned is $-\frac{p.y^-}{p.y^+}$).

Both of these measures fail to a large extent to meet the criterion that we gave at the end of Section 6: they are not only connected with given resources and standards of living but they reflect changes of the tax system and any other feature of the economic organization.

More specific criticisms could indeed be made: the variation of real national income accompanying a technological improvement, for example, could be negative (if there is no change in resources and if the equality $x + y = z^0$ holds, $p.(x^1 - x^0) = p.(y^0 - y^1)$. It is then obvious that "$y^0 \in Y^{0 min}$, $y^1 \in Y^{1 min}$, p is one normal to $Y^{0 min}$ at y^0" does not imply "$p.(y^0 - y^1) \geq 0$"). Even if this happens only in exceptional cases, it throws doubts on the reliability of this measure. The choice has therefore to be made between starting from a definition conceptually acceptable which does not lead to an immediate numerical evaluation but for which approximation processes can be devised and starting from a definition leading to an easy evaluation but conceptually unsatisfactory.

10 Technological Change in the Small

In this last section we assume that the conditions for the use of the tools of calculus are fulfilled. The formulas that we will obtain for infinitesimal variations of the variables can of course be looked upon as approximation formulas for small finite variations. This calculus study will serve other purposes: it will show how in the small 1) the gain attributable to technological change is known from technological data only, 2) the variation of standard of living can be imputed to the different factors of economic evolution. It will show also how the abstract concepts introduced in the preceding sections can be handled.

Let $e(y_1, y_2, \alpha) = 0$ be the equation of the curve $Y^{0 min}$, where α is a numerical parameter. Technological change corresponds to a variation of α, the problem is to find the corresponding variation of ρ.

The curve $Z^{o\min}$ is determined by the equations

(1) $$s_1(x_{11}, x_{21}) = s_1^o, \quad s_2(x_{12}, x_{22}) = s_2^o$$

(2) $$e(y_1, y_2 \alpha) = 0$$

(3) $$\frac{\frac{\partial s_1}{\partial x_{11}}}{p_1} = \frac{\frac{\partial s_1}{\partial x_{21}}}{p_2} \text{ equal by definition to } \sigma_1, \quad \frac{\frac{\partial s_2}{\partial x_{12}}}{p_1} = \frac{\frac{\partial s_2}{\partial x_{22}}}{p_2} \text{ equal by}$$

definition to σ_2

(4) $$\frac{\frac{\partial e}{\partial y_1}}{p_1} = \frac{\frac{\partial e}{\partial y_2}}{p_2} \text{ equal by definition to } \epsilon,$$

where (3) and (4) express the existence of an intrinsic price system p_1, p_2 at every point of $Z^{o\min}$ with respect to which consumption-units behave as if they maximized their satisfactions, and production-units as if they maximized their profits.(6)

6 See Cowles Commission Discussion Paper, Economics No. 297.

To determine ρ we have the additional relations

(5) $$x_{11} + x_{12} + y_1 = \rho z_1^o, \quad x_{21} + x_{22} + y_2 = \rho z_2^o.$$

We first take the differentials of the relations (1), (2), (5).

(1') $$\frac{\partial s_1}{\partial x_{11}} dx_{11} + \frac{\partial s_1}{\partial x_{21}} dx_{21} = 0 \qquad \frac{\partial s_2}{\partial x_{12}} dx_{12} + \frac{\partial s_2}{\partial x_{22}} dx_{22} = 0$$

(2') $$\frac{\partial e}{\partial y_1} dy_1 + \frac{\partial e}{\partial y_2} dy_2 + \frac{\partial e}{\partial \alpha} d\alpha = 0$$

(5') $$dx_{11} + dx_{12} + dy_1 = d\rho\, z_1^o \qquad dx_{21} + dx_{22} + dy_2 = d\rho\, z_2^o.$$

Using (3) and (4), one can transform (1') and (2') respectively into

(6) $$p_1\, dx_{11} + p_2\, dx_{21} = 0, \qquad p_1\, dx_{12} + p_2\, dx_{22} = 0$$
$$p_1\, dy_1 + p_2\, dy_2 = -\frac{1}{\epsilon} \frac{\partial e}{\partial \alpha} d\alpha.$$

Multiplying the two equations (5') respectively by p_1 and p_2 and adding them one finds on the left side the same result as by adding the three equations (6), therefore the two right-hand members are equal.

$$d\rho\,(p_1 z_1^o + p_2 z_2^o) = -\frac{1}{\varepsilon}\,\frac{\partial e}{\partial \alpha}\,d\alpha .$$

Such is the expression of the gain attributable to the technological change represented by $d\alpha$. The right-hand member is known from technological data only: if we interpret for a moment in the third equation (6) our differentials as (small) finite variations, the value of the first member is known as soon as one knows <u>any</u> point of the new curve $Y^{1\,min}$, say $\bar{y}_1^1$, $\bar{y}_2^1$. One can then compute the differences $\bar{y}_1^1 - y_1^o$ and $\bar{y}_2^1 - y_2^o$ and multiply them by p_1, p_2. It is by no means necessary to predict the point (y_1^1, y_2^1) which will be actually observed to give an estimation of the gain.

If we take up the more general case where the standard of living, the technological possibilities, the degree of efficiency ρ of the economic organization, and the physical resources all vary, the differentials of (1), (2), (5) take the form

$$\frac{\partial s_1}{\partial x_{11}}\,dx_{11} + \frac{\partial s_2}{\partial x_{21}}\,dx_{21} = ds_1^o, \quad \frac{\partial s_2}{\partial x_{12}}\,dx_{12} + \frac{\partial s_2}{\partial x_{22}}\,dx_{22} = ds_2^o$$

$$\frac{\partial e}{\partial y_1}\,dy_1 + \frac{\partial e}{\partial y_2}\,dy_2 + \frac{\partial e}{\partial \alpha}\,d\alpha = 0$$

$$dx_{11} + dx_{12} + dy_1 = d\rho\, z_1^o + \rho\, d z_1^o; \quad dx_{21} + dx_{22} + dy_2 = d\rho\, z_2^o + \rho\, d z_2^o.$$

The same transformation as before gives

$$\left(\frac{ds_1^o}{\sigma_1} + \frac{ds_2^o}{\sigma_2}\right) - \frac{1}{\varepsilon}\,\frac{\partial e}{\partial \alpha} = d\rho\,(p_1 z_1^o + p_2 z_2^o) + \rho\,(p_1\, d z_1^o + p_2\, d z_2^o)$$

which makes a synthesis of the four factors of economic evolution quoted.

<u>More generally</u> let $e_j(y_j, \alpha_j) = 0$ be the equation of the surface Y_j^{min}; ρ is determined by the relations

I $\quad s_i(x_i) = s_i^o$ II $\quad e_j(y_j, \alpha_j) = 0$

III $\quad \frac{\partial s_i}{\partial x_i} = \sigma_i p$ IV $\quad \frac{\partial e_j}{\partial y_j} = \mathcal{E}_j p$

V $\quad \sum_i x_i + \sum_j y_j = \rho z^o$

where III and IV express the existence of an intrinsic price-vector p at every point of $Z^{o\,min}$; σ_i, $\mathcal{E}_j$ are numerical coefficients expressing that the vectors $\frac{\partial s_i}{\partial x_i}$ and p on one hand, $\frac{\partial e_j}{\partial y_j}$ and p on the other are collinear.

A variation of the α_j induces a variation of ρ whose differential can be readily computed:

$$\frac{\partial s_i}{\partial x_i} dx_i = 0 \qquad \frac{\partial e_j}{\partial y_j} dy_j + \frac{\partial e_j}{\partial \alpha_j} d\alpha_j = 0$$

$$\sum_i dx_i + \sum_j dy_j = d\rho z^o.$$

Using III and IV one finds

$$p \cdot dx_i = 0 \qquad p \cdot dy_j = -\frac{1}{\mathcal{E}_j} \frac{\partial e_j}{\partial \alpha_j} d\alpha_j$$

and therefore

$$p \cdot z^o \, d\rho = -\sum_j \frac{1}{\mathcal{E}_j} \frac{\partial e_j}{\partial \alpha_j} d\alpha_j.$$

In this relation p, $\mathcal{E}_j$, $\frac{\partial e_j}{\partial \alpha_j}$ refer to the <u>virtual</u> economic situation associated with the point z^* (see 8) and not the actual observed situation. Each term of the right-hand sum depends only on technological data in the j-th production-unit.

Similarly if s^o, the α_j, z^o and ρ vary their changes are related implicitly by

$$\frac{\partial s_i}{\partial x_i} dx_i = ds_i^o, \quad \frac{\partial e_j}{\partial y_j} dy_j + \frac{\partial e_j}{\partial \alpha_j} d\alpha_j = 0$$

$$\sum_i dx_i + \sum_j dy_j = d\rho z^o + \rho \, dz^o$$

i.e., by

$$p \cdot dx_i = \frac{ds_i^o}{\sigma_i} \qquad p \cdot dy_j = \frac{1}{\mathcal{E}_j} \frac{\partial e_j}{\partial \alpha_j} d\alpha_j$$

$$\sum_i p \cdot dx_i + \sum_j p \cdot dy_j = d\rho \; p \cdot z^o + \rho \; p \cdot dz^o .$$

The explicit relation between the four factors of economic evolution is therefore

$$\sum_i \frac{de_i^o}{\sigma_i} - \sum_j \frac{1}{\varepsilon_j} \frac{\partial e_j}{\partial \alpha_j} d\alpha_j = d\rho \; p \cdot z^o + \rho \; p \cdot dz^o$$

where σ_i, ε_j, $\frac{\partial e_j}{\partial \alpha_j}$, p refer to the virtual situation associated with z^*.