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Note on a Proposition of Welfare Economics

Optimum Amount of Capital in a Stationary Economy

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1 - Introduction.

So far welfare economics has been mainly concerned with a noncapitalistic economy. The introduction of several time periods in the model is usually conceived as equivalent to a multiplication of the number of goods taken into consideration. But this procedure is insufficient at least in two respects; namely, it does not give any answer to the following questions:

1 - Does it make any sense to try to sacrifice present satisfaction for better future satisfaction; and, if any, what could be the best policy in this respect?

2 - Is there any optimum amount of capital equipment; and, if any, what is the best way to reach this optimum?

The purpose of this note is to contribute to the answer of the second question. In order to be free of any difficulty arising from the possibility of improving the present welfare by the consumption of capital, the economy is assumed to be in a stationary state, i.e., all the variables take the same values at each time period; in particular at the end of any year the capital equipment should be the same as at the beginning.

Intuitively the problem may be seen as follows: The net national product is equal to the difference between the gross national product and the necessary replacement of capital. It seems to be worth increasing the capital equipment as long as the gross marginal product of capital is greater than the necessary marginal replacement; i.e., as long as the net marginal product of capital is positive. This has been asserted by Professor J. E. Meade [3]. Dealing with the same problem Professor M. Allais [1] reached the following conclusion: "Among all the stationary states, an optimum is characterized by a zero interest rate in the production sector. In the following pages a new and more general demonstration of the same proposition is produced, using the mathematical technique which was introduced by Debreu [2] in his study on the coefficient of resource-utilization.

In section 3, the economic set-up is defined; it consists essentially of three kinds of economic units: consumption-units, production-units and investment-units each one of them being characterized by a different function. The essential part is section 4 in which the following rather trivial proposition is proved: The economic resources at the disposal of the society will be maximized if the investment-units do not discount anything for the use of capital goods above what is necessary for their replacement (in that case the economic state will be said z-optimal). The rest of this note is concerned with the implications of this result for the consumption-units and the formulation of an optimum theorem using the definition of optimum common in welfare economics (s-optimum). This is effected in the three following steps:

1 - In a stationary state, a given level of satisfaction s is associated with a set $\mathcal{K}(s)$ which must contain the annual consumption vector.

2 - In order to prove that a stationary state is s-optimal if and only if it is z-optimal and satisfies some other conditions, it is necessary to introduce two rather mild continuity axioms.

3 - The main theorem is finally stated in section 7. Section 8 is devoted to a tentative picture of an economy working under the conditions outlined in this theorem.

2 - Notation.

The notation has been chosen so as to agree with that used by Debreu [2]

Vectors are denoted by ordinary letters, scalars by Greek letters and sets by thick capitals.

$u \lneq v$ means that no component of u is greater, and at least one is smaller, than the corresponding component of v.

v is said to be a maximal element of $\mathbb{U}$ if $v \in \mathbb{U}$ and there is no $u \in \mathbb{U}$ such that $v \lneq u$.

$\mathbb{R}_n$ is the n-dimensional Euclidean space. There are in the system ℓ commodities ($\mathbb{R}_\ell$), m consumption-units and h time periods.

Sets in $\mathbb{R}_{2\ell}$ are characterized by a star ($\mathbb{Z}^* \subset \mathbb{R}_{2\ell}$).

Vectors and sets in $\mathbb{R}_{\ell h}$ are characterized by a prime ($x' \in \mathbb{X}' \subset \mathbb{R}_{\ell h}$).

$\mathbb{V} = \mathbb{U}_1 + \mathbb{U}_2$ means: any $v \in \mathbb{V}$ may be written as $v = u_1 + u_2$ where $u_1 \in \mathbb{U}_1$ and $u_2 \in \mathbb{U}_2$.

$v = \pi_{\mathbb{R}_\ell} u$ means that v is the usual Cartesian projection of u on the space $\mathbb{R}_\ell$ of its ℓ first coordinates.

3 - Economic set-up.

The time is divided into periods of equal length.

Any commodities which are left over from a preceding time period are said to be "capital goods" of the present time period. The capital is characterized by a vector c in the commodity-space $\mathbb{R}_\ell$.

There are three kinds of economic units:

(1) - consumption-unit, characterized by a subscript i(i = 1, 2,...,m), whose activity is represented by a consumption-vector $x_i' \in \mathbb{R}_{\ell h}$. To each of them is associated a numerical function $s_i : \mathbb{R}_{\ell h} \to \mathbb{R}_1$ such that: $s_i(x_i'^{0}) < s_i(x_i'^{1})$ if $x_i'^{1}$ is preferred to $x_i'^{0}$.

(2) - <u>production-unit</u>, characterized by a subscript $j (j = 1, 2, \ldots, n)$ whose activity is represented by an input-vector y_j. There is a set $\mathbb{Y}_j \subset \mathbb{R}_\ell$, dependent on the technical knowledge, such that $y_j \in \mathbb{Y}_j$ and any vector in $\mathbb{Y}_j$ is a possible input-vector for the j-th production-unit.

(3) - <u>Investment-unit</u>, characterized by a subscript $K (K = 1, 2, \ldots, q)$ whose activity is represented by a resource-capital vector (z_K, c_K). There is a set $\mathbb{Z}_K^* \subset \mathbb{R}_{2\ell}$ such that $(z_K, c_K) \in \mathbb{Z}_K^*$ and any vector in $\mathbb{Z}_K^*$ is a possible resource-capital vector for the K-th investment-unit.

The economic motivations behind the definition of consumption-units and production-units have been given by Debreu [2] and will not be reproduced here. The introduction of the investment-units results from the analysis of the factors which determine the amount of resources at the disposal of the society in each time period.

Suppose given the capital vectors at the beginning, $c(t)$, and at the end, $c(t + 1)$, of the period. The resource-vector z may be written as:

$$z = z_I + z_{II} - z_{III}$$

where z_I represents the permanent resources

z_{II} the services of capital $c(t)$

z_{III} the gross investment which is necessary if, given $c(t)$ and z_{II}, the capital vector is to be $c(t + 1)$ at the end of the period.

In short, z is constrained to a set $\mathbb{Z}$ in $\mathbb{R}_\ell$. This set depends on $c(t)$ and $c(t + 1)$. In particular if the economic state is stationary, $c(t) = c(t + 1)$, and the possible vectors (z, c) form a set $\mathbb{Z}^*$ of $\mathbb{R}_{2\ell}$. Clearly $\mathbb{Z}^*$ depends on the natural wealth (permanent resources) and on the technical knowledge.

The investment units correspond to a decomposition of $\mathbb{Z}^*$ into sets $\mathbb{Z}_K^*$ such that

$$\sum_{K=1}^{q} \mathbb{Z}_K^* = \mathbb{Z}^* .$$

The resource vector is a result of the activity of the investment-units but it is given as far as the activity of the consumption and production-units is concerned. Put:

$$x' = \sum_{i=1}^{m} x'_i \qquad y = \sum_{j=1}^{n} y_j \qquad z = \sum_{K=1}^{q} z_K \qquad c = \sum_{K=1}^{q} c_K$$

$$\mathbb{Y} = \sum_{j=1}^{n} \mathbb{Y}_j \qquad \mathbb{Z}^* = \sum_{K=1}^{q} \mathbb{Z}^*_K \qquad x = \pi_{R_\ell} x' .$$

Between x, y and z there holds:

$$x + y \leqq z$$

which means that the total net consumption of the whole economy can come only from the available physical resources.

Let us recall that an economic state is said to be stationary if all the variables involved take the same values at each time period (except the time). In this note stationary states only are considered; this will often be taken for granted in the following.

Let us adopt the following axiom:

<u>Axiom 1.</u> All the $\mathbb{Z}^*_K$ are convex and closed, $\mathbb{Z}^*$ is closed.

The economic set-up is represented in Figure 1.

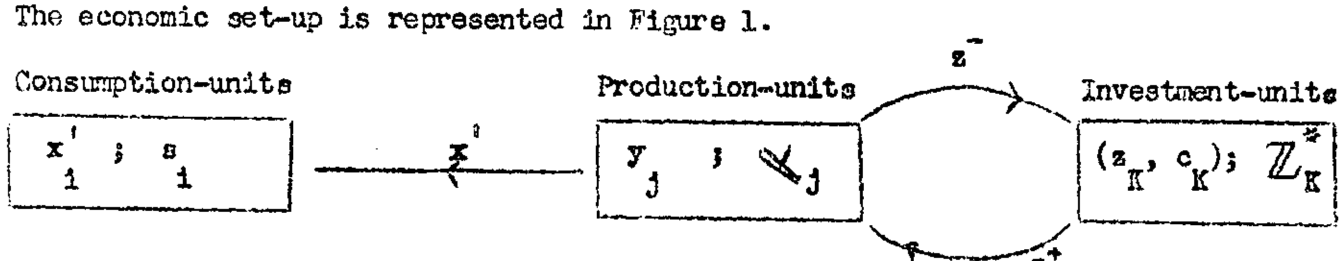

Fig. 1.

Although the present economic set-up could be subject to a large variety of interpretations, let us tentatively suggest the following: Consider a non-monetary socialist economy in which each consumer is a consumption-unit, each firm a production-unit. The investment units are governmental agencies which are the only ones to have the right to hold capital goods. The production-units rent the capital goods from the investment-units and must sell all their annual production either to the consumers or to the governmental agencies.

The assumption that the sets $\mathbb{Z}_K^*$ are closed is obviously not restrictive as far as economic meaning is concerned. It is not clear whether or not the convexity axiom is very restrictive. In order to understand its meaning it can be considered that for a given capital vector, c_K, z_K is the difference between the gross annual revenue (which could generally be subject to the law of decreasing return) and the necessary annual replacement of capital (which is strongly related to the length of the period of production).

4 - The z-optimum theorem.

In this section the investment-units only are considered. Theorems 1.a and 1.b give the condition under which z is maximal.

Theorem 1.a. A necessary and sufficient condition for z^o to be maximal is the existence of a price vector $p \geq 0$ in $\mathbb{R}_\ell$ such that, (z_K, c_K) being constrained by $(z_K, c_K) \in \mathbb{Z}_K^*$, $p\, z_K$ reaches its maximum at z_K^o, for all K.

Theorem 1.b. A necessary and sufficient condition for (z^1, c^1) to be maximal among all the possible (z, c^1) is the existence of a price vector (p, r) in $\mathbb{R}_{2\ell}$, such that $p \geq 0$ and (z_K, c_K) being constrained $(z_K, c_K) \in \mathbb{Z}_K^*$, $p\, z_K + r\, c_K$ reaches its maximum at (z_K^1, c_K^1).

Figure 2 may be helpful for the reader who wants to understand the meaning of these theorems.

Theorem 1.b is not our main concern here but it is likely to clarify somewhat the meaning of Theorem 1.a. Nevertheless it is not necessary to give its proof which can easily be stated along the same lines as the following:

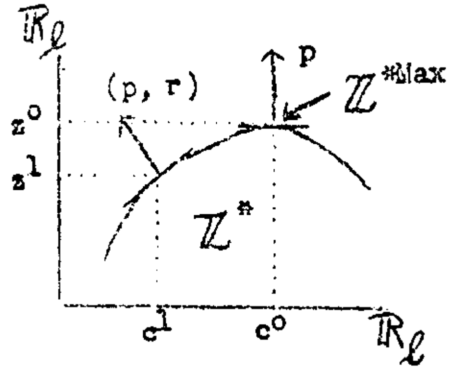

Fig. 2.

Proof of Theorem 1.a.

Define in $\mathbb{Z}^*$ a subset $\mathbb{Z}^{*Max}$ such that: if $(z^o, c^o) \in \mathbb{Z}^{*Max}$ there is no $(z, c) \in \mathbb{Z}^*$ with $z \geq z^o$.

The following propositions are equivalent:

(1) $(z^o, c^o) \in \mathbb{Z}^{*Max}$

(2) $z \geq z^o$ implies $(z, c) \notin \mathbb{Z}^*$ for all c.

(3) $\mathbb{P} \cap \mathbb{Z}^* = 0$ where $\mathbb{P} = \left\{(z, c) / z \geq z^o\right\}$

(4) There is a plane Π separating $\mathbb{P}$ and $\mathbb{Z}^*$ and containing

$$\Pi^o = \left\{(z^o, c) / c \in \mathbb{R}_\ell\right\}$$

(5) There is a vector $p \in \mathbb{R}_\ell$ ((p, 0) normal to Π) $p \geq 0$ such that:

$$(z, c) \in \mathbb{Z}^* \text{ implies } p(z - z^o) \leqq 0$$

(6) There is a positive vector $p \in \mathbb{R}_\ell$ such that:

$$(z_K, c_K) \in \mathbb{Z}^*_K \text{ implies } p(z_K - z^o_K) \leqq 0 \text{ for all K.}$$

(7) There is a positive vector $p \in \mathbb{R}_\ell$ such that, (z_K, c_K) being constrained by $(z_K, c_K) \in \mathbb{Z}^*_K$, $p\, z_K$ reaches its maximum at z^o_K, for all K.

Proof of (3) $\Longrightarrow$ (4). The existence of a separating plane results from the fact that $\mathbb{Z}^*$ and $\mathbb{P}$ are convex and disjoint. Π contains Π^o because if there were some $(z^o, c^1) \notin \Pi$ there would be a sphere around (z^o, c^1) not intersecting Π (closed) and therefore some (z^1, c^1) and (z^2, c^2) on the same side of Π with $z^1 \geq z^o$ and $z^o \geq z^2$; z^1 would be in $\mathbb{P}$ and z^2 in $\mathbb{Z}^*$, contradicting the hypothesis that Π is a separating plane.

Proof of (5) $\Longrightarrow$ (6). Suppose that for some K^o, $p(z_{K^o} - z^o_{K^o}) > 0$ with $(z_{K^o}, c_{K^o}) \in \mathbb{Z}^*_{K^o}$. It would be possible to take $(z, c) \in \mathbb{Z}^*$ such that $p(z - z^o) > 0$, puting $z = z^o$ for all $K \neq K^o$.

Here it should be noticed that Theorem 1 does not say anything about the existence of the maximal vector z^o. It is theoretically possible that such a vector does not exist at all. But this evenutality may be rejected on economic grounds: for a given technological knowledge the amount of capital equipment could not be expanded continuously without reaching sooner or later an optimum beyond which any new increase would be a loss rather than a gain.

The economic interpretation of Theorem 1.a is clear. The planning authority ought to choose a set of prices and the investment-units to maximize the revenue they get from the lease of capital; doing this they should not discount anything for the total amount of capital they are holding.

According to Theorem 1.b, on the contrary, a discount of the value of each capital good should be done on the basis of some interest rate ($-r_j$). A different set of interest rates would induce a different amount of capital equipment, but in any case the resource-vector would be **maximal** for <u>some</u> amount of capital. The fact that there does not seem to be a constant relation between the price p_j and the interest rate $-r_j$ relative to the same commodity should not surprise us because the capital vector c corresponding to (p, r) may be noneconomic not only in its absolute value but also in its structure. This will not be explored here.

5 - Consumer choices in a stationary state.

In Section 3, the activity of the consumption-units has been defined by a vector x'_i in $\mathbb{R}_{\ell h}$; i.e., by the consumption during the present and h-1 future time periods. This may seem an unnecessary complication. In a stationary state the consumption during a future time period should be equal to the present consumption. Actually the confusion arises from the fact that consumption-units and consumers do not play the same role in the model.

Indeed, one part of our problem is to see if the result of the preceding section agrees with the time preference of the consumers. If, in the model, the satisfaction of the consumer had been made only a function of his present consumption, or if his consumption had been assumed constant over time, we would have simply been begging the question.

Hence the following device is introduced: In a given time period each consumer is identified with a consumption-unit, but it is not necessary that this identification remains the same over time; i.e., consumer i_1 may be identified with consumption-unit i_α in one time period and with consumption-unit i_β in another time period. The activity of the consumption-unit is characterized by the vector x_i' which defines the present and future consumption of the consumer who is identified with the i-th consumption-unit in the present time period.

We will now show that, in order to get a given level of satisfaction $s \in \mathbb{R}_m$, x should be in a set $\mathbb{Z}(s) \in \mathbb{R}_\ell$.

s_i has been defined as a function:

$$s_i : \mathbb{R}_{\ell h} \longrightarrow \mathbb{R}_1$$

The inverse image of a point s_i^o in $\mathbb{R}_1$ is a set $\mathcal{X}_i'(s_i^o) \subset \mathbb{R}_{\ell h}$ such that: $s_i(x_i') = s_i^o$ implies $x_i' \in \mathcal{X}_i'(s_i^o)$.

In the same way, to a given $s \in \mathbb{R}_m$ corresponds a set $\mathcal{X}'(s) \subset \mathbb{R}_{\ell h}$ such that $\mathcal{X}'(s) = \sum_{i=1}^{m} \mathcal{X}_i'(s_i)$. And necessarily: $x' \in \mathcal{X}'(s)$.

As the economy is in a stationary state the same total quantities of each commodity are consumed in each time period; i.e., the vector x' should have its ℓ first coordinates equal to the ℓ coordinates between the $(\lambda \ell + 1)$th and the $[(\lambda + 1)\ell]$th (where $1 \leq \lambda < h$). Let us call $\mathbb{H}'$ the set of the vectors $x' \in \mathbb{R}_{\ell h}$ which have this property.

$$x' \in \mathbb{H}' \text{ and } x' \in \mathcal{X}'(s) \text{ imply:}$$

$$x' \in \mathbb{H}' \cap \mathcal{X}'(s)$$

and

$$x = \Pi_{\mathbb{R}_\ell} x' \in \Pi_{\mathbb{R}_\ell} [\mathbb{H}' \cap \mathcal{X}'(s)] = \mathcal{X}(s).$$

Let us notice that $x \in \mathcal{X}(s)$ would have been an immediate result if we had assumed $h = 1$. The assumption of stationary state makes possible the treatment of the more general case along the same lines as in this simple case.

In Section 3 it has been stated:

$$x + y \leqq z \qquad \text{and} \qquad y \in \mathbb{Y}$$

$x \in \mathbb{X}(s)$ implies therefore $z \in \mathbb{Z}(s)$ where $\mathbb{Z}(s)$ is obtained from $\mathbb{Y} + \mathbb{X}(s)$ by adjunction of all $z \geq z^o \in \mathbb{Y} + \mathbb{X}(s)$.

6 - Two continuity axioms.

In Section 4 it has been shown that the economic state is z-optimal if (z, c) is contained in some $\mathbb{Z}^{*Max}$ or z contained in some $\mathbb{Z}^{Max} = \Pi_{\mathbb{R}_\ell} \mathbb{Z}^{*Max}$; in Section 5 it has been shown that in order to get a given satisfaction s, z should be contained in some $\mathbb{Z}(s)$. Our purpose is now to prove that the economic state is s-optimal if and only is the configuration of Fig. 3 is realized. But in order to do so we need to introduce two continuity axioms.

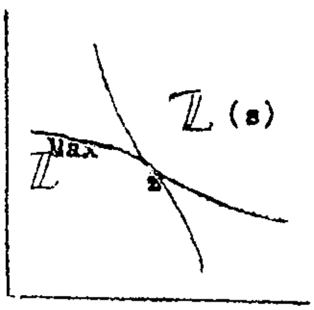

Fig. 3

Before doing so let us define the space $\mathcal{U}$ of all closed and convex sets $\mathbb{A}$ in $\mathbb{R}_\ell$ such that Vol $\mathbb{A} \neq 0$.

Define in $\mathcal{U}$ the following metric:

$$\sigma(\mathbb{A}_1, \mathbb{A}_2) = \text{Vol}(\mathbb{A}_1 \triangle \mathbb{A}_2) = \text{Vol}[(\mathbb{A}_1 \cup \mathbb{A}_2) - (\mathbb{A}_1 \cap \mathbb{A}_2)].$$

And consider the function: $\varphi : \mathbb{R}_m \to \mathcal{U}$ defined by:

$$\varphi(s) = \mathbb{Z}(s).$$

The fact that $\mathbb{Z}(s)$ is closed and convex results from the axioms formulated by Debreu [2] which are taken as granted here. The fact that Vol $\mathbb{Z}(s) \neq 0$ results from its definition.

We formulate now the two continuity axioms:

Axiom 2 - The functions $s_i : \mathbb{R}_{\ell h} \to \mathbb{R}_1$ are continuous.

Axiom 3 - The function $\varphi : \mathbb{R}_m \to \mathcal{U}$ is continuous.

Axiom 2 is the traduction of the fact that when x_i changes a little, $s_i(x_i)$ changes just a little.

Axiom 3 goes in the opposite direction, it could be deduced from the continuity of s_i^{-1}, but this raises quite a few difficulties and the meaning of the axiom is clear enough by itself.

Axioms 2 and 3 make possible the proof of the two following lemmas which are a first step in the proof of the s-optimum theorem:

<u>Lemma 1.</u> If $z^o \in \mathcal{Z}(s^o)$ and $z^o \notin \mathcal{Z}^{Min}(s^o)$; there is some $s \geq s^o$ such that $z^o \in \mathcal{Z}(s)$; where $\mathcal{Z}^{Min}(s)$ is the boundary of $\mathcal{Z}(s)$. (Cf. Fig. 4.)

<u>Proof.</u> Let $\rho(s, s^o)$ be the ordinary distance function in $\mathcal{R}_m$ and $\mathcal{S}_\varepsilon(z^o)$ be the sphere of radius $\varepsilon > 0$ around z^o in $\mathcal{R}_\ell$.

As $\mathcal{Z}(s^o)$ is closed, there is some $\varepsilon > 0$ such that:

$$\mathcal{S}_\varepsilon(s^o) \subset \mathcal{Z}(s^o).$$

As φ is continuous, there is some $\delta > 0$ such that:

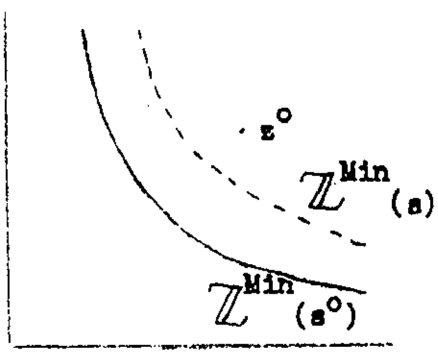

Fig. 4

$\rho(s, s^o) < \delta$ implies

$\sigma[\varphi(s), \varphi(s^o)] < \frac{1}{2} \operatorname{Vol}[\mathcal{S}_\varepsilon(z^o)]$.

But $\rho(s, s^o) < \delta$ implies $z^o \in \mathcal{Z}(s)$ because $z^o \notin \mathcal{Z}(s)$ would imply:

$\operatorname{Vol}[\mathcal{Z}(s) \triangle \mathcal{Z}(s^o)] > \frac{1}{2} \operatorname{Vol}[\mathcal{S}_\varepsilon(z^o)]$

(convexity).

There is obviously some $s \geq s^o$ such that $\rho(s, s^o) < \delta$, which completes the proof.

<u>Lemma 2.</u> $s^1 \geq s^o$ implies $\mathcal{Z}^{Min}(s^o) \cap \mathcal{Z}(s^1) = 0$ (if $s^1 \geq s^o$ Figure 5 is impossible).

<u>Proof.</u> We want to prove that $z^1 \in \mathcal{Z}(s^1)$ implies that there is some $z \leq z^1$ with $z \in \mathcal{Z}(s^o)$.

$s^1 \geq s^o$ implies $s^1_i > s^o_i$ for some i. Let us write $s^1_i - s^o_i = \varepsilon_i > 0$.

It is enough to prove that there is some $x'_i \leq x'^1_i$ with $x'_i \in \mathcal{X}'_i(s^o)$ and $(x'^1_i - x'_i) \in \mathcal{H}'$.

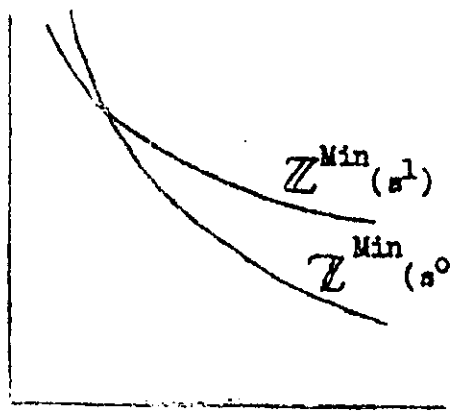

Fig. 5

The continuity of $s_i : \mathbb{R}_{\ell h} \longrightarrow \mathbb{R}$, implies that there is some $\delta > 0$ such that $x_i' \in \mathcal{S}_\delta' (x_i'^1)$ implies $|s_i(x_i') - s_i^1| < \varepsilon_i$ and therefore $s_i(x_i') > s_i^o$ and $x_i' \in \mathcal{X}_i'(s_i^o)$. But there is obviously some x_i' in $\mathcal{S}_\delta (x_i'^1)$ such that $x_i' \leq x_i'^1$, and $x_i'^1 - x_i' \in \mathbb{H}'$, which completes the proof.

7 - The s-optimum theorem.

We are now able to prove in a precise way the result announced at the beginning of the preceding section. This is the object of the next lemma.

Let us first define:

$$\mathbb{Z}_o(s) = \mathbb{Z}(s) \cap \Pi_{\mathbb{R}_\ell} \mathbb{Z}^*$$
$$\mathbb{Z}^M(s) = \mathbb{Z}^{Min}(s) \cap \mathbb{Z}^{Max}.$$

Lemma 3. The necessary and sufficient condition for a stationary state characterized by s^o to be s-optimal among all the possible stationary states is that:

$$\mathbb{Z}_o(s^o) = \mathbb{Z}^M(s^o)$$

Proof. 1) The condition is necessary:

a) Suppose that there is some $z^o \in \mathbb{Z}_o(s^o)$ such that $z^o \notin \mathbb{Z}^{Min}(s^o)$ (cf. Fig. 6). By Lemma 1 there would be some $s \geq s^o$ such that $s^o \in \mathbb{Z}(s)$, contradicting the hypothesis that s^o is optimal.

b) Suppose that there is some $z^o \in \mathbb{Z}_o(s^o)$ such that $z^o \notin \mathbb{Z}^{Max}$ (cf. Fig. 7). There is some $z \geq z^o \in \mathbb{Z}(s^o)$ with $z \in \mathbb{Z}^{Max}$. This implies $z \in \mathbb{Z}(s^o)$ and $z \notin \mathbb{Z}^{Min}(s^o)$, contradicting by Lemma 1 the hypothesis that s^o is optimal.

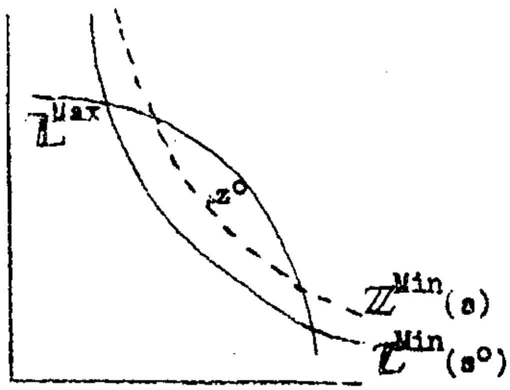

Fig. 6

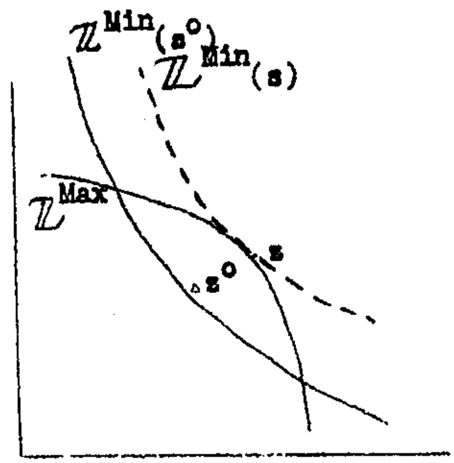

Fig. 7

2) The condition is sufficient (Fig. 8 is impossible with $s^1 \geq s^0$).

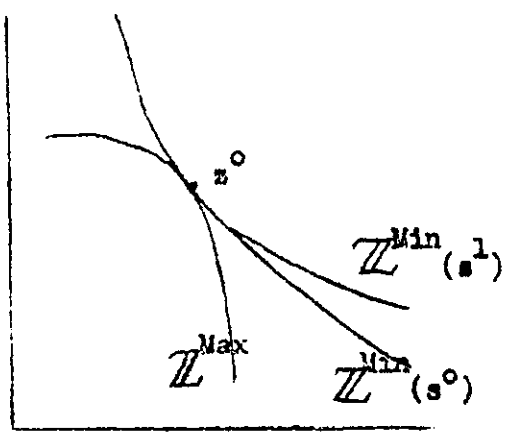

Fig. 8

Suppose $s^1 \geq s^0$ which obviously implies $\mathbb{Z}(s^1) \subset \mathbb{Z}(s^0)$ $z \in \mathbb{Z}_0(s^1)$ would imply $z \in \mathbb{Z}_0(s^0)$ and $z \in \mathbb{Z}^{Min}(s^0)$ which is impossible by Lemma 2. Hence $\mathbb{Z}_0(s^1)$ is empty and there is no possible vector z corresponding to the vector s^1.

Lemma 3 has the following consequence: If s^0 is maximal, $\mathbb{Z}_0(s^0)$ has at most $\ell-1$ dimensions and is convex. Hence it is contained in a support plane to both $\mathbb{Z}(s^0)$ and $\Pi_{\mathbb{R}_\ell} \mathbb{Z}^*$. The requirements implied by the fact that z^0 is in a support plane to $\Pi_{\mathbb{R}_\ell} \mathbb{Z}^*$ have been given by Theorem 1. Let us now see what is the meaning of a support plane to $\mathbb{Z}(s^0)$.

The vector $p \geq 0$ normal to the support plane must be normal to a support plane of $\mathbb{Y}$, and to a support plane of $\mathbb{X}(s^0)$ (cf. Debreu [2]). By the definition of $\mathbb{X}(s^0)$, there is a vector p' normal to $\mathbb{X}'(s^0)$ whose projection p'' on $\mathbb{H}'$ is such that $\Pi_{\mathbb{R}_\ell} p''$ is colinear to p.

Let p'_{st} be the coordinate of p' corresponding to the s-th commodity and to the time period t; and p_s be the coordinate of p corresponding to the s-th commodity. The following relation holds:

$$p_s = \frac{1}{h} \sum_{t=1}^{h} p'_{st}.$$

The vector p', normal to $\mathcal{X}'(s^o)$ must be normal to each $\mathcal{X}'_i(s^o_i)$ (cf. Debreu [2]). This leads to the statement of Theorem 2:

<u>Theorem 2.a.</u> A necessary and sufficient condition for a stationary state characterized by $(x'^o_i, y^o_j, z^o_K, c^o_K)$ to be s-optimal among all the possible stationary states is the existence of two price vectors $p \in \mathbb{R}_\ell$, $p' \in \mathbb{R}_{\ell h}$ and of a set of numbers $\alpha_i (i = 1, 2, \dots, m)$ such that:

i) for each commodity s, $p_s = \frac{1}{h} \sum_{t=1}^{h} p'_{st} \geq 0$

ii) x'_i being constrained by $p' x'_i \leq \alpha_i$, $s_i(x'_i)$ reaches it maximum at x'^o_i, for every i.

iii) y_j being constrained by $y_j \in \mathcal{Y}_j$, $p\, y_j$ reaches it minimum at y^o_j, for every j.

iv) (z_K, c_K) being constrained by $(z_K, c_K) \in \mathcal{Z}^*_K$, $p\, z_K$ reaches its maximum at z^o_K, for every K.

In the same way, the following theorem can be proved:

<u>Theorem 2.b.</u> A necessary and sufficient condition for a stationary state characterized by $(x'^1_i, y^1_j, z^1_K, c^1_K)$ to be s-optimal among all the stationary states characterized by the same structure of capital equipment c^1, is the existence of two price vectors $(p, r) \in \mathbb{R}_{2\ell}$, $p' \in \mathbb{R}_{\ell h}$ and a set of numbers $\alpha_i (i = 1, 2, \dots, m)$ such that:

i) for each commodity s, $p_s = \frac{1}{h} \sum_{t=1}^{h} p'_{st} \geq 0$

ii) x'_i being constrained by $p' x'_i \leq \alpha_i$, $s_i(x'_i)$ reaches its maximum at x'^1_i, for every i.

iii) y_j being constrained by $y_j \in \mathcal{Y}_j$, $p\, y_j$ reaches its minimum at y^1_j, for every j.

iv) (z_K, c_K) being constrained by $(z_K, c_K) \in \mathcal{Z}^*_K$, $p\, z_K + r\, c_K$ reaches its maximum at (z^1_K, c^1_K), for every K.

8 - Economic interpretation of the s-optimum theorem.

Although Theorem 2.a and 2.b are complete by themselves, it is perhaps useful to draw the picture of an economy working according to the requirements of Theorem 2.a. It is clear that such a picture in order to be simple, must be highly unrealistic but it can nevertheless be helpful if one wants to see intuitively the results and implications of the preceding sections.

Let us go back to the nonmonetary socialist economy outlined at the end of Section 3 and notice that there is no uncertainty in such an economy. Let the time period be the year.

At the beginning of every year a constant number of new consumers enter into the economic life where they will stay for fifty years, exactly. They are given fifty satisfaction functions s_{i_1} .... $s_{i_{50}}$ and a sequence of fifty numbers (positive or negative) α_{i_1} .... $\alpha_{i_{50}}$. They can buy at given prices the commodities in government stores or order them for a future time period. They can in the same way commit themselves for present or future work (at given wage rates). But in any case their net expenses during their t-th year must be at most equal to α_{i_t}.

The government investment agencies behave according to the rules outlined at the end of Section 4 and the firms according to the well-known rules.

The planning authority has to choose a triple set of prices:

1 - The prices at which the investment agencies should discount their capital holdings (these should be zero in the case of Theorem 2.a).

2 - The prices at which the investment agencies should lease their capital goods to the firms and the firms should sell their products to the government stores.

3 - The prices at which the government stores should sell their merchandise to the consumers, these prices may vary according to the date of delivery.

In this choice the planning agency has to keep in mind some requirements: For a given commodity the price in the productive sector should be equal to the average price in government stores (averaged over the fifty coming years). Moreover the system of prices as a whole should correspond to some equilibrium. This last requirement may be a hard task to accomplish in practice.

Let us finally summarize the results which might be of value in economic theory:

1 - A zero interest rate in the productive sector has certain optimal properties.

2 - If the holding of capital is not allowed for the consumer, the interest rates in the consumption market may be different from the interest rate existing in the production sector. So far, the maximization of economic welfare does not seem to imply any particular structure of interest rates in the consumption market.

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COWLES COMMISSION DISCUSSION PAPER

Supplement to Economics No. 2007A

by Edmond Malinvaud

February 27, 1951

1 - Introduction.

In Economics No. 2007 the conditions for an optimum amount of capital have been given for the case of a stationary economy. However the solution of the problem was incomplete in at least two respects:

1 - In the final optimum theorems (Theorem 2.b for instance) no unique interest rate exists although there are different prices for the same item according to the date of delivery, and a price for the lease of a commodity together with a price for the sale of the same commodity. In ordinary economics there exists a scalar interest rate which determines all prices of any commodity given its present sale price. Hence the following question: Is this fact the cause of an economic loss, or, on the contrary, is a "unique" interest rate, in some sense, a requirement of welfare economics?

2 - Modern societies are essentially progressive and the properties of stationary states cannot be of any interest for us unless we know how they are modified when we no longer assume that they are stationary.

In the following pages an answer is given to the first question and in Section 4 some tentative comments concerning the second one are formulated. The treatment of the uniqueness of interest rate appears to be quite different according to the sector which is involved, hence it is broken down into two sections: Unique interest rate in the production sector (price of lease versus price of sale) and unique interest rate in the consumption sector (price for present delivery versus price for future delivery).

2 - Unique interest rate in the production sector.

The purpose of this section is to show that a unique interest rate in the production sector has some optimal property. According to the results of Economics No. 2007 an s-optimum is realized only if a z-optimum is realized; hence we can look only here for z-optima. We need the following definition:

Definition. A stationary state characterized by (z^1, c^1) is said to be non-admissible if there is some other stationary state (z, c) such that:

i. - $z \geq z^1$

ii. - It is possible to pass from (z^1, c^1) to (z, c) in one time period, during which the disposable resource vector is equal to z^1.

Loosely speaking, a stationary state is said nonadmissible if it is possible without any present cost to change something so as to be better off in all the future years.

According to our notation, condition ii may be written:

$$(z^1 + c - c^1, c^1) \in \mathbb{Z}^* .$$

Indeed, during the intermediate period and with exactly the same physical structure of the economy the resource vector would be $z^1 + c - c^1$ if the capital vector had to be equal to c^1 at the end of the period, which means that going from (z^1, c^1) to (z, c) while enjoying z^1 during the intermediate period, is possible if and only if $(z^1 + c - c^1, c^1) \in \mathbb{Z}^*$.

Hence (z^1, c^1) is admissible if and only if:

(1) $$\boxed{z \geq z^1 \text{ and } (z, c) \in \mathbb{Z}^* \text{ implies } (z^1 + c - c^1, c^1) \notin \mathbb{Z}^*.}$$

We want now to prove the following theorem:

Theorem 3. A necessary and sufficient condition for (z^1, c^1) to be admissible is the existence of a price vector $p \geq 0$ in $\mathbb{R}_\ell$ and of a scalar $\rho \gtrless 0$ such that (z_K, c_K) being constrained by $(z_K, c_K) \in \mathbb{Z}^*_K$, $p\, z_K - \rho\, p\, c_K$ reaches its maximum at (z^1_K, c^1_K) for all K.

Proof of Theorem 3.

Condition (1) implies obviously the following:

$$z \geq z^1 \Longrightarrow (z, c^1) \notin \mathbb{Z}^*.$$

This is equivalent to saying that there is a support plane Π to $\mathbb{Z}^*$ containing (z^1, c^1) and characterized by (p, r) normal to Π. This is also equivalent to the existence of parallel support planes to the $\mathbb{Z}_K^*$ at (z_K^1, c_K^1).

Consider now the three following sets in $\mathbb{R}_\ell$:

i) $\mathbb{Z}_{c^1}$ set of all z such that $(z, c^1) \in \mathbb{Z}^*$.

$\mathbb{Z}_{c^1}$ is convex and admits at z^1 a support plane normal to $p \geq 0$.

ii) $\mathbb{Z}_{z^1}$ set of all c such that $(z^1, c) \in \mathbb{Z}^*$.

$\mathbb{Z}_{z^1}$ is convex and admits at c^1 a support plane normal to r.

($\mathbb{Z}_{z^1}$ is the set of all c which make possible $z \geqq z^1$)

iii) $\tilde{\mathbb{Z}}$ set of all c such that $(z^1 + c - c^1, c^1) \in \mathbb{Z}^*$.

$\tilde{\mathbb{Z}}$ is obtained from $\mathbb{Z}_{c^1}$ by the translation $c^1 - z^1$. Hence $\tilde{\mathbb{Z}}$ is convex and admits at c^1 a support plane normal to $p \geq 0$.

(Cf. Fig. 9)

($\tilde{\mathbb{Z}}$ is the set of all c which can be obtained without any sacrifice.)

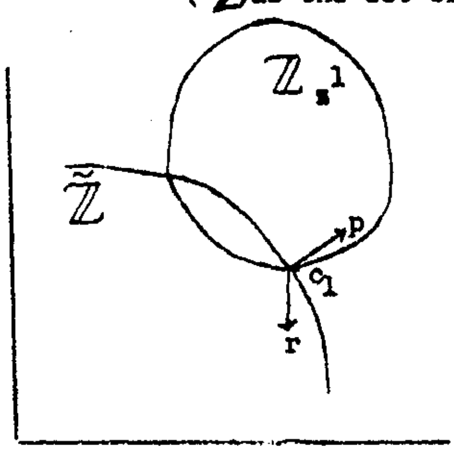

Fig. 9

We will now show that condition

(1) implies $r = -\rho\, p$ with $\rho \geqq 0$.

Condition (1) becomes

(2) $z \geq z^1$ and $(z, c) \in \mathbb{Z}^*$ implies $c \notin \tilde{\mathbb{Z}}$.

Consider the two following cases:

i) $\tilde{\mathbb{Z}} \cap \mathbb{Z}_{z^1}$ has a nonempty interior. We want to prove that z^1 is optimal in the sense of Theorem 1.a., i.e., there is no (z, c) with $z \geq z^1$ and $(z, c) \in \mathbb{Z}^*$.

This is surely true if $c \notin \mathbb{Z}_{z^1}$, by definition of $\mathbb{Z}_{z^1}$.

This is also true for any $c \in \tilde{\mathbb{Z}}$ by condition (2).

In order to prove that it is true for all $c \in \mathcal{Z}_{z^1}$, take any c^2 in the interior of $\mathcal{Z}_{z^1} \cap \tilde{\mathcal{Z}}$ and suppose there is some $c \in \mathcal{Z}_{z^1}$ with $(z, c) \in \mathcal{Z}^*$ and $z \geq z^1$.

By the convexity of $\mathcal{Z}^*$, for any $0 < \alpha < 1$:

$$(z^{\alpha}, c^{\alpha}) = \alpha(z, c) + (1 - \alpha)(z^1, c^2) \in \mathcal{Z}^*.$$

It is possible to choose α small enough but positive so as to have:

$$c^{\alpha} \in \tilde{\mathcal{Z}} \cap \mathcal{Z}_{z^1} .$$

But $\alpha > 0$ implies: $z^{\alpha} = \alpha z + (1 - \alpha)z^1 \geq z^1$ contradicting $c^{\alpha} \in \tilde{\mathcal{Z}}$.

Hence, if $\tilde{\mathcal{Z}} \cap \mathcal{Z}_{z^1}$ has a nonempty interior, z^1 is maximal in the sense of Theorem 1.a. and $r = 0$; the condition of Theorem 3 is satisfied with $\rho = 0$.

ii) $\tilde{\mathcal{Z}} \cap \mathcal{Z}_{z^1}$ has no interior. We know that it is convex, hence it is contained in some support plane to both $\tilde{\mathcal{Z}}$ and $\mathcal{Z}_{z^1}$. Hence there is some ρ such that there is at (z^1, c^1) a support plane $(p, -\rho\ p)$ to $\mathcal{Z}^*$. As p and r are directed towards the outside of $\tilde{\mathcal{Z}}$ and $\mathcal{Z}_{z^1}$ we have: $\rho > 0$.

Theorem 3 follows directly.

3 - Unique interest rate in the consumption sector.

We have seen that according to Theorem 2 an optimum is characterized by a set of prices p_{st} on the consumption market. p_{st} is the price of the commodity s available during the time period t. We would like to know under which condition the set of prices is obtained from a definite pattern of interest rates, or, in other words, when the following holds:

$$\frac{p_{s,t}}{p_{s,1}} = \alpha_t \quad \text{for all s.} \tag{3}$$

Moreover we know that

$p_s = \frac{1}{h} \sum_t p_{st}$, hence (3) may be written as:

(4) $$\boxed{p_{s,t} = p_s \beta_t}$$

This is clearly a property of the vectors normal to the set $\mathcal{X}'$, hence it has to be a property of the normals to each set $\mathcal{X}'_i$, because the support planes of $\mathcal{X}'$ are parallel to support planes of $\mathcal{X}'_i$.

Hence a definite set of interest rates exists on the consumption market if and only if the support planes to each $\mathcal{X}'_i$ satisfy (4). Loosely speaking, we will say in this case that there is a set of psychological interest rates.

The existence of such psychological interest rates seems to be very uncertain. In particular, if it were realized, the relative importance of the different goods would remain the same over time, whatever the actual value of x'.

This does not mean that a set of prices which fulfills condition (4) is not compatible with optimality. In fact it is quite possible that such a set of prices satisfies the condition of Theorem 2. But nothing so far implies that the set p_{st} should fulfill (4). Here we may again point out that the consumption units do not hold any commodity from one time period to another.

By now, however, we should be worried by the following question:

When we look for welfare propositions in a capitalistic economy, we usually proceed to a crude generalization of the main theorem of welfare economics by a suitable multiplication of the number of commodities. By this procedure we are led to say that a state cannot be optimal unless the price and interest vectors are the same in both the production and consumption sectors. Here, on the other hand, we have an optimum theorem whose requirements are definitely weaker in this respect. What is the reason for this difference?

The answer is to be found in the assumption of a stationary economy: in the usual approach the capital vector at the end of each future time period is determined so as to maximize the present satisfaction of the consumption units.

In our case the capital vector has to be maintained at a constant level with respect to time.

The choice between an optimum stationary economy in our sense and an economy which maximizes the present satisfaction of the consumption units is a philosophical question beyond our domain. I feel, however, that the search for some kind of optimum amount of capital is usually conceived along the same lines as are used here.

4 - Optimum amount of capital in a progressive economy.

This section consists entirely of loose considerations concerning the case of a progressive economy. The tentative character of all that will be said should be emphasized at the beginning. However this note would not be complete if it did not suggest at least intuitively how the results obtained in the case of a stationary economy can be used for practical policy in the modern "progressive" societies.

The characteristic feature of a progressive economy is for us a permanent technical progress, which means that new possibilities of production are introduced and that the structure of the productive sector keeps changing from year to year.

1 - Consider first the problem from the point of view of the national accounting and suppose that the economic progressive state is associated with a constant rate of interest ρ. The national consumption in a given year C_t is equal to the difference between the gross national product G_t and the gross investment which is the sum of the net investment I_t and of the replacement of the existing capital A_t:

$$C_t = G_t - A_t - I_t.$$

It has been shown earlier that $G_t - A_t$ is maximal if $\rho = 0$ and decreases when ρ increases. Moreover in our model I_t is determined by the condition that the change in the capital equipment is just appropriate to maintain constant the rate of interest. How I_t changes when ρ changes, depends on the nature of the modifications in the technical possibilities. Hence the optimal property of a zero interest rate seems to disappear.

The following remarks suggest tentative lines along which this new problem could be studied.

2 - Our model may be redefined with the following modifications:

All variables are functions of time. We write c_t, z_t ...

$\mathcal{Y}$ is made equal to the empty set, which is not restrictive as this can easily be seen. Define the net revenue vector w_t and a set $\mathbb{Z}_t^*$ such that

$$(w_t, c_t) \in \mathbb{Z}_t^* \subset \mathbb{R}_{2\ell}.$$

The resource vector for the time period t is deduced from w_t by:

$$z_t = w_t + c_t - c_{t+1}.$$

In other words, (z_t, c_t) is constrained by:

$$(z_t + c_{t+1} - c_t, c_t) \in \mathbb{Z}_t^* .$$

3 - A progressive economic state is defined by a sequence $\left\{z_t, c_t\right\}$ with $t = 1, 2, \ldots, h, \ldots$.

A sequence is said <u>admissible</u> if the following conditions are fulfilled.

i) (w_t, c_t) is on the boundary of $\mathbb{Z}_t^*$, for all t.

ii) The support plane (p_t, r_t) to $\mathbb{Z}_t^*$ at (w_t, c_t) is such that $r_t = -\rho\, p_t$; where ρ is a nonnegative scalar independent of t.

The fact that ρ is supposed independent of t makes the definition of admissibility somewhat stronger than in Section 2. But some requirement of that kind seems to be necessary if one wants to consider only states associated with smooth economic expansion.

An admissible sequence $\left\{z_t^1, c_t^1\right\}$ is said greater than another admissible sequence $\left\{z_t^2, c_t^2\right\}$ if:

$$z_t^1 \geqq z_t^2 \text{ for all t and } z_t^1 \geq z_t^2 \text{ for at least one t.}$$

We will write $\left\{z_t^1, c_t^1\right\} \geq \left\{z_t^2, c_t^2\right\}$.

The truncated sequence $\left\{z_t, c_t\right\}_h$ is the subsequence obtained from $\left\{z_t, c_t\right\}$ by the condition $t > h$.

4 - The preceding concepts may be used in the solution of the two following simplified examples under the strong following conditions:

i) Only the sequences are considered such that p_t remains the same over time. $(p_t = p)$

ii) On the boundary of $\mathbb{Z}_t^*$ there is one and only one point where the support plane has the orientation $(p, \ -\rho\, p)$.

These conditions are hardly justifiable and could probably be removed, but our purpose here is to reach without too much pain some result, likely to make the problem clearer.

<u>1st example.</u>

Suppose $\mathbb{Z}_t^*$ is such that $\mathbb{Z}_{t+1}^* = \mathbb{Z}_t^* + \mathbb{Z}_*$.

Associated to a sequence $\left\{z_t^1, c_t^1\right\}$ there is some: (w_1^1, c_1^1) net revenue-capital vector during the first time period (uniquely determined), and some

$(a^1, b^1) \in \mathbb{Z}_*$: added revenue capital vector for each time period (uniquely determined).

$$\left\{z_t^1, c_t^1\right\} \geq \left\{z_t^2, c_t^2\right\} \text{ means:}$$

$$(w_1^1, - b^1) + t\, a^1 \geq (w_1^2 - b^2) + t\, a^2 \qquad \text{for all } t$$

(where strict inequality holds for at least one t).

w_1 and a are optimal when $\rho = 0$ hence the following results:

i) Corresponding to any admissible sequence $\left\{z_t^1, c_t^1\right\}$ associated with some $\rho^1 > 0$, there is an admissible sequence $\left\{z_t^o, c_t^o\right\}$ such that $a^o \geq a^1$; if the components of $a^o - a^1$ are not zero when the corresponding components of $b^o - b^1$ are negative, then there is some h such that $\left\{z_t^o, c_t^o\right\}_h \geq \left\{z_t^1, c_t^1\right\}_h$.

ii) If $(a, b) \in \mathbb{Z}_*$ implies $b = 0$, the admissible sequences associated with a zero interest rate are optimal. (Cf. Fig. 10.)

2nd example.

Suppose $\mathbb{Z}^*_{t+1} = \alpha \mathbb{Z}^*_t$ where α is some positive number. In the same way as in the preceding example we can see that:

$$\left\{z^1_t, c^1_t\right\} \geq \left\{z^2_t, c^2_t\right\} \text{ means:}$$

$$w^1_1 - (\alpha - 1)\, c^1_1 > w^2_1 - (\alpha - 1)\, c^2_1 .$$

An optimum admissible sequence is obtained if and only if $w_1 - (\alpha - 1)c_1$ is maximal; hence it is associated with an interest rate equal to $\rho = \alpha - 1$. This is exactly the rate of expansion of the economy. (Cf. Fig. 11.)

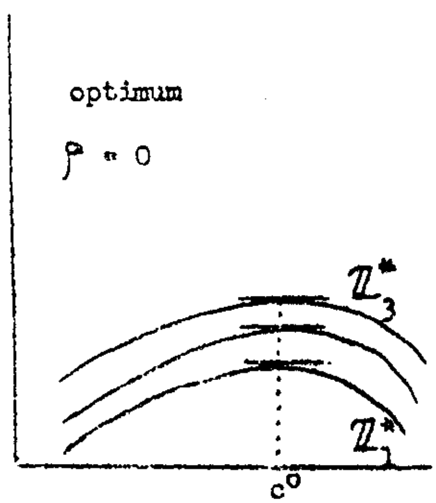

Fig. 10

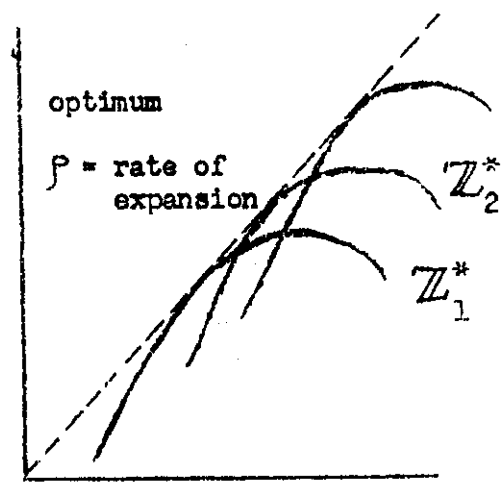

Fig. 11