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Notes on a Continuous Transportation Model[1]

by

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1. This paper gives some additions to Cowles Commission Discussion Paper 293, which are among themselves only loosely connected. The notations are the same as in the paper referred to, except for g which is replaced by q, and the sign of λ which is reversed in order to have Lagrange constraints as additional terms. The notations and basic equations are repeated briefly to make the following understandable, independent of Cowles Commission Discussion Paper 293.

We consider an area A with a boundary B, all functions being continuous (and even to the second order differentiable) ones therein.

x_1, x_2	are Cartesian point-coordinates,
$k = k(x_1, x_2)$	the local cost of transportation of the unit of a commodity C for the unit distance,
$v = v(x_1, x_2)$	the transportation vector field representing direction and quantity of shipments through each point (x_1, x_2) of A.
$q = q(x_1, x_2)$	the amount of C to be brought to or from (x_1, x_2), according to the sign of q. For brevity, q is called "production - program."

(1) Acknowledgement is made to Professor T. C. Koopmans whose papers LPC 401 (Cowles Commission Discussion Paper 214), LPC 410, 411A and Chapter XIV of Activity Analysis of Production and Allocation, (1951), have largely been drawn on.

Some results of Cowles Commission Discussion Paper 293 are:

1.1 If the production program is given and the transportation cost function $k(x_1, x_2)$ is independent from back movements of carrier equipment, then the transportation vector field is known by

(1) $\quad \operatorname{div} v - q = 0$

(2) $\quad k \frac{v}{|v|} + \operatorname{grad} \lambda = 0 \qquad$ or equivalently[2]

(2a) $\quad \operatorname{curl} k \frac{v}{|v|} = 0 \qquad$ and a certain boundary condition, e.g., $v = 0$ on B.

1.2 Under competitive conditions and after exclusion of unconnected sub-areas (to or from which no C is shipped) the parameter λ can, after proper choice of an integrational constant, be identified with the local price of C.

For (1) we have then

(1a) $\quad \operatorname{div} v - q(x_1, x_2, \lambda(x_1, x_2)) = 0.$

1.3 Equations (1a), (2) can be extended to several commodities:

(1b) $\quad \operatorname{div} v_i - q_i(x_1, x_2; \lambda_1, \lambda_2, \ldots, \lambda_n) = 0$

(2b) $\quad k \frac{v_i}{|v_i|} + \operatorname{grad} \lambda_i = 0$

2. The first remark aims at an alternative formulation of the minimum problem underlying Cowles Commission Discussion Paper 293, which was to find v, subject to $\operatorname{div} v - q = 0$, so as to minimize the transportation cost-total

(3) $\quad K = \iint k\,|v|\,dx_1\,dx_2$ or in Lagrangian notation

(3a) $\quad \iint \left[k\,|v| + \lambda(q - \operatorname{div} v)\right] dx_1\,dx_2 .$

Consider (3a) for a solution v of (1), (2). By partial integration of $\lambda \operatorname{div} v$ one has

$$\iint \left[k\,|v| + \lambda q - \lambda \operatorname{div} v\right] dx_1\,dx_2 = \iint (k\,|v| + \lambda q)\,dx_1\,dx_2$$

$$- \oint (\lambda v)_n\,ds + \iint \operatorname{grad} \lambda \cdot v\,dx_1\,dx_2$$

(2) (2a) reveals more overtly that (2) imposes a real condition on v and does not merely introduce an additional variable λ.

where the subscript n denotes the vector component normal to the boundary, taken positive to the interior of A, and ds is the line element on B. The contour integral vanishes because of the boundary condition $v = 0$ on B. With (2) and (1) it follows then easily that

(4) $$K = \iint \lambda q \, dx_1 \, dx_2 \qquad \text{for a solution } v.$$

Since for any constant C

$$\iint C \, q \, dx_1 \, dx_2 = C \iint q \, dx_1 \, dx_2$$

and this integral was shown to be zero in Cowles Commission Discussion Paper 293,[3] the integral (4) is invariant against addition of a constant to λ (which has to be allowed for, since λ is only determined up to an arbitrary integrational constant by (2)).

To extremise $\iint \lambda q \, dx_1 \, dx_2$ with respect to λ has a definite meaning only after the admissible set of λ has been specified. From the results in Cowles Commission Discussion Paper 293 we should expect that a reasonable "potential" function λ is to be the gradient function of a vector field, and related also to the transportation cost. We may prescribe for λ that it fulfills an equation

(5) $$k \frac{Y}{|Y|} + \operatorname{grad} \lambda = 0$$

where Y can be any vector field. We also will need a boundary condition but leave it open yet. An extremum of $\iint \lambda q \, dx_1 \, dx_2$ is then found by variation with respect to λ of

(4a) $$I = \iint \left[\lambda q + u \cdot \left(k \frac{Y}{|Y|} + \operatorname{grad} \lambda \right) \right] dx_1 \, dx_2$$

where u is a vector of Lagrange parameters, scalarly multiplied with the vector sum in the bracket.

The Euler equation of this is

(6) $$q - \operatorname{div} u = 0$$

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(3) Equation 9, Page 5, Cowles Commission Discussion Paper 293.

Integrating in I partially and introducing (6) we have

$$(4b) \qquad I = \iint\left[\lambda q + k\, u \cdot \frac{Y}{|Y|} - \lambda \operatorname{div} u\right] dx_1\, dx_2 + \oint (u\,\lambda)_n\, ds =$$

$$\iint k\, u\, \frac{Y}{|Y|}\, dx_1\, dx_2 + \oint (u\,\lambda)_n\, ds\ .$$

Here the contour integral will depend on the boundary conditionsfor λ. If we choose them so as to give this integral a fixed value, e.g., zero, we are reduced to extremising the first integral. It suggests itself to choose the unit vector $\frac{Y}{|Y|}$ so as to point everywhere in the direction of u. We have, in fact, to operate on Y, since Y is the only variable depending on λ. (By means of Equation (5)). Put then

$$(7) \qquad Y = \alpha \cdot u \qquad \text{where } \alpha \text{ is a positive scalar function in } x_1,\ x_2\ .$$

$$I = \iint k\, u\, \frac{Y}{|Y|}\, dx_1\, dx_2 = \iint k\, u\, \frac{\alpha\, u}{|\alpha\, u|}\, dx_1\, dx_2 = \iint k\, |u|\, dx_1\, dx_2\ .$$

From (6), and (5) and (7) it is seen that u satisfies equation (1) and (2). u is identical with the flow-vector v. And therefore

$$(8) \qquad \underset{\lambda}{\text{Extr.}} \iint \lambda q\, dx_1\, dx_2 \quad = \iint k\, |u|\, dx_1\, dx_2 \quad = \underset{v}{\text{Min}} \iint k\, |v|\, dx_1\, dx_2$$

$$\text{subject to } k\,\frac{Y}{|Y|} + \operatorname{grad} \lambda = 0 \qquad \text{with } \operatorname{div} u - q = 0,\ \ k\,\frac{u}{|u|} + \operatorname{grad}\lambda = 0 \qquad \text{subject to } \operatorname{div} v - q = 0$$

At the beginning of this section it has been shown, that

$$(9) \qquad \iint\left[(k\,|v| + \lambda(q - \operatorname{div} v)\right] dx_1\, dx_2 = \iint \lambda q\, dx_1\, dx_2$$

$$\text{with } k\,\frac{v}{|v|} + \operatorname{grad} \lambda = 0$$

And from Cowles Commission Discussion Paper 293 - equation (2) of this paper - it is known, that the left hand of (9) is equal to

$$(10) \qquad \underset{v}{\text{Min}} \iint\left[k\,|v| + \lambda(q - \operatorname{div} v)\right] dx_1\, dx_2$$

Now it is trivial that $\underset{v}{\text{Min}} \iint\left[k\,|v| + \lambda(q - \operatorname{div} v)\right] dx_1\, dx_2 \leq$

$$\underset{v}{\text{Min}} \qquad \iint\left[k\,|v| + \lambda(q - \operatorname{div} v)\right] dx_1\, dx_2\ .$$

subject to

$q - \operatorname{div} v = 0$

The left side is, according to (10), (9), equal to $\iint \lambda q \, dx_1 \, dx_2$, the right side, because of (8), equal to

$$\underset{\lambda}{\text{Extr.}} \iint \lambda q \, dx_1 \, dx_2 \, .$$
$$\text{subject to } k \frac{Y}{|Y|} + \text{grad } \lambda = 0$$

Hence
$$\iint \lambda q \, dx_1 \, dx_2 \leq \underset{\lambda}{\text{Extr.}} \iint \lambda q \, dx_1 \, dx_2 \quad \text{subject to } k \frac{Y}{|Y|} + \text{grad } \lambda = 0 \, .$$

or, with (8),

$$(11) \qquad \underset{v}{\text{Min}} \iint k \, |v| \, dx_1 \, dx_2 \quad = \quad \underset{\lambda}{\text{Max}} \iint \lambda q \, dx_1 \, dx_2$$
$$\text{subject to div } v - q = 0 \qquad \text{subject to } k \frac{Y}{|Y|} + \text{grad } \lambda = 0$$

In reviewing the course of arguments one sees that $\underset{\lambda}{\text{Max}} \iint \lambda q \, dx_1 \, dx_2$ is in fact uniquely determined through the choice of $\frac{Y}{|Y|}$ in (4a), directed to maximization of I. From the fact that $Y = \propto u$ maximizes I relative to all extremities it could not already be concluded, that it results in a maximum. It might for instance only yield a $\underset{\lambda}{\text{Max}}$ Min I.

What is, apart from a mathematically (perhaps) not uninteresting duality in vector-flow and price functions, gained by equation (11)? This, it may be ventured, is the fact that the problem of finding the optimum utilization of a transportation system (with certain restrictions as to its transportation cost structure) can be formulated in terms of the locational potential λ alone, without making reference to the resulting transportation vector - field. That includes a generalization of the notion of the locational potential. It was found possible to define a potential function as the gradient potential of any (curl-free) vector field, the vector of which may only have a length indicating the local transportation cost - the alternative set-up of the problem is, to conclude: To find among the class of (two times differentiable) functions $\lambda(x_1, x_2)$ satisfying the condition

2.1 $\operatorname{grad} \lambda + k \frac{Y}{|Y|} = 0$ for suitable Y,

2.2 $\oint (u \lambda)_n \, ds = 0$ for all u with $\operatorname{div} u - q = 0$,

such a λ that maximizes $\iint \lambda q \, dx_1 \, dx_2$, the weighed sum of the locational potential.

3. The equivalence

(4) $\iint \lambda q \, dx_1 \, dx_2 = \iint k \, |v| \, dx_1 \, dx_2$ in the case of solution λ, v supplies an interpretation of the term $\lambda(q - \operatorname{div} v)$ in integral (3a). Call, for the moment, $q - \operatorname{div} v = r$. r is of the same character as q or $\operatorname{div} v$, a "source density".

$\iint \lambda r \, dx_1 \, dx_2$ represents, like $\iint \lambda q \, dx_1 \, dx_2$, a transportation cost-sum. These costs are due to the necessity, to remove the surplus (or deficiencies), that an unrestricted v-field would cause everywhere. For the cargoes delivered, div v that is, would not, in the general case, match with the quantities q, imposed by the production program. The difference is just equal to r, and the cost of its abolishment

$$\iint \lambda r \, dx_1 \, dx_2 = \iint (q - \operatorname{div} v) \, dx_1 \, dx_2 \quad .$$

Equation (4) also provides us with a means to take account of such cost factors in transportation that arise from required shipments of empty carrier equipment.

If the production program is given for each commodity and each point, then this cost factor does not influence the direct ways of shipment. The back transportation of empty carriers can be set up separately, since with the quantities of production the required quantities of empty carriers in each point are also given. We have the same problem as for the transportation of any other commodity, and equations (1), (2) suggest the solution. Let k_i, $i = 1, \ldots, n$, denote the direct transportation cost for the unit of commodity i, and k_{n+1} the same for the unit of empty carrier equipment. The total cost integral is then, with due consideration of the side conditions:

(12) $$\iint \sum_{i=1}^{n+1} k_i \, |v_i| + \lambda_i (q_i - \operatorname{div} v_i) \, dx_1 \, dx_2$$

where q_{n+1} is equal to $\operatorname{div}(\sum_{i=1}^{n} - t_i v_i)$ and t_i stands for the amount of carrier equipment required to transport the unit of commodity i. (12) is formally identical with equation (18) in Cowles Commission Discussion Paper 293. This solves the case of a given production program.

In the case of a variable (price-dependent) program we note the following:

From the new interpretation of k (direct shipment costs) it is clear that the inter-local price-differences of commodity i are no longer equal to the inter-local λ_i - differences, since these are based on the direct transportation cost k_i alone. Now the total cost of empty shipments is, by aid of (4) equal to $\int \lambda_{n+1} q_{n+1}\, dx$, where

$$q_{n+1} = \operatorname{div} v_{n+1} = \operatorname{div} \sum_{i=1}^{n} t_i v_i = -\sum t_i \operatorname{div} v_i = -\sum_{i=1}^{n} t_i q_i .$$

It follows that a unit of i shipped from (or to — according to the sign of q_i) a point (x_1, x_2) causes an indirect (empty shipment) cost of $-\lambda_{n+1} t_i$, provided that the integration constant in λ_{n+1} together with those in λ_i, $i = 1, \ldots, n$, have been properly chosen; that is, such that $\iint q_i\, dx_1\, dx_2 = 0$ for $i = 1, \ldots, n$.

With $\lambda_i = \lambda_i - t_i \lambda_{n+1}$ as local price-functions and

$$q_i = q_i(x_1, x_2; \lambda_1 - t_1 \lambda_{n+1}, \ldots, \lambda_i - t_i \lambda_{n+1}, \ldots, \lambda_n - t_n \lambda_{n+1})$$

$$q_{n+1} = -\sum_{i=1}^{n} q_i t_i$$

we have then a system of equations, corresponding to (20) of Cowles Commission Discussion Paper 293, which takes proper account of the cost arising from necessitated empty carrier movements.

4. Instead of assuming the local excess production $q(x_1, x_2)$ to be given and fixed all over A, as was done in the First Section of Cowles Commission Discussion Paper 293, suppose now that $|v|$, the transportation capacity be limited at any point to the extent that its full use can be taken for granted in any optimal system.

It is obvious that equation

(2) $\qquad k \dfrac{v}{|v|} + \operatorname{grad} \lambda = 0$

(1) $\qquad q - \operatorname{div} v = 0$

must hold also in this case, although their interpretation is to be different. For the economy of shortest connections as expressed in (1), (2) must be maintained. Instead of k, however, $|v|$ is the given function and k is variable. Due to this change, there are no boundary conditions existing. Hence (1), (2) constitute necessary, but insufficient, conditions. The solution is a family of "efficient" k and λ functions. It is conceivable to impose another optimum condition on the boundary. This would lead to a variation problem, which can however not be formulated before the system (1), (2) has been solved in terms of an arbitrary boundary constraint, a problem that we do not pursue here.

5. There is a case of some generality in which the solution of the equation system (1), (2) is well known. That is when transportation costs are strictly proportional to traffic density with a coefficient independent of the locality.

(13) $$k(x_1, x_2, v_1, v_2) = c\ |v|\ .$$

For then

(1) $$k \frac{v}{|v|} + \operatorname{grad} \lambda = c\ v + \operatorname{grad} \lambda$$

(2) $$\begin{aligned} \operatorname{div} v - q &= \operatorname{div} c \operatorname{grad} \lambda - q \\ &= c \Delta \lambda - q = 0 \end{aligned}$$

the equation system reduces to the Poisson equation

(14) $$\Delta \lambda - 2\pi\mu = 0$$

$$\text{with } \mu = \frac{q}{2\pi c}\ .$$

Its solution is known to be

$$\lambda = \iint \log \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2} \cdot \mu(\xi_1, \xi_2)\, d\xi_1\, d\xi_2.$$

Where μ_1, hence q_1, can be quite arbitrary, but may not contain λ itself. It should be noted that for $k(x_1, x_2) = \text{const.}$, a solution of similar simplicity does not exist.

The locational potential is in this case also a potential function in the mathematical sense of the word.