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Resource Allocation and Statistical Decision Functions[1]

by

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11. Minimax Interpretation of ρ.

In Section 9 we were led to consider the expression $\frac{p \cdot z}{p \cdot z^o}$ where $z \in \mathbb{Z}^{min}$ and p is one of the normals to $\mathbb{Z}^{min}$ at z. It was proved that its maximum is reached at z* (and possibly at other points) collinear with z^o, and that its value is ρ, the ratio of z* to z^o. This is but a part of a more complete theory that we present now.

We assume that the quantity of every commodity varies continuously but we drop the convexity hypothesis and we look for

$$\underset{z \in \mathbb{Z}}{\text{Min}} \quad \underset{p \in \mathbb{P}'}{\text{Max}} \quad \frac{p \circ z}{p \circ z^o}$$

where $\mathbb{P}'$ is the closed positive orthant, origin excluded. z being given,

$$\underset{p \in \mathbb{P}'}{\text{Max}} \quad \frac{p \circ z}{p \circ z^o} = \underset{h}{\text{Max}} \quad \frac{z_h}{z_h^o}, \quad (2)$$

which may be infinite.

If $z \in \mathbb{Z}$, $\frac{z_h}{z_h^o} \geqq \rho$ for at least one h, otherwise one would have $z < z^*$ contradicting the fact that $z^* \in \mathbb{Z}^{min}$. Therefore $\underset{p \in \mathbb{P}'}{\text{Max}} \frac{p \circ z}{p \circ z^o} \geqq \rho$ whatever be z in $\mathbb{Z}$; it is equal to ρ if, and only if, z = z* (again an immediate consequence of "$z^* \in \mathbb{Z}^{min}$"). In other words $\underset{z \in \mathbb{Z}}{\text{Min}} \; \underset{p \in \mathbb{P}'}{\text{Max}} \; \frac{p \circ z}{p \cdot z^0} = \rho$; it is reached for z^* and p arbitrary in $\mathbb{P}'$. If p is chosen (say by some central agency) in $\mathbb{P}'$ so as to make $\frac{p \circ z}{p \circ z^0}$ as great as possible, and if z is chosen in $\mathbb{Z}$ so as to make this expression as small as possible [and this amounts to choosing y_j in $\mathbb{Y}_j$ (resp. x_i in $\mathbb{X}_i$) so as to make $p \circ y_j$ (resp. $p \circ x_i$) as small as possible for every j (resp. i)], the economic system is led to z^* and the final value of the expression is ρ. The order in which the operations Max and Min are carried out is of utmost importance.

If the set $\mathbb{Z}$ is convex (this property of $\mathbb{Z}$ has been studied in Sections 6 and 9), this order becomes <u>indifferent</u>. In effect let us look for

$$\underset{p \in \mathbb{P}'}{\text{Max}} \; \underset{z \in \mathbb{Z}}{\text{Min}} \; \frac{p \cdot z}{p \circ z^0} .$$

p being given, $\underset{z \in \mathbb{Z}}{\text{Min}} \; \frac{p \cdot z}{p \cdot z^0}$ is reached for a point of $\mathbb{Z}^{min}$ (and possibly other points of $\mathbb{Z}$), we can therefore restrict ourselves to the case where $z \in \mathbb{Z}^{min}$ and p is a normal to $\mathbb{Z}^{min}$ at z. The problem of finding the maximum of $\frac{p \circ z}{p \cdot z^0}$ in these conditions is precisely the problem we solved in Section 9. The maximum ρ is reached at z^* (and possibly at other points of $\mathbb{Z}^{min}$), the corresponding p^* being any one of the normals to $\mathbb{Z}^{min}$ at z^*.

To sum up,

$$\rho = \underset{z \in \mathbb{Z}}{\text{Min}} \; \underset{p \in \mathbb{P}'}{\text{Max}} \; \frac{p \circ z}{p \cdot z^0} = \underset{p \in \mathbb{P}'}{\text{Max}} \; \underset{z \in \mathbb{Z}}{\text{Min}} \; \frac{p \circ z}{p \circ z^0} ,$$

the set of saddle points of the function $\frac{p \circ z}{p \circ z^0}$ is the product [1-13] of

--the set of z where $\underset{z}{\text{Min}} \; \underset{p}{\text{Max}} \; \frac{p \circ z}{p \cdot z^0}$ is reached: it is composed of z^* only

--the set of p where $\underset{p}{\text{Max}} \; \underset{z}{\text{Min}} \; \frac{p \circ z}{p \cdot z^0}$ is reached: it is composed of the normals p^* to $\mathbb{Z}^{min}$ at z^*. (3)

If z^0 is maximal, the value of the minimax is of course 1.

12. Isomorphism With the Theory of Statistical Decision Functions.

If none of the components of z^0 is null, we can, by an appropriate choice of the units, make them all equal to 1. The expression $\frac{p \circ z}{p \circ z^0}$ then takes the form $\frac{p}{\sum_h p_h} \circ z$; we put $\bar{p} = \frac{p}{\sum_h p_h}$, price vector normalized in such a way that the sum of its components is 1, and we have finally the very simple form $\bar{p} \cdot z$ where $z \in \mathbb{Z}$, $\bar{p} \in \bar{\mathbb{P}}$ the simplex defined by $\sum_h \bar{p}_h = 1$, $\bar{p}_h \geqq 0$.

We have proved that, $\mathbb{Z}$ being convex,

$$\rho = \underset{z \in \mathbb{Z}}{\text{Min}} \ \underset{\bar{p} \in \bar{\mathbb{P}}}{\text{Max}} \ \bar{p} \circ z = \underset{\bar{p} \in \bar{\mathbb{P}}}{\text{Max}} \ \underset{z \in \mathbb{Z}}{\text{Min}} \ \bar{p} \circ z .$$

The saddle points are z^*, all of whose components are equal to ρ, associated with any normal $\bar{p}^*$ to $\mathbb{Z}^{min}$ at z^*. They appear to be the result of the antagonistic activities of a central agency choosing $\bar{p}$ in $\bar{\mathbb{P}}$ so as to <u>maximize</u> $\bar{p} \circ z$ and of production-units (resp. consumption units) choosing y_j (resp. x_i) in $\mathbb{Y}_j$ (resp. $\mathbb{X}_i$) so as to <u>minimize</u> $\bar{p} \circ z$. (4)

On the other hand, a simple case of the theory of statistical decision functions can be presented in the following way.(5) Let $F(x)$ be the cumulative distribution function of a random variable ξ (probability that $\xi < x$); F is merely known to be an element of a finite set $(F_1, \ldots, F_\iota, \ldots, F_\nu)$. The statistician is faced with the choice of a decision d in a set D. With every pair F_ι, d is associated a number $r \geqq 0$ called risk, expressing what it costs to use d when F_ι is true.

$r(F_\iota, d)$ can be more conveniently written $r_\iota(d)$; it is thus clear that to each d corresponds a risk-vector $r(d)$ $(r_1, \ldots, r_\iota, \ldots, r_\nu)$ of the space $\mathbb{R}_\nu$. The image of the set D by the function $r(d)$ is a set $\mathbb{R}$ of $\mathbb{R}_\nu$, and the initial problem of choice of d in D can be replaced by the problem of choosing a point r in $\mathbb{R}$. In the usual framework of the theory (including the use of randomized decisions: d_1 and d_2 being two decisions one can choose d_1 with the probability α and d_2 with the probability $1-\alpha$), $\mathbb{R}$ is closed and convex. If $r^1 \leq r^2$, r^1 is better than r^2, (whatever be the true F_ι, $r(F_\iota, d_1) \leq r(F_\iota, d_2)$, the strict inequality holding for at least one ι), and the choice of r is therefore restricted to $\mathbb{R}^{min}$. Let us make the further assumption that the straight line whose equations are $r_1 = r_2 = \dots = r_\nu$ meets $\mathbb{R}$ and meets it for the first time (when one is moving away from the origin) at a point r* of $\mathbb{R}^{min}$ (fig. 4): this assumption is not significant for the theory of statistical decision functions but the isomorphism can be brought out in this case.

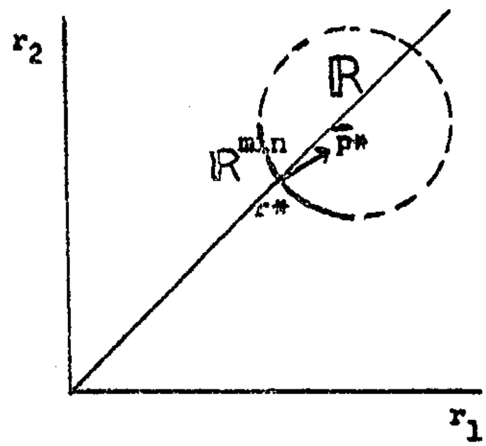

Figure 4

The principle of minimum of the maximum risk amounts to taking $\underset{r}{\text{Min}}\ \underset{\nu}{\text{Max}}\ (r_1, \dots, r_\nu)$, it leads to the selection of r*. Indeed if $\bar{p}$ is a vector whose ν components p_ι satisfy: $\bar{p}_\iota \geq 0$ and $\Sigma\ \bar{p}_\iota = 1$, the Min Max operation mentioned is equivalent to $\underset{r}{\text{Min}}\ \underset{\bar{p}}{\text{Max}}\ \bar{p} \cdot r$: the formal analogy with our former concepts is obvious, z and $\mathbb{Z}$ have been replaced by r and $\mathbb{R}$, the normalized price-vector $\bar{p}$ by the vector $\bar{p}$ whose interpretation will be given in a moment. We have proved in 11 that the operation $\underset{r}{\text{Min}}\ \underset{\bar{p}}{\text{Max}}\ \bar{p} \cdot r$ leads to r*.

We proved also that the operations Min and Max can be inverted. $\underset{\bar{p}}{\text{Max}}\ \underset{r}{\text{Min}}\ \bar{p} \cdot r$ has now the following interpretation: $\bar{p}$ is a probability-vector, $\bar{p}_\iota$ being the *a priori* probability that F_i is true, the statistician minimizes the expected risk and $\underset{r}{\text{Min}}\ \bar{p} \cdot r$ gives the Bayes solution relative to the *a priori* distribution $\bar{p}$, it is a point of $\mathbb{R}^{min}$ (conversely every point of $\mathbb{R}^{min}$ is a Bayes solution for a properly chosen $\bar{p}$). Therefore r^* is the Bayes solution relative to $\bar{p}^*$ (one of the normals to $\mathbb{R}^{min}$ at r^*); $\bar{p}^*$ is the *a priori* distribution which gives the greatest value to the minimum expected risk, i.e., the least favorable *a priori* distribution.

In the same way that prices were historically first considered as primary data and later only as an indirect theoretical construction with optimal properties, the controversial concept of an *a priori* distribution first taken at its face value is here considered as an indirect construction with intrinsic optimal properties.

The formal analogies between the theories of zero-sum two-person games, statistical decision functions, and resource allocation are valuable since a result obtained in any one of them can have an interesting counterpart in the two others; the differences between their philosophies should however, by no means, be overlooked. In a game we have a clear cut case of naturally antagonistic interests: one player tries to make his gain as great as possible, the other tries to make his loss as small as possible. In a statistical decision problem, according to A. Wald's words, "whereas the experimenter wishes to minimize the risk $r(F, d)$, we can hardly say that Nature wishes to maximize $r(F, d)$. Nevertheless, since Nature's choice is unknown to the experimenter, it is perhaps not unreasonable for the experimenter to behave as if Nature wanted to maximize the risk. But, even if one is not willing to take this attitude, the theory of games remains of fundamental importance for the problem of statistical decisions, since ... it leads to basic results concerning admissible decision functions and complete classes of decision functions" [2-1.6]. In the resource allocation problem the central agency determining $\bar{p}$ is not inert and its behavior, which is at our command, can be chosen precisely to conflict fully with the behavior of the various economic units.

Footnotes

(1) This paper has been written as a continuation of Economics 297. Concepts and notations are therefore exactly the same but for a slight modification: s^m has been replaced by s^*.

(2) Some ratios $\frac{s_h}{s_h^o}$ might be of the form $\frac{o}{o}$; they would be disregarded in the operation described by the right-hand number.

(3) If we were interested only in the fact that the operations Min and Max can be inverted we could give the very short following proof: choose s^* and one of the p^* and show that this is a saddle point [1-13]. This is indeed immediate but hardly enlightening.

(4) The structure of the set $\mathcal{Z}$ makes these antagonistic activities formally different from a zero-sum two-person game in the von Neumann - Morgenstern [1-17] sense.

(5) The theory of which this paragraph and the three next ones give a summary is developed in greater detail and generality in the basic work of A. Wald [2]. Its geometric interpretation was pointed out by J. Wolfowitz at the 1950 Chicago meeting of the Econometric Society in a paper, "Some Recent Advances in the Theory of Decision Functions," which is unfortunately not available in printed form.

References

[1] J. von Neumann - O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, 1947.

[2] A. Wald, Statistical Decision Functions, New York, John Wiley and Sons, 1950.