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Measurable Utility and Social Welfare

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In a recent paper Arrow [1] has shown the impossibility of constructing a social welfare function with certain desirable properties (Conditions I-V and Axioms I and II in Arrow's paper) if the function is to be based on orderings of social states according to individual tastes. It seems natural to inquire whether a basic difficulty encountered might not be due to the attempt to determine the social welfare function from individual orderings rather than from individual utility functions. The latter would seem to contain more relevant information and reservations about using them, that one might formerly have held, seem to me to be fairly well answered if one confines himself to utility measures that satisfy the von Neumann and Morgenstern axioms [2]. Individual utility functions used in this paper are to be understood to satisfy these axioms.

A social welfare function is suggested below which satisfies conditions corresponding closely to the Arrow conditions with one modification. I believe this modification makes the modified conditions more rather than less acceptable. I do not propose that the suggested welfare function represents a final social desideratum, in fact I point out some possible objections to the function and suspect that there may be others. I hope, however, that it does illustrate the suggestion that welfare functions based directly on individual utility measures offer more possibilities than those based on individual orderings of the available alternatives.

The social welfare function to be considered is

(1) $W(x) = \prod_{i=1}^{n} \bar{U}_i(x)$ where

$W(x)$ is the social welfare value of social state x[1], $\bar{U}_i(x)$ is the utility of the i^{th} individual in social state x if this utility is zero or positive and is zero if the i^{th} individual's utility is negative. The utility measure of the i^{th} individual ($U_i(x)$) is determined up to a linear transformation by the assumption that it satisfies the von Neumann-Morgenstern axioms. If an origin and a scale are specified, the measure is completely determined. It is here assumed that origins have been determined, the problem of determining them is briefly considered later. Once origins have been determined, it can easily be seen that the individual scales are irrelevant in the application of $W(x)$--i.e., $W(x) > W(y)$ for one set of scales implies $W(x) > W(y)$ for all.

The measure, $W(x)$, is suggested by Nash's recent article [3] on, "The Bargaining Problem." Nash argues that a reasonable solution to a two-person bargaining problem in which individuals freely exchange information about their possibilities for bargaining and their utilities is the solution that maximizes $U_1 \cdot U_2$ in the positive quadrant of a plane in which distances along the two axes represent utilities achieved by the two individuals. This solution follows from the following assumed ([3], p. 159) conditions for a solution --

Let c(S) represent a solution point in a set S of achievable combinations of U_1 and U_2.

(N1) If α is a point in S such that there exists another point β in S with the

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1. Possible definitions of a social state are discussed by Arrow [1], p. 331. For the present it will be considered that a social state is a specification of the goods and services available to and rendered by each of the n individuals in a society and that each individual's utility depends only on the goods and services available to and rendered by him. Possible consequences of relaxing this restriction are not examined here.

property $U_1(\beta) > U_1(\alpha)$ and $U_2(\beta) > U_2(\alpha)$, then $\alpha \neq c(S)$.

(N2) If the set T contains the set S and c(T) is in S, then c(T) = c(S).

(N3) S is said to be symmetric if there exist utility operators U_1 and U_2 such that when $U_1 = a$, $U_2 = b$ is contained in S, $U_1 = b$, $U_2 = a$ is also contained in S. If S is symmetric and U_1 and U_2 display this, then c(S) is on the line $U_1 = U_2$.

If the set S is assumed to be convex and compact, it follows that the solution is the one which maximizes $U_1 \cdot U_2$. The argument can readily be generalized to more than two dimensions giving $\prod_{i=1}^{n} U_i$ as the quantity to be maximized. In his problem, Nash takes as a natural zero point for each individual utility function that utility which the individual would enjoy if no bargaining took place. Arguments in favor of Nash's conditions and his zero points for the utility measures do not necessarily apply in the present context. The reference is introduced mainly to indicate how I happened to consider the welfare function, W(x). However, his conditions extended to n persons do seem to me to have considerable appeal in the present context. N1 is necessary in any case. N2 says that if a state x is socially more desirable than any other in a set T, it should also be more desirable than any other in a set S where $x \in S$, $S \subset T$. A similar idea proposed by Arrow is called by him independence of irrelevant alternatives and is discussed below. N3 essentially says that individuals whose utility functions are indistinguishable shall be treated alike.

To verify that W(x), with one exception, satisfies conditions corresponding to the Arrow conditions is trivial but requires that part of his notation be introduced. He lets R_i represent the ordering of states x, y, z,.... by the i^{th} individual and R the social ordering derived from $R_1, \ldots, R_n$. The conditions are stated for a society of two individuals. He has --

Condition 1: The social welfare function is defined for every admissible pair of individual orderings, R_1, R_2.

To this corresponds —

Condition 1': $W(x)$ is defined for every admissible set of individual utility measures, $U_1, U_2, \ldots, U_n$.

Similarly —

Condition 2: If an alternative social state x rises or does not fall in the ordering of each individual without any other change in those orderings and if x was preferred to another alternative y before the change in individual orderings, then x is still preferred to y.

Condition 2': If an alternative social state x rises or does not fall in the utility measure of each individual, then $W'(x) \geq W(x)$ where $W'(x)$ indicates the welfare value of x after the change.

Definition 4: A social welfare function will be said to be imposed if for some pair of distinct alternatives x and y, x is preferred or equal to y in the social ordering for any set of individual orderings R_1, R_2.

Condition 4: The social welfare function is not to be imposed.

Definition 4': $W(x)$ is imposed if for some pair of distinct alternatives x and y, $W(x) \geq W(y)$ for any set of individual utility functions $U_1, \ldots, U_n$.

Condition 4': Same as 4.

Definition 5: A social welfare function is said to be "dictatorial" if there exists an individual i such that for all x and y, x is preferred to y in R if x is preferred to y in R_i regardless of the orderings of all other individuals.

Condition 5: The social welfare function is not to be dictatorial.

Definition 5': A social welfare function is said to be dictatorial if for all x and y, $U_i(x) > U_i(y)$ implies $W(x) > W(y)$ regardless of the utility functions of all other individuals.

Condition 5': Same as 5.

That W(x) satisfies 1', 2', 4', 5' seems obvious. Consider Condition 3: Let R_1, R_2 and R_1', R_2' be two sets of individual orderings. If, for both individuals i and for all x and y in a given set of alternatives S, x is preferred or equal to y by individual i in R_i if and only if x is preferred or equal to y in R_i', then the social choice made from S is the same whether the orderings are R_1, R_2 or R_1', R_2'. This is called independence of irrelevant alternatives. In this form I regard its desirability as questionable. If we imagine two cases and say that in Case I, individual 1 barely prefers x to y while for individual 2 the preference for y over x is a matter of life and death; while in Case II, individual 1 has a desperate preference for x over y and individual 2 barely prefers y to x, then it would seem to be a nice property of a social welfare function that it could make y preferable to x in Case I and x preferable to y in Case II. Condition 3 clearly says more than that the social ordering is independent of irrelevant alternatives. It also says that the social ordering is independent of the strength of individual desires within given individual orderings. This is a necessary and, to me, undesirable consequence of basing the social welfare function on individual orderings rather than utilities. It is here that the procedure seems to discard relevant information.

A condition of independence of irrelevant alternatives for the case where the welfare function depends on utilities could be stated --

Condition 3': The welfare value of any social state x is independent of the other states with which x is compared.

W(x) clearly satisfies this condition.

It might be argued whether or not W(x) involves inter-personal comparison of utility -- I would say that it does and that it is thus not an exception to Arrow's conclusion that the only "satisfactory"[2] social welfare functions not involving inter-personal comparisons are either imposed or dictatorial. While $W(x)$ has the desirable properties indicated above, questions might well be raised about other of its properties. The ordering of social states under $W(x)$ clearly depends on the choice of zero points for the individual utility functions. If we were to seriously consider applying $W(x)$, this choice would have to be specified. Various proposals might be made and the choice among them would be largely an ethical problem. By analogy with the Nash problem, we might consider letting $U_i(x) = 0$ where x is the existing social state. This would resemble in some respects the compensation principle with compensation actually carried out. The only social states that would have nonzero social value would be those in which every individual were improved somewhat. Some have objected to the compensation principle on the ground that it attaches excessive ethical importance to the status quo and the same objection could be raised against taking existing utilities as zero points in defining $W(x)$.

An alternative would be to imagine an individual state so horrible that we would be willing to regard our social system a failure so long as it placed any individual in this state. This utility attached to this minimal individual state by each individual could be taken as the zero point of his utility function. Something like this would seem to be suggested by the fact that $W(x) = 0$ whenever $U_i(x) \leqq 0$ for some i.

However the zero points are chosen, one prospective weakness of a function like $W(x)$ is its vulnerability to individual caprice or perversion. Suppose, for example,

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2. A "satisfactory" social welfare function is one that --

(a) is defined for every admissible set of individual preferences.

(b) determines one of the following relations between any two social states x and y--x is preferred to y, y is preferred to x, or x is equivalent in preference to y.

(c) determines a transitive preference relation.

we have chosen a very undesirable individual state and said that it should have zero utility for each individual. Suppose we now have an unusual individual who regards all states as equally undesirable unless he is given wealth equal say to 90 percent of the world's resources. Adoption of $W(x)$ will require that we give him such resources to maximize social welfare. At present, society limits the freedom of choice of certain unusual individuals more than that of normal individuals and some such modified treatment might be developed for a social welfare function. This would probably not be easy.

It is interesting to note that $W(x)$ does not satisfy the postulates for an individual preference function. In particular, $W(x) = W(y)$ does not imply that $W(1/2\ x + 1/2\ y) = W(x)$ where $1/2\ x + 1/2\ y$ is a prospect that x will occur with probability 1/2 and y will occur with probability 1/2. This may not be a serious objection.

If we imagine that a social welfare function based on individual utility measures has been developed and accepted, we might raise a question as to how it would be applied. We would want to maximize $W(x)$ subject to certain restrictions. The nature of our technology and available resources would certainly be one restriction or set of restrictions. Clearly we would not expect to determine each individual's utility function and then calculate the maximum $W(x)$. We might hope to learn enough about individual utility functions by inductions such as those of Friedman and Savage [4] or experiments such as those of Mosteller to say which of various public or private policies can be expected to lead to higher social welfare as indicated by our function. The reasoning might resemble that invoked to show that the weak Pareto conditions are satisfied in a purely competitive economy.

There might be some choice as to whether certain considerations should be incorporated into the welfare function or into the restrictions. Suppose, for example, that someone criticizes W(x) on the ground that there is an ethical presumption that an individual's rewards should be an increasing function of his contributions and that this is left out of W(x)[3]. If we wished to take this into account, we could either say that the individual utility functions should contain as arguments other peoples rewards and contributions as well as their own. To the extent that the ethical presumption above was recognized by the members of society it would influence their utility functions, W(x), and the social state selected. Another alternative would be to leave W(x) independent of these considerations and to formulate rules about the permissable relations between rewards and contributions. These would then be imposed as constraints along with the technical constraints mentioned above.

I feel undecided about the prospects for developing a useful social welfare function along the lines suggested in this paper. I do believe that the prospects should be pursued somewhat farther and that the preliminary examination sketched above combined with Arrow's work makes the prospect for a useful social welfare function based on individual utility measures rather more hopeful than a function based on individual orderings.

[1] Kenneth J. Arrow, "A Difficulty in the Concept of Social Welfare," Journal of Political Economy, vol. LVIII, no. 4, p. 328.

[2] John von Neumann and Oskar Morgenstern, Theory of Games and Economic Behavior, Princeton: Princeton University Press, 1944.

[3] John F. Nash, Jr., "The Bargaining Problem," Econometrica, vol. 18, no. 2, p. 155.

[4] Milton Friedman and L.J. Savage, "The Utility Analysis of Choices Involving Risk," The Journal of Political Economy, vol. LVI, no. 4, page 279.

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3. To the extent that people work harder if their rewards are an increasing function of their contributions, the desirability of this relation is included in the technical constraints.

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I. In the first paragraph of Economics 2002 the appropriateness of the von Neumann and Morgenstern utility measure as a basis for a social welfare function is taken for granted. While it does have some intuitive appeal for me, I now feel that its use in this context does require additional justification. Such justification is not attempted here.

In what follows an attempt is made to formalize some of the ideas underlying the discussion in the other paper. The idea of a principle that defines individual utility functions up to an arbitrary non-decreasing linear transformation is retained and underlies some of the conditions adopted in section II below. These conditions are modified in section III by assuming that, in addition to the above principle, we also have a principle for selecting origins for individual utility functions, thus defining them except for an arbitrary scale factor. It is also assumed that the origins are specified in such a fashion that it is regarded as intolerable that any individual's utility should be as low as zero.

Let a social state be given by a matrix, x, let a typical element x_{ij} represent the amount of the jth commodity received (or contributed if $x_{ij} < 0$) by the ith individual. Let $x_i, i=1,2,\ldots n$, represent the ith row of x, i.e. the amounts of all commodities received or contributed by the ith individual. Let S represent a set of possible social states.

$U_i(x_i)$ is the utility function of the ith individual. U(x) is a vector

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1. I have benefited from brief discussions with Hurwicz, Savage, and Slater.

of the utilities of all individuals. For a specified state x_o, $U(x_o)$ may be regarded as a point in an n-dimensional Euclidean space which we shall call the utility space. $U(S)$ represents the set of all points in the utility space corresponding to elements of S, a set in the space of social states.

Two utility functions, say U_i and U_j, are said to be indistinguishable if $U_i(x_i) = \alpha + \beta U_j(x_i)$ for all x_i where α and β are constants. If $\alpha = 0$, $\beta = 1$ then U_i and U_j are said to be indistinguishable and normalized.

The following properties of U and S are assumed -

(1.1) Indistinguishable utility functions are also normalized.

(1.2) $U(S)$ is compact and convex.

II. A set of conditions that I believe would be reasonable to propose for a welfare function, $W(x)$, based on utility measures that are determined up to a linear transformation are the following -

(2.1) $W(x)$ is real valued and is defined for all U,S with properties (1.1) and (1.2).

(2.2) $U(x) \geqq U(y)$ implies $W(x) \geqq W(y)$.

(2.3) $U(x) \geqq U(y)$ implies $W(x) > W(y)$.

(2.4) The order of $W(x)$ and $W(y)$ is unaffected if $U(x)$ is replaced by $v(x) = a + b' U(x)$ where a is a vector of constants and b' is a vector of positive constants.[2]

(2.5) There exists an $x^s \in S$ such that $W(x^s) \geqq W(x)$ for all $x \in S$.

(2.6) If U_i, U_j are indistinguishable, then $U_i(x_i^s) = U_j(x_j^s)$.

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2. If U_i, U_j are indistinguishable then we will have to have $a_i = a_j$, $b_i = b_j$ or $v(x)$ will not satisfy (1.1).

I conjecture that there is no function that satisfies these conditions.

III. To reflect the situation in which we assume that origins for the individual utility functions have been specified in such a fashion that we regard only the positive orthant of the utility space as relevant, the following condition on S might be added -

(1.3) $x \in S$ implies $U(x) > 0$.

Condition (2.4) is altered to read -

(2.4') The order of $W(x)$ and $W(y)$ is unaffected if $U(x)$ is replaced by $v(x) = b' U(x)$. The other conditions of sections I and II continue to hold.

Under these modified conditions, it is possible to show that -

(3.1) x^s is an element of S that maximizes $\prod_{i=1}^{n} U_i(x)$.

U(S) is said to be symmetric if permuting the elements of any point in U(S) yields a point also in U(S). It will first be shown that if U(S) is symmetric, (3.1) holds. Consider the special case in which all of the individual utility functions are indistinguishable and arbitrary permutations of the rows of S yield x's that are also in S. U(S) will then be symmetric and by (2.6) $U_i(x_i^s) = U_j(x_j^s)$ for all i, j. Let k be the maximum of U_i subject to $U_i = U_j$ $i,j = 1,\ldots,n$. By (2.3) $U_i(x_i^s) = k$. Let U(T) be the portion of the utility space bounded by the axes and the plane $\sum_{i=1}^{n} U_i = nk$. The point $U_i = k$ $i = 1,2,\ldots,n$ maximizes $\prod_{i=1}^{n} U_i$ over U(T). Since U(S) is convex $U(S) \subset U(T)$ and $U_i = k$ maximizes $\prod_{i=1}^{n} U_i$ over U(S).

Thus (3.1) holds for the special case.[3] But since by (2.2) W(x) depends only on U(x), (3.1) holds for any symmetric U(S).

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3. The remainder of the proof is but a slight modification of that given by Nash.

Now consider the point that maximizes $\prod_{i=1}^{n} U_i$ for any U(S). Transform this point to the point (1,1,...,1) by a transformation of the form v = b'U. Clearly (1,1,...,1) maximizes $\prod_{i=1}^{n} v_i$ both in v(S) and v(T) where the latter is the portion of the v - space bounded by the axes and the plane $\sum_{i=1}^{n} v_i = n$. Since v(T) is symmetric, this point maximizes W over v(T). Since $v(T) \supset v(S)$, W is also maximized over v(S). By (2.4') the same point in S will maximize W in the original utility set U(S).