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Risky Assets and Optimal Tax Policy

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This paper outlines parts of a study which is concerned with governmental strategies to influence entrepreneural decision.

We deal with a simplified economic situation in which the government and a single firm are the acting units. The firm is assumed to choose its investments among various assets, subject to a borrowing limit, in such a way as to maximize expected net profits. The government is supposed to tax the firm's profits, subject to some conditions on the tax schedule, specifying, say, a) that a fixed tax revenue is obtained, b) that the firm is induced to follow a certain investment policy, viz. expand investment in risky assets.

1. We define the following concepts:

1.1. The set A of admissible assets is the set of vectors a, $a = (a_1, \ldots, a_m)$ such that $0 \leq a$ and a restriction $k(a) = 0$ is satisfied.

1.2. The probability density function of a (gross) income-vector $Y = (Y_1, \ldots, Y_n)$ for a given asset vector a is a continuously differentiable function $\varphi = \varphi(a,Y)$.

1.3. The set of admissible tax functions, denoted by $Y - u(Y)$, is the set of real valued, sectionally continuous, sectionally smooth functions of Y with u satisfying the following conditions:

1.3.1. u is defined on the interval of the reals $\ell \leq Y \leq s$ with finite, properly chosen ℓ and s,

1.3.2. $|u| \leq |Y|$ and is monotonically nondecreasing $u' \geq 0$.

1.3.3. The tax revenue $\mathcal{E}[Y - u(Y)]$ equals a given number c.

1.4. u of 1.3 may be properly called net profit. We suppose entrepreneurs' strategy to be the maximization of the expected value $v = v(a)$ of net profit on the admissible set of a.

1.5. With any admissible function u we associate a value a' of the first component of the asset vector a in the following fashion: a' is the minimum of components a_1 on the (closed) set of vectors a* for which v attains the absolute maximum, that is $v(a^*) \geq v(a)$, all admissible a. In other words, a' is the least amount of asset a_1 induced by a tax function Y - u.

1.6. The present purpose may then be defined as to find such functions u that will maximize the asset a' associated with them. However it will turn out, that we have to content ourselves with criteria that enable the finding of such functions while the precise nature of these functions remains undetermined. In another class of cases it appears not even possible to prove the existence of a positive a' associated with a suitable function u. Finally there are cases in which there exists no positive a' associated with an admissible u. It seems then that in terms of this general approach, the set of all problem solutions u cannot be completely specified. In order to obtain necessary and sufficient conditions on u, recourse must be had to more specific models.

2. We shall limit the subsequent consideration to the case of 2 assets only, a risky and a safe one, say plant and bonds, respectively. As will be indicated in the final section this does not involve a significant loss of generality, since the arguments may be easily extended.

The following notations are being used from here on:

a the asset 1, formerly a_1, with risky returns (plant) measured in units of price ($).

b the safe asset 2.

k the borrowing limit, giving rise to a constraint $a + b = k$.

Z the gross return from asset 1.

z the expected value of Z.

y the gross return, equal to the expected gross return, from asset 2.

q the linear productivity of asset 2.

φ the probability distribution of returns Z from assets a such that $\varphi(a,Z)dZ$ denotes the probability of getting a gross return Z from the amount a of asset 1.

Y the aggregate gross return from both assets.

m_Y its expected value.

$u = u(Y)$ the net profit-after-taxes.

$v = v(a,b)$ the expected net profit from an investment of a,b in assets 1,2.

ℓ, s the infimum and supremum of both Y and Z which may without loss of economic generality be assumed finite.

c the tax revenue level.

From these definitions the following identities are immediate:

(1) $y = q \cdot b = q(k-a)$

(2) $z = \int_{\ell}^{s} Z \; \varphi(a,Z)dZ$

(3) $Y = Z + q \cdot (k-a)$

(4) $$m_Y = \int_{\ell}^{s} Z \; \varphi(a,Z)dZ + q \cdot b = \int_{\ell}^{s} Y \; \varphi(a,Z)dZ$$
$$= \int_{\ell-qb}^{s-qb} Y \; \varphi(a, Y-qb)dY = \int_{\ell}^{s} Y \; \varphi(a, Y-qb)dY$$

the latter by the definitions of ℓ, s.

(5) $v(a,b) = \int_{\ell}^{s} u(Y)\ \varphi(a, Y-qb)dY = \int_{\ell}^{s} u(Z+qb)\ \varphi(a,Z)dZ$

(6) $\int_{\ell}^{s} \frac{d\varphi}{da}\, dZ = \frac{d}{da} \int_{\ell}^{s} \varphi\, dZ = 0$

(7) $\int_{\ell}^{s} \frac{d\varphi}{dZ}\, dZ = \varphi(s) - \varphi(\ell) = 0$

The economic laws of returns are expressed in

(8) $\frac{\partial\, v(a,b)}{\partial a} > 0 \qquad \frac{\partial\, v(a,b)}{\partial b} = q > 0$

(9) $\frac{\partial^2 v(a,b)}{\partial a^2} \leq 0$.

3. Note first, that in general not any asset holding $a*$ is inducible by some properly chosen tax. For it may happen, that $m_Y(a*, k-a*) < 0$. Then by assumption 1.3.2 $\int u\varphi\, dZ \leq \int Y\varphi\, dZ$ and hence $v(a*, k-a*) < 0$. And a tax may not induce entrepreneurs to bear a permanent loss rather than to close down. Subsequently we assume that $m_Y(a*, k-a*) \geq 0$. Since entrepreneurs are assumed to maximize profits (assumption 1.4) any tax pertaining to an investment of $a*$ ($< k$) must satisfy a necessary condition.

(10) $\left.\frac{dv(a,\ k-a)}{da}\right|_{a=a*} = 0$; if $a* = k$ we have instead $\left.\frac{dv(a,\ k-a)}{da}\right|_{a=k} \geq 0$.

With (5)

$$\frac{d}{da} \int_{\ell}^{s} u(Y)\ \varphi[a, Y-q(k-a)]dY = \int_{\ell}^{s} u(Y) \left[\frac{\partial\varphi}{\partial a} + q\,\frac{\partial\varphi}{\partial Y}\right] dY = 0.$$

It is useful to have this criterium in a slightly modified form. By partial integration of the brackets one obtains

$$\frac{dv}{da} = u(s) \int_{\ell}^{s} \left[\frac{\partial\varphi}{\partial a} + q\,\frac{\partial\varphi}{\partial Z}\right] dZ - \int_{\ell}^{s} \frac{du}{dY} \int_{\ell}^{s} \left[\frac{\partial\varphi(a,t)}{\partial t} - q\,\frac{\partial\varphi(a,t)}{\partial t}\right] dt\ dZ.$$

By (6), (7) the first term vanishes and we have

(11) $\frac{dv(a,\ k-a)}{da} = \int_{\ell}^{s} u' \int_{\ell}^{z} \left[-\frac{\partial\varphi}{\partial a} - \frac{\partial\varphi}{\partial Y}\right] dt\ dZ = \int_{\ell}^{s} u'\ \rho(a,Z)dZ = 0$

where we have put $\rho = \int_{\ell}^{Z} \left[-\frac{\partial\varphi}{\partial a} - \frac{\partial\varphi}{\partial Y}\right] dt$.

The necessary condition for a net profit function u to induce an investment of $a = a*$ is thus that

$$(11.a) \qquad \int_{\ell}^{s} u'(z)\ \rho(a,Z)dZ \begin{cases} = 0 \text{ if } a* < k \\ \geq 0 \text{ if } a* = k \end{cases} .$$

Clearly this condition is independent of the absolute value of u'. For with u' also $\alpha u' (\alpha > 0)$ will satisfy it. The following considerations will be entirely based on equations (11).

4. In this section we suppose that the absolute maximum of gross returns m_Y is obtained under a positive investment a_0 in asset 1. We shall show that equations (11) contains a device for finding a tax function that will increase investment in asset 1 beyond that point, and that this tax will not necessarily be a degressive one (with $u'' \geq 0$).

Assume then that

$$(12) \qquad \left.\frac{dm_Y(a,\ k-a)}{da}\right|_{a=a_0} = 0 . \qquad\qquad a_0 > 0$$

From (9) it follows that

$$(13) \qquad \left.\frac{dm_Y(a,\ k-a)}{da}\right|_{0 \leq a \leq a_0} \geq 0 .$$

In terms of $\rho(a,Z)$ equation (12), (13) mean that

$$(14) \quad \left.\int_{\ell}^{s} \rho(a,Z)dZ\ \right|_{0 \leq a \leq a_0} \geq \left.\int_{\ell}^{s} \rho(a,Z)dZ\right|_{a=a_0} = 0$$

with $\rho(a,\ell) = \rho(a,s) = 0$. Hence if not $\rho \equiv 0$, a trivial case which we may exclude, there is a Z with $\frac{\partial \rho}{\partial Z} > 0$. Consider the partition of the interval $[\ell,s]$ into a nonempty set E_1 of intervals with $\frac{\partial \rho}{\partial Z} > 0$ and the set E_2 of intervals with $\frac{\partial \rho}{\partial Z} \leq 0$.

Choose $u'' > 0$ on E_1 and $u'' \leq 0$ on E_2. For the following we need an inequality due to Tschebyscheff (cf. Goursat, <u>A Course of Analysis</u>, vol. 1,

p. 257); if $\alpha' \geq 0$, $\beta' \geq 0$ then $\int dt \int \alpha\beta\, dt \geq \int \alpha\, dt \int \beta\, dt$. Thus $\int_{E_1} u' \rho\, dZ + \int_{E_2} u' \rho\, dZ > 0$. Hence $\left.\frac{dv}{da}\right|_{a=a_0} > 0$. If u' is chosen sufficiently close to a function γY, γ = const. then the sufficient conditions (14) remain unviolated, by continuity, and the tax function expanding investment beyond a_0 is constructed. Since the Tschebyscheff inequality is reversible — if $\alpha' \leq 0$, $\beta' \geq 0$ then $\int dt \int \alpha\beta\, dt \leq \int \alpha\, dt \int \beta\, dt$ — it is seen that the construction suggested is an efficient one in the sense that, other things being equal, a different choice of u" on the intervals considered would yield a smaller value of $\frac{dv}{da}$.

This method may still apply beyond the point where the sufficient condition $\left.\frac{dv}{da}\right|_{a \leq a*} \geq 0$ ceases to hold, but it does not carry beyond $\bar{a}$ with $\rho(\bar{a}, Z) \leq 0$ identically in Z. As further mentioned below (Section 6) we do not have sufficient conditions to decide on the existence of an absolute maximum of v beyond that point.—The construction of u in this section will not generally yield a degressive tax (with $u'' \geq 0$). There are examples, where a degressive tax will even decrease risky investments, as the following case shows:

As before suppose $\int_{\ell}^{s} \rho(a_0, Z)\, dZ = 0$, $\rho(\ell) = \rho(s) = 0$.

Let E_1 with $\rho' > 0$ be the interval $\ell \leq Z \leq m$

and E_2 with $\rho' \leq 0$ the interval $m \leq Z \leq s$.

Choose $u'' = 0$ on E_1; $u'' \geq 0$, but not $u'' \equiv 0$ on E_2, so that $u'' \geq 0$ everywhere.

Now

$$\int_{E_1} u' \rho\, dZ = u'(m) \int_{E_1} \rho\, dZ; \quad \int_{E_2} u' \rho\, dZ \leq \frac{\int_{E_2} u'\, dZ \int_{E_2} \rho\, dZ}{\int_{E_2} dZ} < u'(m) \int_{E_2} \rho\, dZ$$

because $\int_{E_2} \rho\, dZ \leq 0$ and $u'(m) \leq u'(Z)$; $Z \in E_2$.

Hence $\int_{E_1 \cup E_2} u' \rho\, dZ < u'(m) \int \rho\, dZ = 0$, $\left.\frac{dv}{da}\right|_{a=a_0} < 0$

and $\dot{Y}$ - u degressive.

5. In this section we make the weaker assumption that there is a Z, $\ell < Z < s$ such that $\rho(0,Z) > 0$. This is certainly implied by the assumption of 4. For $\left.\frac{dm_Y}{da}\right|_{a=0} = \int_\ell^s \rho(0,Z)dZ \geq \int_\ell^s \rho(a_0,Z)dZ = 0$ secures the existence of some Z with $\rho(0,Z) > 0$.

By continuity of ρ it follows that there exists a whole environment $0 \leq a \leq a^*$ such that $\rho(a,Z) > 0$ on an interval E_0: $Z_1 \leq Z \leq Z_2$.

Choose $u' > 0$ on E and $u' = 0$ (or rather sufficiantly close to zero $0 < u' < \mathcal{E}$) otherwise. Clearly $\int_\ell^s u' \rho(a,Z)dZ > 0$ in $0 \leq a \leq a^*$ and the necessary and sufficient conditions for $v(a^*,k-a^*)$ to be the absolute maximum on $0 \leq a \leq a^*$ are satisfied. Hence there is at least one tax function possible that induces a positive investment in asset 1. Obviously these considerations do not carry far. What may be said in addition is contained in the next section.

6. If $\rho(0,Z) \leq 0$ for all Z $\ell \leq Z \leq s$, then our approach does not offer sufficient conditions for the existence of a tax function associated with a positive investment a*. Note however that if there is any tax function at all pertaining to a positive asset a*, then u must satisfy the necessary conditions for a relative maximum of v. $\int u' \rho \, dZ = 0 \quad u' \geq 0$. Hence, if $\rho(a,Z) \leq 0$ for all $a \geq \bar{a}$, then there is no u possible for any $a^* \geq \bar{a}$.

Of course the condition of 4 is not satisfied in the present case since $\left.\frac{dm_Y}{da}\right|_0 = \int_\ell^s \rho(0,Z)dZ < 0$. Hence there would be no investment in asset 1 at all in absence of taxation or under a proportional tax.

7. Assume that a function u has been found that yields an investment a* in the first asset. Under what conditions does it also comply with the postulate 1.3.3 to yield a given revenue?

Obviously $\mathcal{E}(Y - u) = m_Y - v$. Now with v all functions $\alpha v (\alpha > 0)$ will satisfy the necessary and sufficient conditions based on equation (11). Thus if and only if $c < m_Y$ there is a sufficiently small α with $c = m_Y - \alpha v$. $m_Y > c$ is therefore the only effective limit to the tax revenue. But this is a restriction on the admissible set of a rather than on u. This restriction reads explicitly $\int_{\ell}^{s}\int_{\ell}^{s}\left[-\frac{\partial\varphi(a,Z)}{\partial a} - q\frac{\partial\varphi}{\partial Y}\right] dt\, dZ \geq c$.

8. With regard to the taxation of losses we may distinguish 3 cases.

8.1. No loss carry-over. Then all results apply evidently.

8.2. Complete carry-over. No tax may be collected as long as losses are not yet made up. When the first gain enters it may be distributed over the loss period and the average be taxed. Then the domain of u is entirely positive and $\varphi(a,Z)$ is continuous in $Z = 0$. Obviously the previous considerations are applicable.

8.3. Partial carry-over. Again there may be no taxation if a loss is being made. Losses may be deducted from profits in a limited period of subsequent years -- then final losses are still possible, the domain of Z contains negative values. But Z = 0 becomes a singularity since the net profit zero (--if losses are deducted from subsequent gains--) attains a positive (non-infinitesimal) probability.

$\int_{\ell}^{s} Y\, d\phi(a,Z) = \int_{\ell}^{s} Y \varphi(a,Z)dZ + \varphi_0 \cdot 0$. Hence $\int_{\ell}^{s}\varphi\, dZ < 1$ and $\frac{d}{da}\int_{\ell}^{s}\varphi\, dZ \neq 0$. The partial integration leading to (11) cannot be made. It appears difficult to build this case into the framework so far presented.

9. In concluding let us briefly sketch the extension of previous considerations to the case of several (risky) assets, and of a more general constraint.

Let the distribution of returns be given by

$$\varphi = \varphi(Y_1, Y_2, \ldots, Y_n; a_1, a_2, \ldots, a_n)$$

and a constraint be imposed on the assets of the general form

$$k(a_1, \ldots, a_n) = 0 \quad \text{with} \quad \frac{\partial k}{\partial a_i} > 0, \qquad i = 1, \ldots, n.$$

An interior maximum of v is characterized by

$$(15) \quad 0 = \frac{\partial}{\partial a_i}\left\{\iiint\ldots\int u\,\varphi\,dY_1\,dY_2, \ldots, dY_n + \lambda k\right\} = \iiint\ldots\int \frac{\partial\varphi}{\partial a_i}\,dY_1, \ldots, dY_n + \lambda\frac{\partial k}{\partial a_i} \quad \text{for all } i.$$

If a_1 is a corner point, then $\iint\ldots\int u\,\frac{\partial\varphi}{\partial a_1}\,dY_1, \ldots, dY_n + \lambda\frac{\partial k}{\partial a_1} \geq 0$

and possibly $\iint\ldots\int u\,\frac{\partial\varphi}{\partial a_i}\,dY_1, \ldots, dY_n + \lambda\frac{\partial k}{\partial a_i} \leq 0$ for some i.

Hence $-\frac{1}{\frac{\partial k}{\partial a_1}}\iint\ldots\int u\,\frac{\partial\varphi}{\partial a_1}\,dY_1, \ldots, dY_n \leq \lambda \leq -\frac{1}{\frac{\partial k}{\partial a_i}}\iint\ldots\int u\,\frac{\partial\varphi}{\partial a_i}\,dY_1, \ldots, dY_n$

or $\iint\ldots\int u\left(\frac{\partial\varphi}{\partial a_1}\cdot\frac{\partial k}{\partial a_i} - \frac{\partial\varphi}{\partial a_i}\frac{\partial k}{\partial a_1}\right)\,dY_1, \ldots, dY_n \geq 0.$

Through the transformation $Y = \sum_{i=1}^{n} Y_i$, $Y_2 = Y_2, \ldots, Y_n = Y_n$

$$\varphi(Y_1, \ldots, Y_n) = \Psi(Y, Y_2, \ldots, Y_n)$$

we obtain $\iint\ldots\int u(Y)\left[\frac{\partial\Psi}{\partial a_1}\cdot\frac{\partial k}{\partial a_i} - \frac{\partial\Psi}{\partial a_i}\frac{\partial k}{\partial a_1}\right]dY_1, \ldots, dY_n \geq 0$

$$i = 2, \ldots, n.$$

Again since the $\frac{\partial k}{\partial a_1}$ are constants with respect to Y, Y_i and

$$\iint\ldots\int \varphi\,dY_1, \ldots, dY_n = 1 \qquad \text{and } \Psi(s,s,\ldots,s) = \Psi(\ell,\ell,\ldots,\ell) = 0$$

the first term, obtained by partial integration, vanishes

$$\int_{\ell}^{s} u\,dY\int_{\ell}^{s}\ldots\int_{\ell}^{s}[\quad]\,dY_2, \ldots, dY_n = u(s)\int_{\ell}^{s}\ldots\int_{\ell}^{s}[\quad]\,dY\,dY_2, \ldots, dY_n$$

$$-\int_{\ell}^{s} u'(Y)\int_{\ell}^{y}\ldots\int_{\ell}^{s}[\quad]\,dY\,dY_2, \ldots, dY_n$$

$$= - \int_{\ell}^{s} u'(Y) \int_{\ell}^{y} \int_{\ell}^{s} \cdots \int_{\ell}^{s} \left[\frac{\partial \Psi}{\partial a_1} \cdot \frac{\partial k}{\partial a_i} - \frac{\partial \Psi}{\partial a_i} \cdot \frac{\partial k}{\partial a_1} \right] dY\, dY_2, \ldots, dY_n$$

$$= \int_{\ell}^{s} u'(Y)\, \rho(Y)\, dY \geq 0$$

and the considerations of Sections 3 through 7 can be employed with little trouble. It also is easily verified that for $n = 2$ and the particular distribution $\psi(a_1\ b_1\ Y_1\ Y_2) = \varphi(a,Y) \cdot \delta(q.b)$ where δ is the Kronecker function with $\int_{\ell}^{z} \delta(qb)dZ = \begin{cases} 0 \text{ if } Z < q.b \\ 1 \text{ if } Z \geq q.b \end{cases}$, the previous case is again obtained.