COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 297

The Coefficient of Resource-Utilization[1]

by Gerard Debreu

November 20, 1950

A numerical evaluation of the dead loss associated with a nonoptimal situation (in the Pareto sense) of an economic system is sought. Use is made of the intrinsic price systems associated with optimal situations of whose existence a noncalculus proof is given. A coefficient of resource-utilization yielding measures of the efficiency of the economy is introduced. The treatment is based on vector sets properties in the commodity space.

1. Introduction

The activity of the economic system we study can be viewed as the transformation by n production-units and the consumption by m consumption-units of ℓ commodities (the quantities of which can or cannot be perfectly divisible). Each consumption-unit, say the i^{th} one, is assumed to have a preference ordering of its possible consumptions, and therefore a numerical index of its satisfaction, s_i. Each production-unit has a set of possibilities defined independently of the limitation of physical resources and of conditions in the consumption sector (depending for example on technological knowledge). Finally the total net consumption of all consumption-units and all production-units for each commodity must be at most equal to the available quantity of this commodity.

If we impose on the economic system the constraints defined by 1) the set of possibilities of each production-unit, 2) the limitation of physical resources, we cannot indefinitely increase the m satisfactions. In trying to do so we would find situations where it is impossible to increase any satisfaction without making at least another one decrease: in any one of these situations all the resources are fully exploited, it can be considered as optimal. When a situation is nonoptimal is it possible to find some measure of the loss involved, indicating how far it is from being optimal? The basic difficulty comes from the fact that each satisfaction function is defined only up to a monotonic increasing function.

For this reason we take up the following dual problem. We impose on the economic system the constraints defined by 1) the set of possibilities of each production-unit, 2) the condition that for each consumption-unit the satisfaction s_i is at least equal to a given value s_i^o. We cannot decrease indefinitely the ℓ quantities of available physical resources. In trying to do so we would find situations where it is impossible to decrease one of them without making at least another one increase: in any one of these situations one has attained the prescribed levels of satisfaction with as little physical resources as possible, it can be considered as optimal. The loss associated with a nonoptimal situation is now a measure of the distance from the actually available complex of resources to the set of optimal complexes; this notion is far more simple than the former one because we are dealing now with <u>quantities of commodities</u>. The two definitions of optimality are equivalent if the saturation cases are excluded.

Using the second definition of optimality we proceed to a noncalculus proof of the intrinsic existence of price-systems associated with the optimal complexes of physical resources, the basic theorem of the new welfare economics. This proof is more general than the usual ones since it does not require the existence of derivatives which indeed do not exist in simple and realistic cases; more complete since it deals with global instead of local properties of maxima or minima; more concise, as the synthetic nature of the problem required it to be; it gives a deeper explanation of the intrinsic existence of prices by its geometric interpretation in the commodity space. These reasons seem to justify the higher level of abstraction on which it is placed.

This proof was based on convexity properties which implied continuity of quantities of commodities; if this assumption of continuity is dropped, the same technique shows that to achieve an optimal situation the use of a (real or virtual) price system is still sufficient but no longer necessary.

We are now prepared to measure the distance from the actually given complex of physical resources to the set of optimal complexes, i.e., the minimum of the distance from the given complex to a varying optimal complex. To evaluate such a distance we multiply for each commodity the difference between the available quantity and the optimal quantity by the price deriving from the intrinsic price system whose existence has been previously proved. We take the sum of all such expressions for all commodities, and we divide by a price index in order to eliminate the arbitrary multiplicative factor affecting all the prices. The minimum is reached for an optimal complex resulting from a reduction of all quantities of the nonoptimal complex in a ratio ρ, coefficient of resource-utilization of the economic system. This number, equal to 1 if the situation is optimal, smaller than 1 if it is nonoptimal, measures the efficiency of the economy and summarizes 1) the underemployment of physical resources, 2) the technical inefficiency of production-units and 3) the inefficiency of economic organization (due for example to monopolies or a system of indirect taxes).

The value of the dead loss associated with a nonoptimal situation can be derived from ρ and the inefficiency of the economy is now described by a certain number of dollars, the value of the physical resources which could be thrown away without preventing the achievement of the prescribed levels of satisfaction. This definition seems to obviate the shortcomings of the older ones.

2. Basic mathematical concepts

Vectors are denoted by bold face lower case types, their components by corresponding ordinary lower case types with a subscript characterizing the coordinate axis. In the finite dimensional Euclidean spaces which we consider we use the following notations for inequalities among vectors:

$$u \geqq v \quad \text{if} \quad u_\iota \geqq v_\iota \quad \text{for every } \iota,$$

$$u > v \quad \text{if} \quad u_\iota > v_\iota \quad \text{for every } \iota,$$

$$u \geq v \quad \text{if} \quad u \geqq v \text{ and } u \neq v.$$

A function $w(u)$ is increasing (resp. non-decreasing) if "$u^2 \geq u^1$" implies "$w(u^2) \geq w(u^1)$ [resp. $w(u^2) \geqq w(u^1)$]."

Sets of vectors are denoted by bold face capital german letters. According to the usual terminology, v is a <u>maximal</u> (resp. <u>minimal</u>) element of $\mathfrak{U}$ if 1) $v \in \mathfrak{U}$, 2) there is no u such that $u \in \mathfrak{U}$ and $v \leq u$ (resp. $v \geq u$). The set of maximal (resp. minimal) elements of $\mathfrak{U}$ is denoted by $\mathfrak{U}^{max}$ (resp. $\mathfrak{U}^{min}$).

The vector sum of a finite number of sets $\mathfrak{U}_\iota$, $\mathfrak{V} = \Sigma_\iota \mathfrak{U}_\iota$, is the set of $v = \Sigma_\iota u_\iota$, $u_\iota \in \mathfrak{U}_\iota$.

A set $\mathfrak{U}$ is convex if "$u \in \mathfrak{U}$, $v \in \mathfrak{U}$, $0 \leqq t \leqq 1$" implies "$tu + (1-t) v \in \mathfrak{U}$."

A set $\mathfrak{U}$ is closed if it contains every point at a zero-distance from $\mathfrak{U}$. All the sets we will consider will be closed (either by assumption or by immediate consequence of postulates) and we will never mention it explicitly again.

A set $\mathfrak{U}^2$ is greater than a set $\mathfrak{U}^1$ if it includes $\mathfrak{U}^1$: $\mathfrak{U}^2 \supset \mathfrak{U}^1$.

3. Description of the economic system

A <u>commodity</u> of the economic system is characterized by a subscript h ($h = 1, \ldots, \ell$). This concept can be given the most various contents: it can be a good or a service, direct or indirect, playing a role in any production or consumption process, for example the training of pilots by some Air Force agency. The quantity of the h^{th} commodity can either vary contınously or be an integral multiple of a given unit. The discontinuous case, which is indeed very widespread, can be easily included in the frame we present as will be shown.

A consumption-unit is characterized by a subscript $i(i = 1, \ldots, m)$; its activity is represented by a consumption-vector x_i of the ℓ dimensional Euclidean commodity-space $\mathcal{R}_\ell$; the components x_{hi} are quantities of commodities actually consumed or negatives of quantities of commodities produced (for a consumer of the classical type the only negative components correspond to the different kinds of labor he can produce). We assume that x_i^1 and x_i^2 being two arbitrary consumption-vectors of the i^{th} consumption-unit, it either "prefers x_i^1 to x_i^2," "thinks x_i^1 equivalent to x_i^2," or "prefers x_i^2 to x_i^1" ($x_i^2 \geq x_i^1$ excluding x_i^1 preferred to x_i^2) with the usual transitivity property. One can therefore construct classes of equivalence (a class of equivalence may happen to contain only one vector), and define a numerical function $s_i(x_i)$, constant in, and only in, each class of equivalence, nondecreasing: this index of the satisfaction of the needs of the i^{th} consumption-unit is defined but for a monotonically increasing function. The m s_i are considered as the components of a vector s of the satisfaction space $\mathcal{R}_m$. The basic features of this reasoning are well known; our purpose was only to reformulate it in a language applicable to more general cases including the discontinuous case. Here again the contents of the concept consumption-unit are left indeterminate, it can be a consumer, a household unit, a governmental agency,...; in an economy provided with a central planning board incarnating a social welfare function there is only one consumption-unit; the whole economic system can be divided in nations among which consumption-units are distributed. The theory to be developed applies to all those cases.

The production activity of the system is represented by the total input-vector $y \in \mathbb{R}_\ell$; the components of y are inputs (net quantities of commodities consumed during the period considered by the whole production sector) or negatives of outputs (defined in a symmetrical way). Constraints such as the limitation of technological knowledge determine the set $\mathbb{Y}$ of possible y. $\mathbb{Y}$ is defined independently of the limitation of physical resources (which will be dealt with later), and of conditions in the consumption sector. The set of efficient vectors in production is $\mathbb{Y}^{\min}$.

A family of sets $\mathbb{Y}_j (j = 1,\ldots,n)$ is a decomposition of $\mathbb{Y}$ if $\mathbb{Y} = \Sigma_j \mathbb{Y}_j$; in other words $y = \Sigma_j y_j$, $y_j \in \mathbb{Y}_j$. The input-vector y_j characterises the activity of the j^{th} production-unit. The concept of production-unit coincides eventually with that of industry, firm, plant,... . This formulation allows for production and consumption of intermediate commodities, even in a circular way, with as many intermediate steps as one wants; it allows, of course, for discontinuities of variables, or, if they are continuous, for nonsmooth surfaces $\mathbb{Y}_j^{\min}$, for the existence of fixed ratios between some variables, etc... . The more usual exposition which amounts to starting from the $\mathbb{Y}_j$ to obtain $\mathbb{Y}$ is valid only if the assumption that $\mathbb{Y}$ is nothing more than $\Sigma_j \mathbb{Y}_j$ is explicitly made. If $y \in \mathbb{Y}^{\min}$, it is necessary but not sufficient that $y_j \in \mathbb{Y}_j^{\min}$ $j = 1,\ldots,n$.

$x = \Sigma_i x_i$ is the total consumption-vector, and $z = x + y$ is the total net consumption of the whole economy (all consumption-units and all production-units), it can come only from the available physical resources: we call it the utilized-physical-resources-vector by opposition to z^o, a vector of $\mathbb{R}_\ell$ whose components are the available quantities of each commodity (natural resources, and services of existing capital for example; new investment would give rise to negative components, and the different kinds of labor to zero components), which we call the utilizable-physical-resources-vector. One has necessarily $z \leqq z^o$.

4. Optimum and loss defined in the satisfaction space.

The constraints imposed on the economic system are

$$y \in \mathcal{Y} \qquad\qquad z \leqq z^o;$$

this determines in $\mathbb{R}_m$ the set $\mathcal{S}$ of attainable s.[2] According to the Paretian criterion, if the goal of the economic system is to make the s_i, which cannot be compared to each other, as great as possible, s^2 is better than s^1 if, and only if, $s^2 \geq s^1$, and s is optimal if, and only if, it is maximal: $s \in \mathcal{S}^{max}$. Any planning board, anxious to satisfy the needs of the consumption-units as well as possible, and confronted with the problem of selecting one s in $\mathcal{S}$, would in fact restrict its choice to $\mathcal{S}^{max}$.

If $s^o \notin \mathcal{S}^{max}$ (Fig. 3a)[3] a dead loss is associated with s^o; its magnitude is, intuitively, the distance from s^o to the set $\mathcal{S}^{max}$ (i.e., the minimum of the distance from s^o to a variable s belonging to $\mathcal{S}^{max}$). The arbitrariness of the functions s_i prevents us from finding a meaningful content for that definition.

5. Optimum and loss defined in the commodity space.

Let us therefore study the following dual problem and consider in $\mathbb{R}_\ell$ $\mathcal{Z}$ the set of z defined by the constraints

$$y \in \mathcal{Y} \quad , \qquad s \geqq s^o.$$

$\mathcal{Z}$ is the set of utilized-physical-resources-vectors which, taking into account the production possibilities $\mathcal{Y}$, enable the economy to achieve at least s^o. Let $\mathcal{X}_i$ be the set of x_i defined by $s_i(x_i) \geqq s_i^o$, and $\mathcal{X} = \Sigma_i \mathcal{X}_i$, since $z = x + y$, $\mathcal{Z}$ is nothing else than $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$.

$\mathcal{Z}^{min}$ is a natural concept: it describes the minimal physical resources required to achieve at least s^o. One sees how z^o can be defined as optimal with respect to $\mathcal{Y}$ and s^o if and only if $z^o \in \mathcal{Z}^{min}$, and how, if $z^o \notin \mathcal{Z}^{min}$(Fig. 3b), the dead loss can be defined as the distance from z^o to the set $\mathcal{Z}^{min}$. This distance can now be meaningful since the coordinates of the commodity space $\mathbb{R}_\ell$ are quantities of commodities.

The definitions of optimum and loss given in 4 and in 5 are not necessarily equivalent but under conditions which amount essentially to excluding the saturation cases[4] "$s^o \in \mathcal{S}^{max}$" is equivalent to "$z^o \in \mathcal{Z}^{min}$." (Fig. 1a and 1d).

6. The optimum theorem.

Let us now assume that the sets $\mathcal{X}_i$, $\mathcal{Y}_j$ are all convex (this implies of course that the quantities of commodities all vary continuously); it follows that $\mathcal{Z} = \Sigma_i \mathcal{X}_i + \Sigma_j \mathcal{Y}_j$ is convex. As for the $\mathcal{X}_i$ the assumption is a classical one and needs no particular comments; as for the $\mathcal{Y}_j$ it may be worth noticing that if one added the two postulates

1) $y_1 \in \mathcal{Y}_j$, $y_2 \in \mathcal{Y}_j$ implies $y_1 + y_2 \in \mathcal{Y}_j$ (additivity postulate)

2) $0 \in \mathcal{Y}_j$

then $\mathcal{Y}_j$ would be a cone.

We now concentrate our attention on a vector $z^o = \Sigma_i x_i^o + \Sigma_j y_j^o$ ($x_i^o \in \mathcal{X}_i$, $y_j^o \in \mathcal{Y}_j$) (Fig. 1). $\mathbb{P}$ being the positive orthant of $\mathbb{R}_\ell$ (the set of vectors of $\mathbb{R}_\ell \geqq 0$), we have the following chain of equivalent propositions:

— $z^o \in \mathcal{Z}^{min}$

— the convex sets $\mathcal{Z}$ and $z^o - \mathbb{P}$[5] have no other point in common than z^o

— there is a plane through z^o separating those two sets

— there is a vector $p > 0$ (normal to the separating plane) such that $z \in \mathcal{Z}$ implies $p.(z - z^o) \geqq 0$

— there is a vector $p > 0$ such that

1) $x_i \in \mathfrak{X}_i$ implies $p.(x_i - x_i^o) \geqq 0$ for every i

2) $y_j \in \mathfrak{Y}_j$ implies $p.(y_j - y_j^o) \geqq 0$ for every j [6]

Proof: $p.(z - z^o) = \Sigma_i\, p.(x_i - x_i^o) + \Sigma_j\, p.(y_j - y_j^o)$, the condition is therefore sufficient; it is necessary for if one term of the right-hand member could be made < 0, as all the other can be made $= 0$, the left-hand member could be made < 0.[7]

Finally 1) is equivalent to

1') $p.(x_i - x_i^o) < 0$ implies $x_i \notin \mathfrak{X}_i$ i.e., $s_i(x_i) < s_i(x_i^o)$ for every i or

1") $p.(x_i - x_i^o) \leqq 0$ implies $s_i(x_i) \leqq s_i(x_i^o)$ for every i.

Interpreting p as a price-vector and defining $a_i = p.x_i^o$ $(i = 1,\ldots,m)$ we have the statement:

<u>The necessary and sufficient condition for z^o to be maximal, or for $z^o = \Sigma_i\, x_i^o + \Sigma_j\, y_j^o$ to be minimal, is the existence of a price-vector $p > 0$ and of a set of numbers a_i $(i = 1,\ldots,m)$ such that</u>

<u>α) x_i being constrained by $p.x_i \leqq a_i$, $s_i(x_i)$ reaches its maximum at x_i^o, for every i</u>

<u>β) y_j being constrained by $y_j \in \mathfrak{Y}_j$, $p.y_j$ reaches its minimum at y_j^o for every j,</u>

a formalization of well-known rules of behavior for consumption-units and production-units.

Given z^o, $\mathfrak{Y}$ and $s^o \in \mathfrak{S}^{max}$, the direction of p is not always uniquely determined:[8] it is only constrained to belong to the set of directions normal to supporting planes for $\mathfrak{Z}$ through z^o which we call briefly the cone of normals. Even if its direction is known, p is determined only up to a multiplication by a positive scalar. Once p is known, the set of m numbers (a_i) is determined.

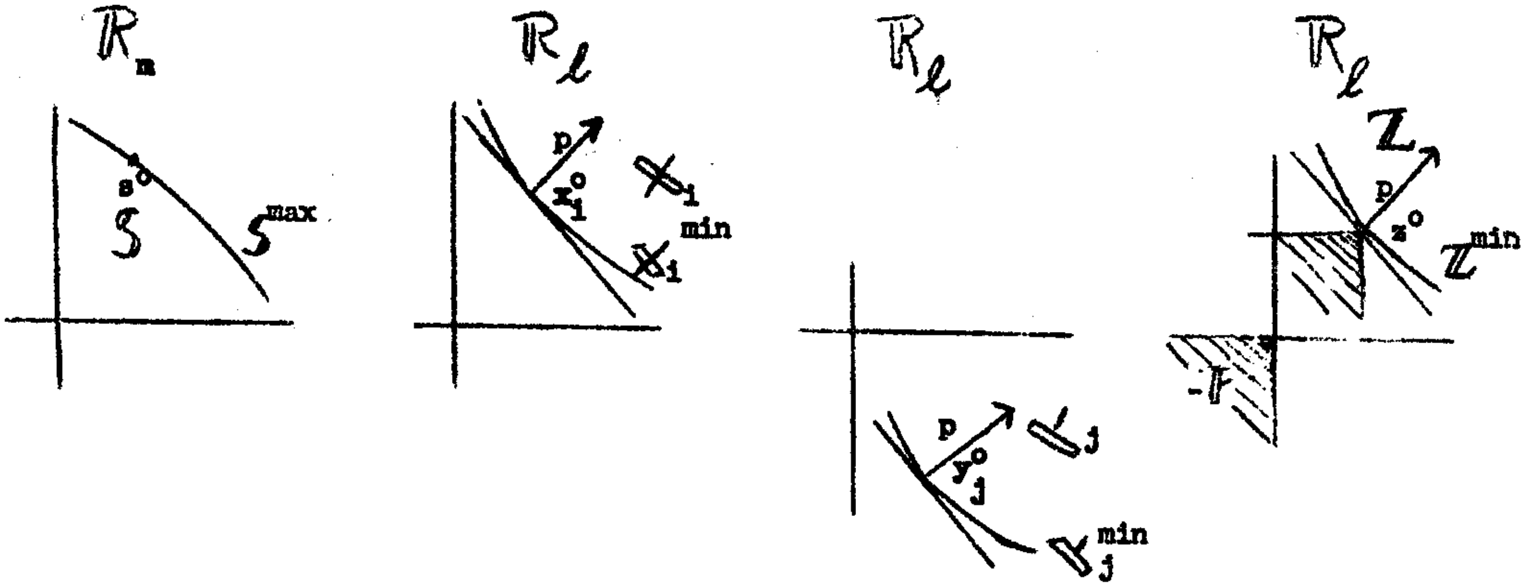

Fig. 1

Given z^o and $\mathcal{Y}$, the different s^o belonging to $\mathcal{S}^{max}$ determine all possible pairs p, (a_i). To attain an arbitrary s^o maximal one can imagine the following procedure: choose (a_i) among its possible values, then find a $p > 0$ such that when

α') every consumption-unit maximizes $s_i(x_i)$ subject to $p \cdot x_i \leq a_i$

β') every production-unit minimizes $p \cdot y_j$ subject to $y_j \in \mathcal{Y}_j$,

the x_i^o and y_j^o thus determined satisfy $\Sigma_i x_i^o + \Sigma_j y_j^o = z^o$.

A proper choice of (a_i) can lead to any given vector $s^o \in \mathcal{S}^{max}$ that one wishes to attain.[9]

7. Historical note.

All proofs of this basic theorem of the new welfare economics published so far were based on the use of the calculus: they required unnecessary restrictive assumptions on the existence of derivatives which cannot be made in the very simple and realistic case of Linear Programming where $\mathcal{Y}$ is a polyhedral cone, moreover, they could at best establish the existence of a local maximal. Indeed they generally limited themselves to the study of first order conditions.

Pareto himself, who defined an optimal s as a maximal s [16], [17] and conceived the set $\mathcal{S}^{max}$ [10] [18], did not establish satisfactorily those conditions in spite of lengthy developments [17]. The gradual improvements brought by Barone [4], Bergson [5], [6], Hotelling [12], [13], Hicks [8], Lange [14], Lerner [15], Allais [1], [2], Samuelson [19], Tintner [23] clarified, made more rigorous, and extended the contents of his writings.

The long and piecemeal treatment consisting in proving that the rates of substitution between any two commodities are independent of the individual, of the industry, etc... failed to comply with the <u>synthetic</u> nature of the problem; moreover, it put the emphasis on the equality of rates of substitution which disappear in most simple cases (polyhedral cones) instead of putting it on the necessary and sufficient existence of a price system (real or virtual), the actually meaningful operational concept. For these reasons the proofs given independently by O. Lange [14] and M. Allais [1], [2] were of particular interest: they were essentially synthetic and some of their Lagrange multipliers could be interpreted immediately as prices, which was done forcefully by M. Allais. However they used an asymmetrical exposition (one individual or one commodity played a particular role) to obtain symmetrical results from symmetrical assumptions; their Lagrange multipliers were a mathematical trick obscuring the more fundamental facts; they had the mentioned weaknesses of calculus proofs.

8. The discontinuous case.

If the quantities of some commodities vary discontinuously, we can, in an attempt to preserve certain properties of convexity, define a <u>quasi-convex</u> set (Fig. 2) as a set which has through each minimal point, at least, one supporting plane. But the assumption that all the $\mathcal{X}_i$ and $\mathcal{Y}_j$ are quasi-convex does not imply that $\mathcal{Z} = \Sigma_i \mathcal{X}_i + \Sigma_j \mathcal{Y}_j$ is quasi-convex. [11]

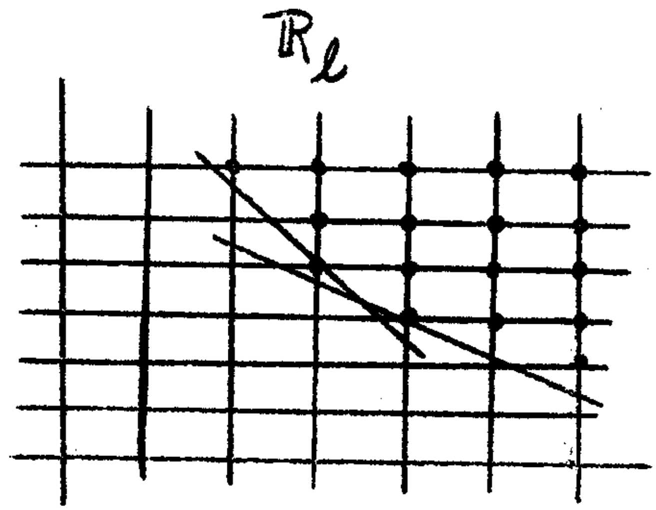

Fig. 2.

In other words, the theorem proved in 6 was based on the additivity property of convexity; quasi-convexity is not additive and the theorem cannot be extended: the existence of a price vector $p > 0$ used according to the rules α) and β) is still sufficient but no longer necessary for z^0 to be maximal.

9. The Coefficient of Resource-Utilization.

We go back therefore to the convexity case and concern ourselves with the measurement of the dead loss associated with a vector $z^0 \notin \mathbb{Z}^{min}$ (Fig. 3). This loss is depicted entirely by and only by the relative positions of z^0 and the set $\mathbb{Z}^{min}$; however, if we want, instead of this complex picture, a simple representation by a number, we define the magnitude of the loss as the distance from z^0 to $\mathbb{Z}^{min}$, i.e., the minimum of the distance from the fixed point z^0 to the point z varying in $\mathbb{Z}^{min}$. To have a distance with an economic meaning we evaluate the vector $z^0 - z$, which represents the nonutilized resources, by the <u>intrinsic</u> price-vector p associated with z, whose existence we proved in 6. We thus obtain $p\cdot(z^0 - z)$. In fact, there can be several p associated with z; it is easy to see that whatever be the one chosen in the cone of normals, the result to be obtained below is <u>the same</u>. No other price vector can be taken for this evaluation for 1) it is quite possible that no price vector exists at all in the concrete economic situation observed (if for example there is no unicity of price for one commodity) and 2) even if there were one, let us say p^o, it would have no intrinsic significance.

Before we engage in a minimization process we must not forget that p is affected by an arbitrary positive multiplicative scalar whose influence we eliminate dividing by the price index $p \cdot z$ or $p \cdot z^o$. It must be remarked that the result to be obtained is again independent of this choice. Indeed the use of $p \cdot z$ has a very intuitive justification: all the points of $\mathcal{Z}^{min}$ have the same "value."

We are led to look for

$$\underset{z \in \mathcal{Z}^{min}}{\text{Min}} \quad \frac{p \cdot (z^o - z)}{p \cdot z} \qquad \text{i.e., for} \quad \underset{z \in \mathcal{Z}^{min}}{\text{Max}} \quad \frac{p \cdot z}{p \cdot z^o}$$

Let z^m be the vector collinear with z^o and belonging to $\mathcal{Z}^{min}$;

$$z^m = \rho \, z^o, \qquad z^m \in \mathcal{Z}^{min}.$$

$$\underset{z \in \mathcal{Z}^{min}}{\text{Max}} \quad \frac{p \cdot z}{p \cdot z^o} = \rho \underset{z \in \mathcal{Z}^{min}}{\text{Max}} \quad \frac{p \cdot z}{p \cdot z^m}$$

But the convexity of $\mathcal{Z}$ insures that

$$p \cdot (z^m - z) \geq 0 \qquad \text{i.e.,} \quad \frac{p \cdot z}{p \cdot z^m} \leq 1;$$

as this ratio is equal to 1 when $z = z^m$, we have

$$\underset{z \in \mathcal{Z}^{min}}{\text{Max}} \quad \frac{p \cdot z}{p \cdot z^o} = \rho \quad ;$$

the maximum is reached at every point $z \in \mathcal{Z}^{min}$ such that a supporting plane through z contains z^m, i.e., at every point z contained in a supporting plane through z^m.

We call ρ defined in the preceding way <u>the coefficient of resource-utilization</u> of the economic system; it is a function $\rho(s^o, z^o, \mathcal{Y})$ describing the efficiency of the economy. In a precise way it is the smallest fraction of the actually available physical resources which would permit the achievement of s^o.

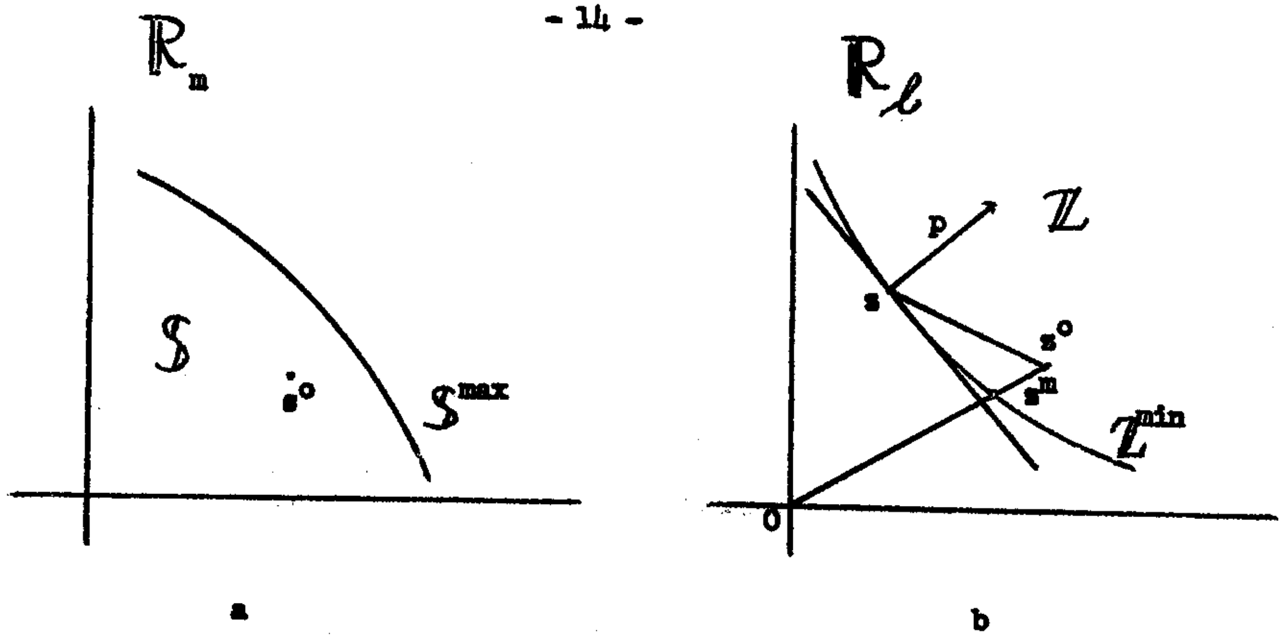

Fig. 3

$\rho = 1$ if and only if s^o is maximal (i.e., z^o minimal).

$\rho < 1$ if and only if s^o attainable is not maximal.

ρ decreases if s^o decreases or if $\mathbb{Y}$ increases for in both cases $\mathbb{Z}$ increases; it is hardly more difficult to see that ρ decreases if z^o increases.

The appelation suggested for ρ has a general content which must be clearly brought out. An economic system has three kinds of resources: 1) its physical resources s^o, 2) its production possibilities $\mathbb{Y}$, and 3) its economic organisation possibilities. If a vector s^o is not maximal there is a loss originating from one or several of the following sources:

1) underemployment of physical resources: unemployment of labor, idle machinery, lands uncultivated by agreement, etc... This is the most obvious, and therefore the least interesting, source of loss. In a very narrow sense a coefficient of resource-utilization would describe only this phenomenon; this is by no means our purpose.

2) Inefficiency in production; $y \notin \mathbb{Y}^{min}$. This kind of loss is already much less obvious but is not, by its very nature, the main concern of the economist.

3) Imperfection of economic organization; if the physical resources are fully

utilized, if the production is perfectly efficient, it is still possible that z^o is not maximal if the conditions of the basic theorem are not satisfied. As it is well known such a case arises, for example, with monopolies or indirect taxation. This kind of loss is the most subtle (in fact may be hardly conceivable to the layman) and therefore the one for which a numerical evaluation is the most necessary.

The coefficient ρ takes into account the three kinds of loss.

The definition of the coefficient of resource-utilization as the ratio of a vector collinear with z^o to z^o can be legitimized in cases more general than the convexity case, and defined in still further general cases, but this would lead to a certain amount of undesirable sophistication.

10. Definitions of the economic loss.

In summary the loss is $z^o - z^m = z^o(1 - \rho)$, its value is $p \cdot z^o(1 - \rho)$, p being the price vector associated with z^m. However, p has no immediate concrete significance and, if a price vector p^o exists in the economic situation actually observed, a more interesting evaluation is probably $p^o \cdot z^o(1 - \rho)$. p^o which was inacceptable in the minimization process leading to z^m is of course acceptable now that an approximate numerical evaluation is sought. Whether one wants the magnitude of the loss due to monopolies (in the absence of other distortions, the total degree of monopoly could be taken as $1 - \rho$), or to a taxation system, the above expression gives the answer under the form of a certain number of billions of dollars. ρ itself, a percentage describing the degree of efficiency of the economy, can be found more useful in some cases.

Since J. Dupuit [7] several definitions of the loss described have been more or less explicitly suggested. A very unsophisticated one is the variation of real national income[12] [9], [20] (according to our notation $p \cdot (x^2 - x^1)$[13]); other ones were directly based on the various notions of consumers' surplus as presented in their modern forms by J. R. Hicks [10], [11]. All of them derived the value of the loss from the comparison of two sets of individual consumptions $(x_1^1,\dots,x_m^1)$ and $(x_1^2,\dots,x_m^2)$ and if those two sets varied in such a way that s^1 and s^2 did not change, the value of the loss varied. This was inconsistent with the paretian philosophy which considers two situations yielding the same vector s as equivalent.[14] Even if this were overcome by the construction of a plausible numerical index of comparison of s^1 and s^2 it would still be unsatisfactory for finding the loss associated with s^1 to compare it with an s^2 arbitrarily selected in $\mathcal{S}^{max}$ instead of comparing it with the set $\mathcal{S}^{max}$.

The treatment of this question by M. Allais [1] avoids this criticism but its exposition and its results rely entirely on the asymmetrical role played by a particular commodity.

References

[1] M. Allais, A la Recherche d'une Discipline Economique, Tome I, Paris, 1943, Chap. IV-E.

[2] M. Allais, Economie Pure et Rendement Social, Paris, Sirey, 1945.

[3] M. Allais, Economie et Intérêt, Paris, Librairie des Publications officielles, 1947, Chap. VI - Annexe III.

[4] E. Barone, "The Ministry of Production in the Collectivist State," in Collectivist Economic Planning, F. A. Hayek, editor, London, Routledge, 1935 (reprinted from Giornale degli Economisti, 1908).

[5] A. Bergson, "A Reformulation of Certain Aspects of Welfare Economics," Quarterly Journal of Economics, Vol. 52, February, 1938, pp. 310-334.

[6] A. Bergson, "Socialist Economics," Chapter 12, in A Survey of Contemporary Economics, H. S. Ellis, editor, Philadelphia, The Blakiston Company, 1948.

[7] J. Dupuit, De l'Utilité et de sa Mesure, Turin, La Riforma Sociale, 1933 (reprints of works published in 1844 and the following years).

[8] J. R. Hicks, "The Foundations of Welfare Economics," Economic Journal, Vol. 49, December, 1939, pp. 696-712.

[9] J. R. Hicks, "The Valuation of the Social Income," Economica, Vol. 7, May, 1940, pp. 105-124.

[10] J. R. Hicks, "Consumers' Surplus and Index-Numbers," The Review of Economic Studies, Vol. 9, Summer, 1942, pp. 126-137.

[11] J. R. Hicks, "The Generalized Theory of Consumer's Surplus", The Review of Economic Studies, Vol. XIII, pp. 68-74.

[12] H. Hotelling, "The General Welfare in Relation to Problems of Taxation and of Railway and Utility rates," Econometrica, July, 1938, pp. 242-269.

[13] H. Hotelling, R. Frisch, Econometrica, April, 1939, pp. 145-160.

[14] O. Lange, "The Foundations of Welfare Economics," Econometrica, July-October, 1942, pp. 215-228.

[15] A. P. Lerner, The Economics of Control, New York, Macmillan, 1944.

[16] V. Pareto, Cours d'Economie Politique, Lausanne, 1897, Vol. II, Nos. 720 to 735 and 1.013 to 1.023.

[17] V. Pareto, Manuel d'Economie Politique, Paris, 1909, VI-32 to 64, IX-35 and Appendice 89 to the end.

[18] V. Pareto, Traité de Sociologie Générale, Paris, 1917-1919, nos. 2.128 to 2.130.

[19] P. A. Samuelson, Foundations of Economic Analysis, Cambridge (Mass.), Harvard University Press, 1947, Chapter VIII.

[20] P. A. Samuelson, "Evaluation of Real National Income," Oxford Economic Papers, Vol. 2, January, 1950, pp. 1-29.

[21] T. de Scitovszky, "A Note on Welfare Propositions in Economics," The Review of Economic Studies, Vol. 9, November 1941, pp. 77-88.

[22] T. de Scitovszky, "A Reconsideration of the Theory of Tariffs," The Review of Economic Studies, Vol. 9, Summer 1942, pp. 89-110.

[23] G. Tintner, "A Note on Welfare Economics," Econometrica, Vol. 14, January, 1946, pp. 69-78.

Footnotes

1. Based on a Cowles Commission Discussion Paper, Economics No. 284 (hectographed), June, 1950, and a paper presented at the Cambridge Meeting of the Econometric Society, August, 1950.

My acknowledgments are due to R. Solow of Massachusetts Institute of Technology, Cowles Commission staff members and their guests, particularly J. Gurland, L. Hurwicz, J. Marschak, M. Slater, J. Leveugle, E. Malinvaud. To nobody is my debt more specific than to T. C. Koopmans, and perhaps greater than to M. Allais whose interest in this kind of question has been the origin of mine.

2. Supplementary constraints such as the existence of some minimum standard of living, $s \geq s^o$, would bring no change into the following analysis.

3. Occasional references will be made to figures. They are all drawn in the two dimensional case and, with one exception, contain only smooth curves while the reasoning deals with a greater number of dimensions, nonsmooth surfaces, and even discrete sets of points. They are therefore mere illustrations loosely connected with the text, but likely to be found useful.

4. This can be done by a few additional simple postulates which we do not discuss in detail for they would lead to very formal developments without throwing more light on the heart of our problem. The saturation case has been considered by K. J. Arrow in the paper, "A Generalization of the Basic Theorem of Classical Welfare Economics," given at the West Coast 1950 summer meeting of the Econometric Society independently of the present paper which was given at the same time at the East Coast meeting.

5. No distinction is made between a vector such as z^o, and the set containing only this vector.

6. Therefore $x_i^o \in \mathcal{X}_i^{min}$, $y_j^o \in \mathcal{Y}_j^{min}$.

7. The geometric interpretation of this is the following: the necessary and sufficient condition for the existence of a supporting plane through z^o for $\mathcal{Z}$ is the existence of a family of parallel supporting planes through the x_i^o (resp. y_j^o) for the $\mathcal{X}_i$ (resp. $\mathcal{Y}_j$). Such is the deep meaning of the optimum theorem to be enunciated in a moment.

8. Unless, of course, $\mathcal{Z}^{min}$ is a smooth surface having only one normal direction at each point.

9. If the conditions of derivability are fulfilled, (β') coincides with the well-known rule "every production-unit produces its output at the smallest possible total cost and sells it at marginal cost."

If $\mathcal{Y}_j$ is a cone, the minimum of $p \cdot y_j$ is zero and (β') coincides with the rule of perfect competition within the j^{th} industry.

10. Here, as in similar cases, the author quoted did not use the language we used, however, the translation should never raise any difficulty.

11. It is easy to build a two-dimensional counter example.

12. But it can be defined only if a price vector with some intrinsic meaning exists at all.

13. The superscripts 1 and 2 will denote the two economic situations compared.

14. Moreover the roles played by the situation (1) and (2) were generally asymmetrical in such a way that inconsistencies pointed out by T. de Scitovszky[21], [22] arose.