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On The Certainty Equivalence and Risk Discount Hypotheses

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I. The Argument Thus Far.

The "risk allowance" (or "risk discount") hypothesis is a special case of the "certainty equivalence" hypothesis. According to the latter, there is a history of future prices such that, if the entrepreneur knew this history with certainty, he would take the same actions as he does under uncertainty. According to the risk allowance hypothesis, each certainty-equivalent price is equal to that price which the entrepreneur believes to be "most likely", plus or minus an allowance for risk.

There have been criticisms of the certainty equivalence and risk allowance hypotheses. In particular Professor Hart has pointed out that "there are many things people do because of uncertainty which they would not do for any certainty whatever".[1] Particular phenomena which, according to Hart, "cannot be adequately explained by certainty equivalence"[1] include "cash holdings, differentiation of asset portfolios, and inventory policy".[1]

This and other[2] objections leave room for a counter argument, in favor of

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1. Albert G. Hart, "Assets, Liquidity, and Investment", American Economic Review, Papers and Proceedings, Vol. 39, May 1949, p. 180.

2. Milton Friedman, "Lange on Price Flexibility and Employment: A Methodological Criticism", American Economic Review, September 1946, particularly pp. 627-630.

certainty equivalence. While some phenomena cannot be explained, some problems cannot be handled by such hypotheses, still, perhaps there are problems and phenomena, involving uncertainty, to which certainty equivalence hypotheses can be usefully applied. Thus, Professor Friedman — a pioneer critic of certainty equivalence (note 2) — has said, in discussing Hart's paper, that "the range of analysis that falls into the certainty-equivalent class is far broader than might at first appear, and so is the range of phenomena capable of explanation by such analysis."[3]

Recently Paul B. Simpson has attempted to show that — in the determination of an optimum production plan — the risk discount hypothesis is implied by the hypothesis that the entrepreneur maximizes utility which depends on the expected value E and variance V of discounted returns: $U=U(E,V)$.[4] This result, if it were true, would be a strong argument in favor of the opinion that certainty equivalent hypotheses can be applied to an important class of problems involving uncertainty.

It will be argued in the present paper that Simpson's conclusions are false. The U(E,V)- hypothesis does not, in general, imply the risk discount hypothesis. In some cases the U(E,V)- hypothesis is inconsistent with any certainty equivalence hypothesis. In other cases the general hypothesis of certainty equivalence is tautological, while the more specific hypothesis of risk discount is inconsistent with the U(E,V) hypothesis. It will be argued further that similar results hold if utility depends not only on E and V, but also on higher moments. And further still, where the U(E,V) and similar hypotheses have implications differing from those of certainty equivalence or risk discount, observation and common sense favor the implications of the former rather than the latter.

3. American Economic Review, Papers and Proceedings, May 1949, p. 196.

4. Paul B. Simpson, "Risk Allowances for Price Expectations", Econometrica, July 1950.

The importance of these considerations lies in the numerous examples of economic thought and empirical research based upon assumptions of certainty equivalence in one form or another. I believe that in most of these cases the arguments against certainty equivalence given below, can be applied with little modification.

II. Simpson's Argument.

We will now summarize Simpson's argument and state one fundamental objection to it.

Our notation will be as follows:

$\bar{p}_1, \ldots, \bar{p}_n$ — expected values of prices

σ_{ij} $i,j=1,\ldots,n$ — covariances among prices

q_i $i=1,\ldots,n$ — quantities sold (+) and bought (-).

$Q(q_1,\ldots,q_n) = 0$ — the production function.

Simpson purports to show that a necessary condition for an optimal production plan is that

$$1)\qquad \frac{\frac{\partial Q}{\partial q_i}}{\frac{\partial Q}{\partial q_j}} = \frac{\bar{p}_i - \left(\frac{1}{V}\right)\left(\frac{dE}{dV}\right)\left(\sum_{\ell} q_\ell \sigma_{\ell i}\right)}{\bar{p}_j - \left(\frac{1}{V}\right)\left(\frac{dE}{dV}\right)\left(\sum_{\ell} q_\ell \sigma_{\ell j}\right)}$$

for all i, j (where $\frac{dE}{dV}$ is the marginal rate of substitution of V and E in U(E,V)). Or if we let

$$2)\qquad p_i^d = \bar{p}_i - d_i = \bar{p}_i - \left(\frac{1}{V}\right)\left(\frac{dE}{dV}\right)\left(\sum_{\ell} q_\ell \sigma_{\ell i}\right)$$

then

$$3)\qquad \frac{\frac{\partial Q}{\partial q_i}}{\frac{\partial Q}{\partial q_j}} = \frac{p_i^d}{p_j^d}.$$

Simpson's central argument is as follows: "The usual condition for the equilibrium of the firm is that the entrepreneur selects a production plan such that the ratios of marginal substitutabilities equal the ratios of prices for each two commodities. Equations [1] express similar conditions, but in place of prices we have expected prices $\bar{p}_i$ and $\bar{p}_j$, plus or minus expressions which are equivalent to Hick's risk allowances. The risk allowance can be considered as a certainty equivalent of choice among probability alternatives."[5]

Let us change the above notation slightly to let p_{it}, q_{it}, $\sigma_{it,jt'}$ etc. be the prices, quantities, covariances, and so on, of goods i, $j=1,\ldots,K$ bought or sold at time t, $t'=1,\ldots,T$. Simpson allows T to be any positive integer (his n equals our KT).

Simpson assumes that p_{it} and $q_{jt'}$ are independent for all i,j,t,t'. But this is not true of the plan an entrepreneur would draw up if $T \geq 2$ and if the entrepreneur maximized $U=U(E,V)$. The entrepreneur would plan a certain bundle of receipts and expenditures $q_{11},\ldots,q_{n1}$ at time 1 at the current (known) prices. For time 2 he would not, in general, plan a particular bundle $q_{12},\ldots,q_{n2}$; but would let his decision as to the particular bundle to choose vary with the information available at that time. In particular he will let $q_{12},\ldots,q_{n2}$ vary with the prices $p_{12},\ldots,p_{n2}$ — not only because these express the costs and returns of the q_{i2}, but also because he may feel that they give information as to prices at time $3,4,\ldots$. Similarly he will plan to allow q_{i3} to depend on information available at time 3. And so on. In other words, he will not plan a bundle $q_{11},\ldots, q_{n1}, q_{12},\ldots, q_{13},\ldots$, but will plan a <u>strategy</u>:[6]

5. <u>ibid</u>., p. 255.

6. This idea of a <u>strategy</u> is that of von Neumann and Morgenstern, <u>Theory of Games and Economic Behavior</u>.

4) $q_{11}, \ldots, q_{n1}$

$$q_{12} = q_{12}\ (p_{12}, \ldots, p_{n2};\ z_{12}, \ldots, z_{S_2 2})$$

$$\vdots$$

$$q_{n2} = q_{n2}\ (p_{12}, \ldots, p_{n2};\ z_{12}, \ldots, z_{S_2 2})$$

$$q_{13} = q_{13}\ (p_{13}, \ldots, p_{n3};\ p_{12}, \ldots, p_{n2};\ z_{13}, \ldots, z_{S_3 3})$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$q_{ij} = q_{ij}\ (p_{1j}, \ldots;\ p_{1j-1}, \ldots;\ p_{12}, \ldots;\ z_{1i}, \ldots\ z_{S_i i})$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

(where the z_{ij} is other information, not originally available, concerning $p_{K\ell}$ $\ell > j$).

Any set of prices (p^d_{it}), known with certainty, will imply as optimum a bundle $q_{11}, q_{21}, \ldots, q_{it}, \ldots, q_{KT}$ rather than a (non-degenerate) strategy. Thus we see that it is not true in general that the plan implied by the U(E,V) - hypothesis is the same as that implied by some set of certainty equivalent prices (in particular p^d of equation 2). We may, however, ask a) Are Simpson's conclusions correct when the entrepreneur can act at only one point in time? b) Does the possibility of letting q_{it} $t \geq 2$ depend on variables known then, affect the choice $q_{11}, \ldots, q_{K1}$? I.e., does Simpson's p^d_i predict the same current action $q_{11}, \ldots, q_{K1}$, even though it does not give the same plan as the U(E,V) - hypothesis? Our answer will be no to both questions.

First let us state in a more formal manner the hypotheses under consideration and our conclusions about them.

III. Formalization of Hypotheses and Conclusions.

Outcome o is a function of the chosen strategy (s) and random variables (p) (in this paper, prices) o = (s,p). o in general may be a vector, but in this paper is a number: discounted profits. We will assume that the entrepreneur has a (subjective) probability distribution P_r (p). Given P_r (p) and o (s,p) we can find the (subjective) probability distribution of o given any s: $f(o|s)$. We assume that s is chosen so as to maximize $U=U\left\{f(o|s)\right\}$. (When o is known with certainty, U is an increasing function of o). To every vector p and function o=o (s,p) there is a range of possible strategies $S = \left\{s\right\}$. We may, in some problems, wish to restrict our consideration to functions o(s,p), distributions P_r (p) and utility functions $U\left\{f(o)\right\}$ which lie in sets $\mathcal{O}$, $\mathcal{P}$, $\mathcal{U}$, respectively. To say that under certain conditions ($\mathcal{O}$, $\mathcal{P}$, $\mathcal{U}$) the certainty equivalence hypothesis is correct is the same as to say that for every (o, P_r, U) $o \in \mathcal{O}$ $P_r \in \mathcal{P}$ and $U \in \mathcal{U}$, there exists a vector p such that $S_1=S_2$, where S_1 and S_2 are sets of strategies with the following properties:

$$U\left\{f(o \mid s_1)\right\} > U\left\{f(o \mid s_o)\right\}$$

for all $s_1 \in S_1$, and all $s_o \in S-S_1$

$$U\left\{f(o \mid s_1)\right\} = U\left\{f(o \mid s_1^*)\right\}$$

for all s_1, $s_1^* \in S_1$

$$U\left\{o=o\ (s_2,\ p)\right\} > U\left\{o=o\ (s_o,\ p)\right\}$$

for all $s_2 \in S_2$, and all $s_o \in S-S_2$

$$U\left\{o=o\ (s_2,\ p)\right\} = U\left\{o=o\ (s_2^*,\ p)\right\}$$

for all s_2, $s_2^* \in S_2$.

p depends on the functions P_r, o and U. The risk allowance hypothesis specifies properties of $p(P_r, o, U)$.

We will let $C(\mathcal{O}\mathcal{P}\mathcal{U})$ stand for the certainty equivalence hypothesis applied under conditions $(\mathcal{O}, \mathcal{P}, \mathcal{U})$. In Simpson's paper U is restricted to be a differentiable function of mean and variance with $\frac{\partial U}{\partial E} > 0$, $\frac{\partial U}{\partial V} < 0$; $\mathcal{O}$ is restricted by the assumption that Q be differentiable and $\mathcal{P}$ by the assumption that all means and variances exist. It has already been noted that under these conditions $C(\mathcal{O}\mathcal{P}\mathcal{U})$ is false. It will be shown that if $\mathcal{O}$ were further restricted to the case where T=1 (i.e., where S includes only allocations and not (non-degenerate) strategies) then C is still false. If we further restrict conditions (still assuming T=1) so that no two points on the production function have the same relative partials $\frac{\partial Q}{\partial q_2} / \frac{\partial Q}{\partial q_1}, \ldots, \frac{\partial Q}{\partial q_n} / \frac{\partial Q}{\partial q_1}$, and the optimum s is unique then a) we have the case which I suspect Simpson had in mind; b) C is true, tautologically; c) $p=p(P_r, o, U)$ is given by equation 2; but d) the function $p(P_r, o, U)$ lacks at least one important property commonly ascribed to a "risk discounted expected price".

When $T \geq 2$, C is false, for no prices believed with certainty would lead to a (non-degenerate) strategy rather than an allocation. We may define a weak certainty equivalence hypothesis $W(\mathcal{O}\mathcal{P}\mathcal{U})$, such that for $W(\mathcal{O}\mathcal{P}\mathcal{U})$ to be true, it is not required that $S_1=S_2$ as for $C(\mathcal{O}\mathcal{P}\mathcal{U})$; it is only required that the sets of initial allocations $\overset{1}{q}_{11}, \ldots, \overset{1}{q}_{n1}$ of S_1 and those $\overset{2}{q}_{11}, \ldots, \overset{2}{q}_{n1}$ of S_2 be the same: $\left\{(\overset{1}{q}_{11}, \ldots, \overset{1}{q}_{n1})\right\} = \left\{(\overset{2}{q}_{11}, \ldots, \overset{2}{q}_{n1})\right\}$. In other words, W requires that there be prices p^W which if held with certainty would lead to the same "current" action (action at t=1) — not necessarily to the same strategy. If conditions are restricted by the assumption that no two points on the production function have the same set of relative partials and s is unique, then W is true, tautologically; but the p^d of equation 2 are not in general the (weak) certainty equivalence prices.

So far we have restricted $\mathcal{U}$ to include only utility functions depending on mean and variance. The proofs which we will give for the assertions above can be extended to cases where $\mathcal{U}$ includes utility functions which contain as arguments almost any other set of moments. The one exception is the case where utility is restricted to depend only on the mean.

IV. The Certainty Equivalence, Risk Allowance, and Mean-Variance Hypotheses.

Let us first consider the case wherein the entrepreneur acts at only one point in time. Let us also suppose that Q is linear:

$$Q = A - \Sigma \alpha_i q_i = 0.$$

This possibility is not excluded by Simpson's (explicit) assumptions. Suppose prices p^d were believed with certainty. Then necessary conditions for $(q_1, \ldots, q_n)$ to be a maximum is that

$$\frac{\alpha_i}{\alpha_1} = \frac{p_i^d}{p_1^d} \qquad i = 2, \ldots, n$$

This gives n-1 equations in no unknowns. Either every point on Q satisfies these equations or no point does. To take a concrete example (which, like all the concrete examples we give, is unimportant to the formal argument but will be important later when we consider the realism of the hypotheses discussed) we may suppose the "entrepreneur" to be a trader who allocates "today" a given amount of assets A, among various securities or commodity "futures" at current prices α_i, and who intends to withdraw from the market "tomorrow", selling his goods or securities at prices p_i. We ignore costs of acquisition or sale (e.g., commissions, taxes) and admit the possibility of short sales (i.e., we do not exclude negative values of q_i). If the ratio of the prices $\frac{p_i}{\alpha_i}$ is the same for all goods and securities then every allocation is as good as any other. If one or more ratios exceed all others, then every allocation can be improved by selling (or selling short) a good with a lower ratio and using the funds to buy one with a higher ratio.

The above conclusions, however, are not the same as those of the hypothesis that utility depends on mean and variance; and therefore, under these conditions $C(\mathcal{O}\mathcal{P}\mathcal{U})$ is false. This is not to say that every (U, o (s, p), P_r) under consideration leads to an S that is not implied as optimum by some (p^d). Rather it says that there is at least one (U, o (s, p), P_r) with this property. This could be demonstrated by simply exhibiting an example of such a case (such as the example on page 11 given for another purpose). More generally let us note that when U is a function of E and V, the necessary conditions for $q_1, \ldots, q_n$ to be a maximum are n equations in n (rather than no) unknowns:

$$\frac{\frac{\partial U}{\partial E}\bar{p}_i + \frac{\partial U}{\partial V}(\Sigma\, \sigma_{ij}\, q_j)}{\frac{\partial U}{\partial V}\bar{p}_1 + \frac{\partial U}{\partial V}(\Sigma\, \sigma_{1j}\, q_j)} = \frac{\alpha_i}{\alpha_1} \qquad i = 2, \ldots, n.$$

and the production function, $Q(q_i) = 0$.

These equations will typically be satisfied by one or more points $(q_1, \ldots, q_n)$, but not by the whole production function.

It may be that none of these points are an optimal strategy, for the optimum strategy may be "at infinity" (i.e., it may be that for any point $(q_1, \ldots, q_n)$ $|q_i| \lesssim K$ for all i, there is another point, $q_1^*, \ldots, q_n^*$ with $|q_j^*| > K$ for some j, such that $U(q_1^*, \ldots, q_n^*) > U(q_1, \ldots, q_n)$. Consider, for example, the case where of the n future prices there are two (say, 1 and 2) with $\sigma_{11} = \sigma_{22} = 0$, $\bar{p}_1 \neq \bar{p}_2$). One set of restrictions which excludes such "infinite solutions" is the following: a) not more than one future price has zero variance b) $U(E, V) \rightarrow \underline{U}$ (where $\underline{U}$ is the minimum value or greatest lower bound of U) if V and E approach infinity in such a way that $\frac{V}{E} \rightarrow \infty$. A more general analysis of the conditions under which "infinite solutions" are ruled out would be interesting, but is not essential to our present argument.

If we do not admit short sales, i.e., if we require $q_i \geq 0$, C is still false. This is not quite a special case of Simpson's assumptions; it violates the differentiability assumptions. But it has an important concrete interpretation: the individual may be thought of as allocating a given quantity of funds among n securities in amounts $q_1, \dots, q_n$ so as to maximize utility which depends on mean and variance of the flow of returns ("the flow of returns" equals $\Sigma q_i Y_i$ where Y_i is the yield on the i^{th} security and is equal to $Y_i = \frac{p_i}{\alpha_i} - 1$). It turns out that in this case no set of prices, held with certainty, would imply that a particular diversified portfolio ($q_i > 0$ for at least two securities) was preferable to all others. If

$$\frac{p_1^d}{\alpha_1} = \frac{p_2^d}{\alpha_2} = \dots = \frac{p_K^d}{\alpha_K} \quad \frac{p_{K+1}^d}{\alpha_K} \geq \dots \geq \frac{p_n^d}{\alpha_n}$$

then any allocation of funds among securities 1 to K is as desirable as any other allocation among securities 1 to K (and is preferred to any allocation including $q_i > 0$ for some $i > K$). In particular, the allocation $q_1 = A$, $q_i = 0$ for $i \geq 2$ is as good as any. The mean-variance hypothesis, on the other hand, implies -- under certain conditions -- that a particular diversified portfolio is preferred to all others, in particular to any allocation $q_1 = A$, $q_j = 0$ $j \geq 2$. See, e.g., the example on page 11.

In the above cases more than one point on the production function has the same relative partials $\frac{\partial Q}{\partial q_2} \Big/ \frac{\partial Q}{\partial q_1}, \dots, \frac{\partial Q}{\partial q_n} \Big/ \frac{\partial Q}{\partial q_1}$.

When the mean-variance hypothesis implies that one point is preferred to another with the same relative partials, no set of prices believed with certainty will have the same implication. If (still assuming the individual acts at only one point in time) we further restrict conditions ($\mathcal{P}, \mathcal{U}, \mathcal{O}$) by the assumption that a) no two points on the production function have the same relative partials and b) the mean-variance hypothesis leads to a unique optimum strategy, then C is true. In fact, no matter how the $q_1, \dots, q_n$ are chosen, even if by random, a set of prices could afterwards be constructed such that if they had been believed with certainty they would have led to the same (q_i). If choice is according to the mean variance hypothesis, then the p_i^d of equation 2 are these certainty equivalence prices.

Simpson speaks of the p_i^d as expected prices minus an allowance for risk. (The allowance being negative or positive depending on whether a purchase or a sale is anticipated at that price.) It is not easy to tell exactly what properties the function $p^d = p^d(P_r, U, o)$ has according to the traditional sources of the risk allowance hypothesis. For example, suppose $\bar{p}_1$ were to change; typically $q_1, \ldots, q_n$ would be changed, V and E would change; and, typically p_2^d would change. Is it consistent with traditional and current notions for the "risk discounted expected value" of p_2 to depend on the value of $\bar{p}_1$? I am not sure about this; and I don't intend to pursue an analysis of texts to find out.

There are, however, properties of $p(P_r, U, o)$ which, it seems to me, are generally recognized by the users of the risk allowance hypothesis. For example, suppose an entrepreneur used a given set of resources "today" to produce two commodities, in amounts q_1 and q_2, which had to be sold "tomorrow" at prices p_1 and p_2. Suppose p_1 and p_2 are independent, that p_1 has a higher mean and lower variance than p_2 then the risk allowance hypothesis implies that $p_2^d \geq p_1^d$.

But this is not, in general, the property of the p_i^d of equation 2. Let us first give a simple, linear counter example, and then generalize the example to the non-linear case.

Suppose $Q = 10,000 - 10q_1 - 9q_2 = 0$ $(q_1, q_2 \geq 0)$; suppose $U = E - V$, $\bar{p}_1 = 11$, $\bar{p}_2 = 12$, $\sigma_{11} = 2$, $\sigma_{22} = 1$, $\sigma_{12} = 0$. If we maximize U subject to $Q = 0$ we find that the optimum allocation is $q_1 = 381$, $q_2 = 687$. $\bar{p}_2 > \bar{p}_1$, $\sigma_{22} < \sigma_{11}$; but since $\frac{\partial Q}{\partial q_2} \Big/ \frac{\partial Q}{\partial q_1} = .9$, and since equation 3 is a necessary condition for an optimum bundle, we therefore have $p_1^d > p_2^d$; contrary to the risk allowance hypothesis.

The above counter example can easily be generalized to the non-linear case. Rather than a linear production function we can postulate one, with a marginal rate of substitution of .9 at the point (381; 687), such that this point is still optimum. We could similarly drop the linear assumption concerning the utility function.

To summarize: If we make assumptions a) and b) on page10, C is true. In fact it is true no matter how (q_i) is chosen. But we must carefully avoid thinking intuitively of the p_i^d as expected prices minus risk allowances.

I can see no way in which looking at the problem in terms of these certainty equivalence prices adds to our insight into behavior under uncertainty or supplies us with a tool of practical economic analysis.

As we saw before when the entrepreneur could act at more than one point in time, C is false. We will now consider whether or not equation 3 implies the same current action, even though it does not imply the same strategy as the mean-variance hypothesis. If we make the assumptions a) and b) on page 10, then there always exists sets of future prices which if held with certainty would lead to the same current action as the mean-variance hypothesis. Are the p_i^d of equation 2 such a set?

Let us consider the following case: The entrepreneur makes sales (+) and purchases (-) $q_{11},\dots,q_{n1}$ at time one at current known prices $p_{11},\dots,p_{n1}$; he similarly makes sales and purchases at time two at prices $p_{12},\dots,p_{n2}$ which become known only at that time. He has a production function

$$Q(q_{11},\dots,q_{n1},\ q_{12},\dots,q_{n2}) = 0,$$

a joint probability distribution $P_r(p_{12},\dots,p_{n2})$ concerning prices at time 2. Profit is $\Pi = \sum_i p_{i1}\, q_{i1} + \sum_i p_{i2}\, q_{i2} = \Pi_1 + \Pi_2$. Utility is $U = U(E\,\Pi, V\,\Pi)$.

Since p_{i1} is known with certainty, i.e., has a subjective variance of zero, $U = U(\pi_1 + \mathcal{E}\pi_2, V\pi_2)$. What can we say about the optimum strategy? At time 2, given his first decision $X_{11}, \ldots, X_{1n}$ and current prices, $p_{12}, \ldots, p_{n2}$, the entrepreneur will simply maximize profits. He will choose $X_{12}, \ldots, X_{n2}$ in such a way that

$$5) \qquad p_{i2} + \lambda \frac{dQ}{dX_{i2}} = 0 \quad \text{for all } i.$$

Given that such will be his action at time 2, we seek the optimum decision at time 1. The entrepreneur must choose $X_{11}, \ldots, X_{n1}$ in such a way that

$$0 = \frac{\partial U + \delta Q}{\partial X_{i1}} = \frac{\partial U}{\partial E}\frac{\partial E}{\partial X_{i1}} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial X_{i1}} + \delta \frac{\partial Q}{\partial X_{i1}}$$

$$= \delta \frac{\partial Q}{\partial X_{i1}} + \frac{\partial U}{\partial E} p_{i1} + \frac{\partial U}{\partial E}\left(\frac{\partial \int \Sigma\, p_{j2} X_{j2}\, d\, P_r(p_{12}, \ldots, p_{n2})}{\partial X_{i1}}\right)$$

$$6) \qquad + \frac{\partial U}{\partial V} \frac{\partial \int \Sigma\Sigma\, p_{j2} p_{k2} X_{j2} X_{k2}\, d\, P_r(p_{12}, \ldots, p_{n2})}{\partial X_{i1}}$$

$$- \frac{\partial U}{\partial V} \frac{\partial \left(\int \Sigma\, p_{j2} X_{j2}\, d\, P_r(p_{12}, \ldots, p_{n2})\right)^2}{\partial X_{i1}}$$

$$= \delta \frac{\partial Q}{\partial X_{i1}} + \left[p_{i1} + \sum_{j=1}^{n} \int p_{j2} \frac{\partial X_{j2}}{\partial X_{i1}}\, d\, P_r(p_{12}, \ldots, p_{n2})\right] \frac{\partial U}{\partial E}$$

$$+ \left[\sum_k \sum_\gamma \int p_{k2}\, p_{\gamma 2}\left(X_{k2} \frac{dX_{k2}}{dX_{i1}} + X_{\gamma 2}\frac{dX_{\gamma 2}}{dX_{i1}}\right) d\, P_r(p_{12}, \ldots, p_{n2})\right] \frac{\partial U}{\partial V}$$

$$- 2 \left[\left(\int \Sigma\, p_{j2} X_{j2}\, d\, P_r(p_1, \ldots, p_n)\right)^2 \left(\Sigma \int p_{j2} \frac{\partial X_{j2}}{\partial X_{i1}}\, d\, P_r(p_{12}, \ldots, p_n)\right)\right] \frac{\partial U}{\partial V}.$$

To evaluate $\frac{\partial X_{j2}}{\partial X_{i1}}$, for given $p_{12}, \ldots, p_{n2}$, we differentiate equation 5) and get

$$7) \qquad \frac{\partial \lambda}{X_{i1}} Q_{X_{j2}} + \lambda \left\{ Q_{X_{j2}X_{i1}} + \sum_k Q_{X_{j2}X_{k2}} \frac{\partial X_{k2}}{\partial X_{i1}} \right\} = 0$$

divide by λ, use $\lambda = \frac{-p_{j2}}{Q_{X_{j2}}}$ and get

8) $$\frac{\partial \lambda}{\partial X_{i1}} P_{j2} + \sum_k Q_{X_{j2}X_{k2}} \frac{\partial X_{k2}}{\partial X_{i1}} = Q_{X_{j2}X_{i1}}$$

Because of the production function we have

9) $$\frac{\partial Q}{\partial X_{12}} \frac{\partial X_{12}}{\partial X_{i1}} + \frac{\partial Q}{\partial X_{22}} \frac{\partial X_{22}}{\partial X_{i1}} + \dots + \frac{\partial Q}{\partial X_{n2}} \frac{\partial X_{n2}}{\partial X_{i1}} + \frac{\partial Q}{\partial X_{i1}} = 0$$

or

10) $$\Sigma P_j \frac{\partial X_{j2}}{\partial X_{i1}} = \frac{\partial Q}{\partial X_{i1}} \lambda$$

All this may be summarized by

11) $$\begin{pmatrix} Q_{X_{12}X_{12}}, & \dots, & Q_{X_{12}X_{n2}} & P_{12} \\ \vdots & & \vdots & \vdots \\ Q_{X_{n2}X_{12}}, & \dots, & Q_{X_{n2}X_{n2}} & P_{n2} \\ P_{12} & , \dots, & P_{n2} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial X_{12}}{\partial X_{i1}} \\ \vdots \\ \frac{\partial X_{n2}}{\partial X_{i1}} \\ \frac{\partial \lambda}{\partial X_{i1}} \end{pmatrix} = \begin{pmatrix} Q_{X_{12}X_{i1}} \\ \vdots \\ Q_{X_{n2}X_{i1}} \\ Q_{i1} \lambda \end{pmatrix}$$

We can solve 11 and substitute in 6. This, together with 5 gives necessary conditions for an optimum strategy.

For our present purposes it is sufficient to note that $X_{11}, \dots, X_{n1}$ as part of the optimum strategy depends upon things which would be irrelevant if the entrepreneur were constrained to choose and stick to a single bundle. The means, variances and covariances used in calculating the optimum bundle are not sufficient to determine the optimal strategy. In particular they typically do not imply that the $X_{11}, \dots, X_{n1}$ of the optimum strategy must be the same as that of the optimum bundle.

V. Generalization.

The previous arguments would not be affected essentially if we assumed that utility depended upon higher central moments $\mu_3, \mu_4, \ldots$, as well as mean and variance $(= \mu_2)$. In this case equations 1 and 2 become

$$1') \quad \frac{\frac{\partial Q}{\partial q_i}}{\frac{\partial Q}{\partial q_j}} = \frac{\bar{p}_i + \frac{1}{\frac{\partial U}{\partial E}} \left\{ \frac{\partial U}{\partial V} \frac{\partial V}{\partial q_i} + \frac{\partial U}{\partial \mu_3} \frac{\partial \mu_3}{\partial q_i} + \ldots \right\}}{\bar{p}_j + \frac{1}{\frac{\partial U}{\partial E}} \left\{ \frac{\partial U}{\partial V} \frac{\partial V}{\partial q_j} + \frac{\partial U}{\partial \mu_3} \frac{\partial \mu_3}{\partial q_j} + \ldots \right\}}$$

$$2') \quad p_i^d = \bar{p}_i + \frac{1}{\frac{\partial U}{\partial E}} \left\{ \frac{\partial U}{\partial V} \frac{\partial V}{\partial q_i} + \ldots \right\}$$

As in the mean variance case, if the individual plans to act at more than one point in time, his plan takes the form of a strategy rather than a bundle. If we consider an individual who can act only at a single point in time: in the linear case we again have n equations in n unknowns rather than one equation in n unknowns; and examples again can be given to show that, in the linear and non-linear cases, the p_i^d have properties which seem inconsistent with the common notions of the risk allowance hypothesis.

Our arguments would break down if utility was assumed to depend only on expected profit (E).

VI. Evaluation of Hypotheses.

Diversification is an important aspect of investment behavior. "Infinite solutions" are not characteristic of even speculative behavior. Certainty equivalence hypotheses, including the risk allowance hypothesis, are clearly inadequate as explanations of investment or speculative behavior. The hypothesis that utility depends upon mean, variance and perhaps higher moments, can not be rejected on these grounds.

The objections which J. R. Hicks made against the risk allowance hypothesis—when he introduced the concept in his Value and Capital—applies to the single action, non-linear case:

> "Further (and this is the most serious weakness of our treatment), the willingness to bear any particular risk (to plan to buy or sell at any particular future date for which prices are uncertain, and to act on that plan) will be appreciably affected by the riskiness involved in the rest of the plan. I can do very little about this on present methods, though some consequences of the interrelations of risks will come to our notice now and then.[7]

The U(E, V)-hypothesis takes into account the "interrelations of risks."

A certainty equivalence hypothesis never leads to a strategy, only to a bundle. The virtue of this bundle is that it would maximize profits if a particular future history of prices were to occur. No consideration is given to the consequences which would result if that particular history did not materialize. In planning action today, no value is attached to flexibility tomorrow. Few people, I believe, would recognize these as properties of planning under uncertainty. They are not the general implications of the mean-variance hypothesis, or the hypothesis that U = U(E, V,...)—to be said to the credit of such hypothesis.

VII. Conclusion.

All this does not prove that certainty equivalence hypotheses can have no applications. It only serves to show that "the range of analysis which falls into the certainty equivalence class is far" narrower "than might at first appear, and so is the range of phenomena capable of explanation by such analysis."

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[7] Value and Capital p. 126.