COWLES COMMISSION DISCUSSION PAPER: ECONOMICS No. 294

Investment Company Behavior Equations

Harry Markowitz

This paper discusses part of a project in which assumptions concerning behavior under uncertainty are combined with institutional descriptions of Investment Companies to form models of Investment Company behavior. The empirically verifiable consequences of these models are sought; in particular, an attempt is made to derive behavior equations to be fitted to and tested by data. The Investment Company seems ideal as a place to start in an empirical study of behavior under uncertainty in general, and financial behavior in particular. The Investment Company has a relatively simple institutional framework and "production function." Here we can study problems of behavior under uncertainty with a minimum of institutional complications.

In Cowles Commission Discussion Paper, Economics 278, it was shown that if (1) the investor allocates amounts $X_1,\ldots,X_n$ among n securities so as to maximize utility (U) which depends upon the expected value (E) of the flow of returns, the variance (V) of this flow, and the amount (A) of assets tied up in securities (i.e., U = U(E, V, A).), and if (2) we know the forms of functions relating subjective means μ_i $(i = 1,\ldots,n)$ and covariances σ_{ij} $(i, j = 1,\ldots,n)$ to observable variables $(Z_1,\ldots,Z_s)$, then we can derive the forms of behavior equations relating $X_1,\ldots,X_n$ to observable variables. If the "belief formation functions" $\mu_i = \mu_i(Z_1,\ldots,Z_s)$, $\sigma_{ij} = \sigma_{ij}(Z_1,\ldots,Z_s)$ are linear in unknown parameters, then the behavior equations of Discussion

Paper 278 are also linear in unknown parameters and may be fitted by the Limited Information method of estimation.[1]

The above mentioned behavior equations are rather complicated, from the point of view of actually fitting them to data. In this paper equivalent but simpler equations will be derived. The results will be extended to cases where utility depends on higher moments as well as mean and variance. We will also consider conditions under which behavior equations describe the allocation of funds in a part of the investors portfolio without taking into account the allocation of funds among or subjection beliefs concerning the rest of the portfolio.

Before we have equations which can be fitted to data we must specify at least the forms of the belief formation functions. The nature of these functions will be considered in a subsequent paper. Our considerations will principally involve a probabilistic formulation of certain aspects of Security Analysis.

As was argued in Discussion Paper 278: if there are n + q securities in which the individual could invest, and if he allocates amounts $X_1 > 0$, $X_2 > 0, \ldots, X_n > 0$, $X_{n+1} = X_{n+2} = \ldots = X_{n+q} = 0$ then the allocation $(X_1, \ldots, X_n, 0\ldots0)$ must minimize (1) $V = \sum_{i=1}^{n+q} \gamma_{ij} X_i X_j$ subject to the constraints (2) $E = \sum_{i=1}^{n+q} \mu_i X_i$, (3) $A = \sum_{i=1} X_i$ and (4) $X_i \geq 0$ for all i. Necessary conditions for this to hold (therefore conditions satisfied by every observed allocation if our hypothesis is correct) are the following:

---

1. There are strong reasons for doubting that the belief formation functions are linear in their parameters. This means that either we must fit polynomial approximations or we must attempt to completely specify the belief formation functions or else we must wait for more general methods of estimation.

$$5.1 \quad \sigma_{11} X_1 + \sigma_{12} X_2 + \dots + \sigma_{1n} X_n + \lambda_1 \mu_1 + \lambda_2 = 0$$

$$5.2 \quad \sigma_{12} X_1 + \sigma_{22} X_2 + \dots + \sigma_{2n} X_n + \lambda_1 \mu_2 + \lambda_2 = 0$$

$$\dots\dots\dots\dots$$

$$5.n \quad \sigma_{1n} X_1 + \dots + \sigma_{nn} X_n + \lambda_1 \mu_n + \lambda_2 = 0$$

$$5.n{+}1 \qquad X_1 \mu_1 + \dots + X_n \mu_n = E$$

$$5.n{+}2 \qquad X_1 + \dots + X_n = A$$

(where λ_1 and λ_2 are Lagrangian multipliers).

Behavior equations may be derived from equations (5) in two steps. First, derive equations which contain only the variables $X_i, \mu_i, \sigma_{ij} (i,j=1,\dots,n)$; second, substitute the belief formation functions for the variables μ_i, σ_{ij}, obtaining equations containing only observable variables and unknown parameters. In Cowles Commission Discussion Paper 278, the first step involved solving the equations 5) for X_i and substituting the right sides of 2) and 3) for E and A. Simpler equations are obtained by eliminating λ_1 and λ_2 in an obvious way: Subtract equation 1 from equation i and obtain

$$6.i) \qquad \sum_{j=1}^{n} X_j (\sigma_{ij} - \sigma_{1j}) + \lambda_1 (\mu_i - \mu_1) = 0 \qquad i = 2,\dots,n$$

Subtract $(\mu_i - \mu_1)$ times equation 6.2) from $(\mu_2 - \mu_1)$ times equation 5.i) and get equations

$$7.i) \qquad \Sigma X_j \left\{ (\mu_2^o - \mu_1^o)(\sigma_{ji} - \sigma_{1i}) - (X_j^o - X_1^o)(\sigma_{2i} - \sigma_{1i}) \right\}$$

for $i = 3,\dots,n$.

This gives us n-2 equations describing the 2-dimensional "efficient surface." If the belief formation equations

$$8_k) \qquad \mu_k = \mu_k (Z_1,\dots,Z_s)$$

$$8'_{ij}) \qquad \sigma_{ij} = \sigma_{ij} (Z_1,\dots,Z_s)$$

are linear in unknown parameters

$\alpha_{k,a}$ $\quad a = 1,\ldots,K_h$

$\beta_{ij,b}$ $\quad b = 1,\ldots,K_{ij}$, respectively, then the behavior equations obtained by substituting equations 8 in equations 7 are linear in parameters $\gamma_{k,a\,.\,ij,b} = a_{k,a} \cdot \beta_{ij,b}$.[2]

Suppose we knew the values of the γ, could we derive those of the α and β? There is some lack of identification; for if we multiplied all parameters α_{ia} by $\delta \neq 0$ and all $\beta_{ij,b}$ by $\frac{1}{\delta}$, all γ's would remain the same. On the other hand it can be easily shown by construction that if one of α's or β's ($\neq 0$) is given an arbitrary value ($\neq 0$), then the values of all the rest can be inferred from the γ's.

Procedures similar to those above will yield behavior equations when (1) utility is assumed to depend on higher moments, or (2) the probability beliefs and utility function are expressed in terms of non-central moments. The Bernoulli expected utility hypothesis is included as a special case, if the utility function is approximately a polynomial over the relevant range of outcomes.

The extension of our results to the case where utility depends on higher moments involves no mathematical difficulty or interest. We will simply note some points which are of interest from the viewpoint of practical curve fitting. Each new moment which is assumed to enter the utility function adds another λ to be eliminated and increases by one the dimensionality of the efficient surface. In the general case, if the belief formation function are linear in parameters then the behavior equations are linear in parameters $\xi_{h,a.i,jb,.lmn,c\ldots} = \alpha_{i,a} \cdot \beta_{ij,b}\, \gamma_{lmn,c} \ldots$. If the first n moments enter the utility function the behavior equations are homogeneous polynomials in the X's of degree $\frac{(n-1)n}{2}$. We must give a-priori values to n-1

2. See note page [2].

unknown parameters (at most one α_{ij}, at most one β_{ij}, etc.) before we can derive the values of the rest from a knowledge of the γ's.

A large Investment Company may hold well over 100 securities; an analysis which took individual account of all of them would be too complex to be carried thru. For this reason we have sought behavior equations concerning part of the portfolio, which did not take into account the allocation of funds among or probability beliefs about securities in the rest of the portfolio. This is the problem of "segregation."

If we make no more assumptions than have already been made, segregation is impossible. For whether or not an allocation of funds within a group is efficient (i.e., whether or not it minimizes total V, given E and A and given the allocation of funds in the rest of the portfolio) depends upon the allocation of funds in and probability beliefs about the rest of the portfolio.

An added assumption which may reasonably be made is the following: Let Y_1 and Y_2 be the yields of two securities in the same industry. Let Y_i be the yield on some security in a different industry. It may be reasonable to suppose that the correlation (ρ_{1i}) between Y_1 and Y_i always equals (ρ_{2i}) that between Y_2 and Y_i. Below we will consider a possible justification of this assumption. For the moment, let us assume that it is true.

Suppose $X_1, \ldots, X_K$ are the securities of enterprises in a particular industry; $X_{K+1}, \ldots, X_n$ are other securities held by the investor. Assume that $\rho_{ik} = \rho_{jk}$ for $i, j \leq K$, $k > K$. We then can write equations 5.1 to 5.K as

9.j $$\sum_{i=1}^{K} \sigma_{ij} X_i + \lambda_1 \mu_j + \lambda_2 = -\sqrt{\sigma_{jj}} \left(\Sigma \rho_{ij} \sqrt{\sigma_{ii}} X_i\right) = -\sqrt{\sigma_{jj}}\, K.$$

We can eliminate the three constants λ_1, λ_2, K in the same way that we eliminated λ_1 and λ_2 above; and by substituting belief formation functions, can derive behavior equations.[3]

Various assumptions as to the way investors believe the vector $(Y_1,\ldots,Y_n)$ is generated will lead to the result that the subjective correlations $\rho_{ih} = \rho_{jk}$ for $j \leq K$, $k > K$. Consider, for example, the following assumptions $Y_i = \mu_i + P_i + \delta_i C$ $\quad \delta_i$ a constant

$$i = 1,\ldots,K$$

$$Y_i = \mu_i + D_i \qquad\qquad i = K+1,\ldots,N.$$

$$E\,P_i = E\,C = E\,D_j = 0$$

$$E\,P_i\,D_j = 0 \qquad\qquad i \leq K,\ j > K$$

$$\frac{E(P_i + \delta_i C)^2}{E(\delta_i C)^2} = \frac{E(P_j + \delta_j C)^2}{E(\delta_j C)^2} \qquad i,j \quad K$$

From this it follows that

$$\rho_{ih} = \frac{E(P_i + \delta_i C)\,\Pi_k}{\sqrt{E(P_i + \delta_i C)^2}\sqrt{E\,\Pi_h^2}}$$

$$= \frac{E(P_j + \delta_j C)\,\Pi_h}{\sqrt{E(P_j + \delta_j C)^2}\sqrt{E\,\Pi_h^2}} = \rho_{jk}$$

Weaker conditions could be imposed upon the above variables;but the above model has a clear cut, somewhat plausible economic interpretation. Deviations of Y_i from its expected value μ_i are ascribed to two sources; an "industry-wide" source and a "personal" source. The industry-wide source includes factors such as changes in product or factor prices; its effect on the yield of firm i is given by δ_i C (where δ_i is a constant which

3. Here we must have $\sqrt{\sigma_{11}} = \sqrt{\sigma_{11}(Z_1,\ldots,Z_s)}$ linear in any unknown parameters if our behavior equations are to be similarly linear.

may differ from firm to firm). The "personal" source may include factors such as market position or skill of management; its effect is P_i. If we assume that the yield of a security of firm outside the industry is independent of the personal component of the yield of a firm within the industry (although it may be correlated with the industry wise component, and the personal components of firms within the industry may be correlated) and if we assume that, between firms in the industry, the variance of the industry wide component is proportional to the total variance of yield, then it follows that $\rho_{ih} = \rho_{jh}$.

Other assumptions as to the generation of $(Y_1, \ldots, Y_n)$ will also lead to the result, $\rho_{ih} = \rho_{jh}$. Such equality of subjective correlation coefficients cannot, however, be accepted a priori, and may in some cases prove unreasonable. It should be remembered that these correlation coefficients describe subjective beliefs not objective reality. The former, one may suspect, is simpler than the latter.

To generalize this discussion to the case where utility depends on higher moments we must define a "generalized correlation coefficient"

$$\rho_{\alpha_1, \ldots, \alpha_n} = \frac{E(\alpha_1 - E\alpha_1)(\alpha_2 - E\alpha_2)\ldots(\alpha_n - E\alpha_n)}{\sqrt[n]{E(\alpha_1 - E\alpha_1)^n}\ \sqrt[n]{E(\alpha_2 - E\alpha_2)^n}\ldots\sqrt[n]{E(\alpha_n - E\alpha_n)^n}}$$

We must assume $\rho_{\alpha_1 \alpha_2 \ldots \alpha_n} = \rho_{\alpha_1^*, \alpha_2 \ldots \alpha_n}$ if $\alpha_1, \alpha_1^* \leq K$ and some $\alpha_i > K$. Each additional moment not only adds a new constant λ_m but also a new constant K_m. Thus if U is assumed to be a function of the first n moments, there are 2n-1 constants to be eliminated.

In this paper, simplified behavior equations and aggregate behavior equations have been derived. The goal has been to obtain equations simple enough to make the fitting of them practicable. Our results can be easily generalized to the case where utility depends on higher moments. Increasing complexity and decreasing content of behavior equations follows as more moments are included in the utility function.