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A Formal Approach to Localisation Theory

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The basic concern of localisation theory, that is to explain the spatial constellation of production and consumption from certain data (as the loci of resources, distribution of population, facilities of communication) may, for the simple case of stationary conditions and a single commodity under consideration, be completely set in terms of price function. Take

$$s(x_1, x_2, p)$$
$$d(x_1, x_2, p)$$
$$k(x_1, x_2)$$

then as the local cost-or supply-function, the local demand-function, and the local transportation-cost function, respectively. In this it is assumed that the cost functions are strictly locally determined and independent from output at all other places; and similar for demand and transportation-costs. Given then these three functions in an area A with a boundary B and a boundary value constraint, expressing the foreign trade conditions in the considered commodity, there should be sufficiently determined:

1. The price distribution $p(x_1, x_2)$ for the commodity, C,

and

2. a vector-field $v(x_1, x_2) = [v_1(x_1, x_2), v_2(x_1, x_2)]$,

giving the amount and direction in which C is transported at any point of A.

By means of $s(x_1, x_2, p)$ and $d(x_1, x_2, p)$ the price distribution $p(x_1, x_2)$

completely determines production and consumption throughout A. From $v(x_1, x_2)$ all movements in C and, by means of $k(x_1, x_2)$, the local expense in transportation costs would be known. In fact, $p(x_1, x_2)$ and $v(x_1, x_2)$ would solve the localization problem with respect to C completely. It might be asked, however, whether $v(x_1, x_2)$ is not already sufficiently determined through $p(x_1, x_2)$, and vice versa. This will easily come out later.

So far the functions involved have not been specified. Henceforth it is assumed that s, d and k are continuous in x_1, x_2, and p. There is no loss of generality in it because every actually occurring functional relation can be approximated by a continuous function to any desired degree. The mathematical convenience of continuity in this case will be obvious. As a boundary condition it shall be assumed that no export nor import in C takes place, that is, that $[v(x_1, x_2)] = 0$ for (x_1, x_2) on B.

I (*)

It will be useful to derive $v(x_1, x_2)$ in the much simplified case of given production and consumption at first, that is to solve the problem with functions

$$s(x_1, x_2)$$
$$d(x_1, x_2)$$
$$k(x_1, x_2).$$

As an illustration of this take a centrally administered economy with fixed production orders and consumption rations in C. How are the transportation ways to be selected most economically? Obviously the total of transportation-cost must be minimal.

There is however a constraint into which the connection between production, consumption, and transportation enters. There is a complete analogy to hydro-dynamics, giving rise to the well known condition

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(*) See page 11, this paper for explanation.

(1) $s(x_1, x_2) - d(x_1, x_2) = \text{div } v\ (x_1, x_2)$, the "continuity-equation."[1]

In the following s and d will occur only in this form. Call $s(x_1, x_2) - d(x_1, x_2) = g(x_1, x_2)$. Then minimization of transportation-costs with regard to constraint (1) leads now immediately to

(2) $$0 = \delta \iint_A \left[k(x_1, x_2) \left| v(x_1, x_2) \right| - \lambda [g(x_1, x_2) - \text{div } v(x_1, x_2)] \right] dx_1 dx_2$$

where λ is a Lagrange parameter.

From the boundary-value condition

(3) $v(x_1, x_2) = 0$ for all (x_1, x_2) on B it follows that

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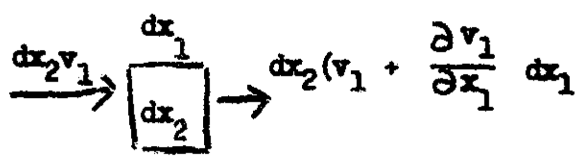
1) Consider the cell $dx_1 dx_2$



The C-stream passing through may be divided in its horizontal and vertical components v_1, v_2 respectively. By multiplying the stream vector with the cell breadth, one gets for the horizontal direction

$$dx_2 \cdot v_1(x_1, x_2) \text{ and } dx_2 \cdot v_1(x_1 + dx_1, x_2) = dx_2[v_1(x_1, x_2) + \frac{\partial v_1(x_1, x_2)}{\partial x_1} dx_2]$$

as incoming and outgoing flow respectively. The difference

$$dx_2 \cdot dx_1 \frac{\partial v_1(x_1, x_2)}{\partial x_1} ,$$

and correspondingly for the vertical stream

$$dx_1 \cdot dx_2 \frac{\partial v_2(x_1, x_2)}{\partial x_2} ,$$

added together gives the amount of additional C, that enters the transported stream while in $dx_1 \cdot dx_2$. This equals to $dx_1 \cdot dx_2 \cdot [s(x_1, x_2) - d(x_1, x_2)]$. Hence

$$s(x_1, x_2) - d(x_1, x_2) = \frac{\partial v_1(x_1, x_2)}{\partial x_1} + \frac{\partial v_2(x_1, x_2)}{\partial x_2}$$

the E_ler equations are applicable:

$$(4)\qquad 0 = \frac{\partial}{\partial v_i}\left(k(x_1, x_2) \cdot \sqrt{v_1^2(x_1, x_2) + v_2^2(x_1, x_2)}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial\left(\frac{\partial v_i}{\partial x_i}\right)}\left[\lambda\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right)\right]\right) \quad i = 1, 2.$$

Hence eventually:

$$(5)\qquad k\frac{v}{|v|} - \text{grad}\,\lambda = 0 \quad (2)$$

where, of course, k, v and λ are functions of (x_1, x_2).

By (5) we are lead to consider the field of

$$(6)\qquad k(x_1, x_2)\,\frac{v(x_1, x_2)}{|v(v_1, x_2)|} = w(x_1, x_2)$$

which is a scalar transformation of our original v-field, that is, it has the same direction as the original field in any point but different vector magnitudes. w is distinguished as a gradient field. This implies because of rot grad $\equiv$ 0, that w is free from curls, that is any circular movement, a plausible statement also from an economic point of view.

Integrating w along a streamline, the vector differential of which being $dx \equiv d(x_1, x_2)$, one has $\int_{x'}^{x''} k(x_1, x_2)\,\frac{v(x_1, x_2)}{|v(x_1, x_2)|}\,dx$, which because dx and v have the same direction, simplifyies to

$$(7)\qquad \int_{x'}^{x''} k\,d\,|x| = \int_{x'}^{x''} k\,d\,s \text{ , where ds is the way-element.}$$

Thus (7) measures the cost of transportation from the point x' to x'' (both of which bound to the same streamline).

On the other hand (7) is, because of (5) equal to $\int_{x'}^{x''} \text{grad}\,\lambda \cdot dx$, which as a

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(2) This is a necessary condition. Sufficiency proofs, to be based upon the four sufficiency theorems in the calculus of variations, would turn out too unhandy. There is a strong economic presumption so that there really <u>is</u> a minimum. Then it must obey (5), so that we may satisfy ourselves.

total differential can be evaluated to $\int_{\lambda(x')}^{\lambda(x'')} d\lambda = \lambda(x') - \lambda(x'')$ whence

(8) $$\int_{x'}^{x''} k \, ds = \lambda(x') - \lambda(x'') \; .$$

The "potential curves" λ of w thus have the remarkable property to indicate the transportation cost between any pair of points in A. It must , however, be kept in mind that in directions against the stream vector the costs are measured as negative.

Finally it may be noted that w and hence v intersect the system of potential curves λ = const. orthogonally in every point of A, an easy consequence of w being the gradient field of the λ equations.(3)

Finally it may be noted, that by virtue of the boundary conditions the Gauss theorem provides for

(9) $$0 = \iint_A \operatorname{div} v \, dx_1 dx_2 = \iint_A g(x_1, x_2) \, dx_1 dx_2$$

which means nothing more than the equality of total production and consumption, of C.

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(3) It may be stated that the obvious analogy of this "transportation field" to vector fields in physics does not extend very far, the main difference being that in physics an expression, similar to k v^2 (instead of k v), the Energie, or more exactly the Hamilton-function, is to be extremized. Thus the resulting equations

$$k \frac{v}{|v|} - \operatorname{grad} \lambda = 0 \; .$$
$$\operatorname{div} v - g = 0$$

have no close relationship to the potential function of hydro-dynamics. There is however some resemblance to optics. Consider for the moment k to be the "local resistance," that is the reciprocal local speed. We then have in v a system of "curves of quickest descent" and equations (1) and (5) are the most general forms of the "law of refraction" in traffic, which in its simpler, linear, form has been treated by Launhard,(1) Palander(2) and v Stackelberg(3).

(1) Launhard, W., Mathematische Begründung der Volkswirtschaftslehre, 1885

(2) Palander, T., Beiträge zur Standortheorie, 1933

(3) v. Stackelberg, H., Das Brechungsgesetz des Verkehrs Weltwirtschaftliches Archiv, 1935.

II

Let $g(x_1, x_2)$ not now be given but be dependent upon a variable p, the local price, which itself is an unknown to be determined. A necessary condition for equilibrium in A with respect to C is that the price difference for C between any two points in A is exactly equal to the least transportation cost, provided that there are no unconnected isolated regions in A. In the latter, the price $p(x_1, x_2)$ would be so as to make $g(x_1, x_2, p(x_1, x_2))$ vanish. Excluding such occurrences for the rest of this paper it appears suggestive to put

(10) $$p(x_1, x_2) = \lambda(x_1, x_2) + C$$

for price differences obey the same conditions as does λ according to (8).

A rigid justification of this comes with the notion that whatever the form of the isopricelines, they must intersect the transportation vectors orthogonally.[4] This holds for any point and hence any vector streamline in A. The system of streamlines now determines that of the orthogonal transversals uniquely. Hence, as the λ - curves are orthogonal to the streamlines they must coincide with the system of isopricelines.

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(4) To prove this assume that a point x and an isopriceline I might be given. We ask for that curve between x and I which makes

$$\int_x^I k(x_1, x_2) \sqrt{1 + \left(\frac{dx_2}{dx_1}\right)^2}\, dx_1 \quad \text{minimal.}$$

It follows from the well known transversality-theorem that the resulting extremale E must intersect I transversally, that is, so as to make

$$F(x, y, y') + (\tilde{y}' - y')\ F_{y'}(x, y, y') = 0,$$

where

$$F = k(x_1, x_2) \sqrt{1 + \left(\frac{dx_2}{dx_1}\right)^2}$$

$$y = x_2 \qquad y' = \frac{dx_2}{dx_1}$$

and $\tilde{y}'$ the slope of I at the intersection, it follows:

$$0 = k(x_1, x_2) \sqrt{1 + \left(\frac{dx_2}{dx_1}\right)^2} + (\tilde{y}' - \frac{dx_2}{dx_1})\ k \frac{\frac{dx_2}{dx_1}}{\sqrt{1 + \frac{dx_2}{dx_1}}}$$

and, as $k(x_1, x_2) \neq 0$

$$1 + \frac{dx_2}{dx_1}\tilde{y}' = 0$$ that is E and I are orthogonal, q.e.d.

This and equation (8) are sufficient for (10).

We then get

(11a) $$k(x_1, x_2) \frac{v(x_1, x_2)}{|v(x_1, x_2)|} - \text{grad } \lambda(x_1, x_2) = 0$$

(11b) $$\text{div } v(x_1, x_2) - g(x_1, x_2, \lambda(x_1, x_2) + C) = 0$$

immediately from (5). For as (5) holds for any given local production and consumption distribution it is valid also when g is again given through λ. A regression to the variation equation (2) need not be made. Equations (11) are then already a solution of the initial problem. The constant C is fixed through (9) which cares for

$$\iint_A g(x_1, x_2, \lambda(x_1, x_2) + C)\, dx_1 dx_2 = 0 .$$

In discussing the above differential equations which written out appear to be

(12) $$\frac{v_i}{\sqrt{v_1^2 + v_2^2}} k(x_1, x_2) - \frac{\partial \lambda(x_1, x_2)}{\partial x_i} = 0 \qquad i = 1, 2$$

(13) $$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} - g(x_1, x_2, \lambda(x_1, x_2) + C) = 0,$$

we may first notice that they constitute a boundary value problem, linear in v_1 and v_2, and quasi-linear in λ. (12) may be easily combined into

(14) $$\left(\frac{\partial \lambda}{\partial x_1}\right)^2 + \left(\frac{\partial \lambda}{\partial x_2}\right)^2 - k^2(x_1, x_2) = 0$$

and (13) brought into

(15) $$\frac{\partial}{\partial x_1}\left(y \frac{\partial \lambda}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(y \frac{\partial \lambda}{\partial x_2}\right) - g(x_1, x_2, \lambda) = 0$$

where $y = \sqrt{v_1^2 + v_2^2}$.

The reduction into a single equation of higher order is not possible. Nor can an explicit solution of (14), (15) be given in a routine way. For the characteristic curves of (15) attain the form

$$\frac{dx_2}{dx_1} = \frac{\partial \lambda}{\partial x_2} : \frac{\partial \lambda}{\partial x_1},$$

which cannot be solved without λ being specified. λ on the other hand depends upon (15) because of the boundary conditions. It might seem, as if this cross-dependency could be avoided through suitable boundary conditions in $\frac{\partial \lambda}{\partial x_1}$ exclusively. This would however lead to a determination of λ by (14) alone, the singularities of which might not coincide with those from (15), so that in the general case, y from (15) cannot be fitted into the λ-frame. Moreover there are no economic data for a choice of boundary conditions in $\frac{\partial \lambda}{\partial x_1}$. The "natural boundary conditions" lead upon λ = const., which is incompatible with any singularities not depending upon (14) alone. The economic interdependency between transportation lines and conditions of production and consumption thus turn out of the equation-structure. The economic interpretation of this may state, that g, the a priori conditions for production and consumption at any point in A, carries on the boundary conditions and the singularities (as centers of production or separating lines between market areas). These once being fixed, however, the detailed structure of the price-field in A is solely an outcome of the transportation conditions. This is almost self-evident in such simple cases as, for instance, when the λ-circles are circles concentric to a singularity S.

III

There are some easily derived generalizations. First: though $k = k(x_1, x_2)$ was hitherto assumed to be independent from the amount of C to be transported, this is by no means required in the present form of equations (14) and (15). It then holds, that

$$(16) \qquad \left(\frac{\partial \lambda}{\partial x_1}\right)^2 + \left(\frac{\partial \lambda}{\partial x_2}\right)^2 - \left[k(x_1, x_2\, y) + y\,\frac{\partial k}{\partial y}\right]^2 = 0$$

$$(15) \qquad \frac{\partial}{\partial x_1}\left(y\,\frac{\partial \lambda}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(y\,\frac{\partial \lambda}{\partial x_2}\right) - g(x_1, x_2, \lambda) = 0$$

as may, without difficulty, be derived from (2). Here both equations contain each of y, λ and the interdependence goes even further than in (14) and (15).

Secondly, Let k now be dependent upon the direction of transportation. We may assume a linear dependency, that is, if u and z are vectors through point x, the cost of transportation over z and u being C_u and C_z respectively, then $C_{u+z} = C_u + C_z$. Instead of scalar k we then have a matrix $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$ and (2) becomes

$$\delta \int\!\!\int_A \left\{ |K v| - \lambda(\text{div } v - g) \right\} dx_1 dx_2 = 0 .$$

This easily gives:

(17) $$\frac{K' K v}{|K v|} - \text{grad } \lambda = 0$$

where K' is the transposed of K. Denote $\frac{K v}{|K v|} = w$ in analogy to (6) and have

(17b) $$K' w - \text{grad } \lambda = 0$$

(17b) reveals that in this case the streamlines do no more intersect the isopricelines orthogonally (though, of course, transversally). It also is no longer true that rot $w \equiv 0$ throughout A. Thus dependency of transportation cost from the direction leads to an essential distortion, of the simple pattern above.

Thirdly: Consider now the case of several commodities to be exchanged throughout A. Denote the respective magnitudes by an index i. $i = 1, \ldots u$.
Integral (2) then becomes:

(18) $$\int\!\!\int_A \left\{ \sum_{i=1}^{u} k_i \, |v_{(i)}| - \lambda_i (\text{div } v_{(i)} - g_i) \right\} dx_1 dx_2$$

Notice that

$$g_i = g_i(x_1, x_2, p_1, \ldots, p_u) = g_i(x_1, x_2, \lambda_1 + C_1, \ldots, \lambda_u + C_u)$$

and

(19) $$\sum_{i=1}^{u} p_i \, g_i = 0$$

identically in A. This latter may be regarded as the local "budget equation." It is no constraint for p_i but rather part of the definition of the g_i.

Variation of (18) yields:

(20) $$k_i \cdot \frac{v_{(i)j}}{|v_{(i)}|} - \frac{\partial \lambda_i}{\partial x_j} = 0 \qquad \begin{array}{l} j = 1, 2 \\ i = 1, \ldots, u. \end{array}$$

as the u-dimensional pendant of (5).

Let commodity C_u now be money and its transportation costs zero (or an independent constant) throughout A. $k_u = 0$. We then have no restriction for v_u except that $\iint_A \operatorname{div} v_{(u)}\, dx_1 dx_2 = 0$, that is that no money is added or lost in A.

$\tilde{v} = \sum_{i=1}^{u-1} v_i p_i$ then represents the total commodity stream through any point weighted by its component values. From (20):

$$(21) \qquad \tilde{v} = \sum_{i=1}^{u-1} \frac{\lambda_i + C_i}{k_i} \operatorname{grad} \lambda_i$$

This is not a gradient vector itself.

Fourth: So far the field vectors applied were to indicate either amount or transportation costs. It is also possible to consider time vectors, such vectors which measure the distances to be reached in time unit from any considered point, $k(x_1, x_2)$. In this case k would mean the reciproce of speed. When time is the main cost factor and a proportional gauge for cost, nothing will be changed in the above consideration and the w-fields represent time as well as cost.

There remains two major limitations of the present attempt. The first one is our requirement that every point should be communicated with the total system through actual transportation of commodities. This was pre-required for $p(x_1, x_2) = \lambda(x_1, x_2) + C$. If this assumption is dropped the present approach leads to a solution in which

(1) $\operatorname{div} v - g = 0$

is hurt at the singularities. That means that commodities would be stocked in certain points and be missing at other ones. It appears possible to give an adequate solution for this case by leaving the boundaries variable, requiring that

$$\left.\begin{array}{ll} (3) & \gamma = 0 \\ (22) & g(x_1, x_2, \lambda(x_1, x_2) + C) = 0 \end{array}\right\} \quad \text{on B}$$

In general we then have no simply connected area in a topological sense. The consideration involved would lead too far apart to be included in this tentative paper.

The most serious shortcoming in this attempt is however the abolishment of time and, related to this, of interlocal effects on production-and consumption-conditions. It is inadequate to assume that g be strictly locally conditioned and a priori given. For the function $g(x_1, x_2, p)$ will vary according to the actual production-and consumption-rates in a neighborhood of x_1, x_2. And it is here that the phenomenon of outer savings, so essential in localization problems, enters. It may be stated then that this approach will be of little predictive power, and thus little theoretical value, unless it can be merged into a dynamical model. Whether it might be possibly applicable to problems of spatial planning appears, so far, even more questionable.

(*) After completion of the manuscript it has been found that the problem in section (I)has been treated for a special case and with much farther carrying results by T. C. Koopmans, "Optimum Utilization of the Transportation System," L.P.C. No. 401. To indicate the connection, Koopmans' problem may be formulated in those terms applied here: Consider the commodity "empty ship space," produced in given quantities at given discrete points (the ports with an excess of cargo received over cargo dispatched) and consumed at other given discrete points (the other ports) in given quantities. What is the most economic transportation of this commodity between these points, when local transportation costs are constant?

λ in the present notion turns out to be the potential function of Koopmans. There is a difference only with respect to the area under consideration, which in Koopmans paper in the closed surface of the globe, so that closed circuits of transportation curves are then possible which is excluded in the present case because of rot $v = 0$.