COWLES COMMISSION DISCUSSION PAPER: ECONOMICS No. 291

THE INVALIDITY OF CLASSICAL MONETARY THEORY

SUMMARY

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October 2, 1950

In two articles which appeared in this journal the micro- and macro-economic versions of classical monetary theory were, respectively, examined.[1] The main conclusion of this analysis was that the classical attempt to dichotomize the economic processes of a monetary economy into a real sector, dependent upon and determining relative prices, and a money sector, dependent upon and determining absolute prices, cannot possibly succeed. In particular, it was argued that this dichotomized theory is either inconsistent, or, at best, indeterminate in the absolute prices. An alternative theory was then suggested[2] which was free of this dichotomization.

These propositions were recently attacked by W. Braddock Hickman,[3] Wassily Leontief,[4] and Cecil G. Phipps[5] in criticisms which appeared in this journal. It is the purpose of this note to answer these criticisms, and

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1. Don Patinkin, "Relative Prices, Say's Law, and the Demand for Money," Econometrica, Vol. 16, April, 1948, pp. 135-54; henceforth referred to as "Demand for Money;" "The Indeterminacy of Absolute Prices in Classical Economic Theory," ibid., Vol. 17, January, 1949, pp. 1-27; henceforth referred to as "Indeterminacy."

2. "Indeterminacy," § 14, pp. 23-27.

3. W. Braddock Hickman, "The Determinacy of Absolute Prices in Classical Economic Theory," ibid., Vol. 18, January, 1950, pp. 9-20; henceforth referred to as "Hickman."

4. Wassily Leontief, "The Consistency of the Classical Theory of Money and Prices," ibid., pp. 21-24; henceforth referred to as "Leontief."

5. Cecil G. Phipps, "A Note on Patinkin's Relative Prices," ibid., pp. 25-26; henceforth referred to as "Phipps."

thus to show that the conclusions of the original analysis remain intact. The material in Karl Brunner's survey of the problem--printed elsewhere in this issue--will also be taken into account.[6]

1. Phipp's criticism is the only one which refers directly to "Demand for Money;" hence it will be most convenient to consider it first. Phipps argues that the assumption that money does not enter the utility function is inconsistent with the assumption that the price of money is one. Hence he claims that system B of "Demand for Money" (p. 145) is based on contradictory assumptions.

The most direct way of refuting Phipps' criticism is by offering a simple historical counter-example: In England, for many years, the guinea existed as a unit of account, without there being any units of this money in existence. Clearly the price of a guinea was one (guinea); yet no one held any quantities of guineas, and no one wanted to. That is, the guinea did not enter the utility function.

It must be emphasized that the validity of this counter-example cannot be attacked by claiming that the guinea had a non-zero value "because the guinea was set equal to twenty-one shillings." The phrase in quotations has meaning only if it were possible to exchange a guinea for twenty-one shillings; for a non-existent guinea it becomes meaningless.

If we examine Phipps' argument to find why and where he went astray, we immediately see that he does not take account of a commodity with zero initial stocks. Thus, in the proof of his first corollary, the crucial sentence is: "Since no useful good would be exchanged for the useless good, the relative price of the useless good is zero." A similar sentence appears in the proof

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6. Karl Brunner, "Inconsistency and Indeterminacy in Classical Economics," below, pp. , henceforth referred to as "Brunner."

of Phipps' second corollary. But the argument of this sentence clearly does not hold if there are no initial stocks of the "useless good" in existence.

The fallacy of Phipps' argument can be demonstrated mathematically by returning to the original analysis of "The Demand for Money." Using the notation of §2 there we have that the equilibrium condition for the excess demand for money is

(1.1) $Z_n(p_1,\ldots,p_n) - \bar{Z}_n = 0,$

where $Z_n(p_1,\ldots,p_n)$ is the demand for money, and $\bar{Z}_n$ is the stock of money in existence. Under the assumption that money does not enter the utility function it was shown[7] that $Z_n(p_1,\ldots,p_n) = 0$ identically in the p_i $(i=1,\ldots,n)$. Phipps now says that, under this assumption, a necessary condition that (1.1) be satisfied is that $p_n = 0$. But this argument is clearly incorrect, since (1.1) can be satisfied if $\bar{Z}_n = 0$. It might be said that this is a trivial case. But it is only in this trivial case that it was ever claimed that system B could be consistent; this is precisely the content of the theorem on page 143 of "Demand for Money;" and it is precisely because of this triviality that system B is rejected there as a description of the real world.

2. The cricitisms of Hickman and Leontief are directed against the "Indeterminacy" article. It seems best to answer their criticisms by reformulating briefly the argument of this article, relegating point-by-point refutation to the footnotes.

First a few words of a general nature. The issue at stake is the correspondence of the classical economic system to the real world; for if it can be shown that this system is either internally inconsistent or indeterminate, it clearly cannot be used as an explanation of reality.[8] On the other hand,

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7. <u>Ibid.</u>, theorem on page 142.

8. Thus I feel that Hickman's assertion (p. 9) that the "Indeterminacy" article "criticizes neither the basic assumptions of the theory nor the theoretical conclusions from the standpoint of their correspondence with the known facts of economic life" is definitely incorrect.

internal consistency and determinacy are only necessary conditions for the acceptability of an economic model. They clearly are not sufficient. We shall not enter into this question of sufficiency; instead, we shall be interested only in determining whether the models examined meet certain necessary conditions for acceptability.

Secondly, one of the issues not at stake is that of the dependence of the classical functions. This is a completely false issue that has been the source of much confusion. Consequently it is well to recognize from the outset that there is no relationship whatsoever between functional dependence, on the one hand, and the consistency of a system of equations, on the other.

What is the meaning of functional dependence? Consider a system of functions $u_i = f_i\ (x_1,\ldots,x_m)\ (i=1,\ldots,n)$. The u_i are said to be functionally dependent if there exists a functional relationship, $F\ (u_1,\ldots,u_n) = 0$, connecting them. It follows that, in general, any time there are more functions than variables, the functions are dependent. For let $n = m + 1$. Then, if the Jacobian of the first m functions does not vanish, it is possible, within a certain neighborhood, to solve out the x_j $(j = 1,\ldots,n-1)$ as functions of the u_j—say, $x_j = g_j\ (u_1,\ldots,u_{n-1})$. Substituting these in the remaining function we obtain $u_n = f_n[g_1(u_1,\ldots,u_{n-1}),\ldots,g_m(u_1,\ldots,u_{n-1})]$ $= G(u_1,\ldots,u_{n-1})$ within a certain neighborhood, so that functional dependence clearly exists. If, however, the number of functions is equal to the number of variables, then there exists functional dependence if, and only if, the Jacobian of the n functions vanishes identically. These are all well known theorems.[9]

As counterexamples establishing the complete absence of any relationship

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9. Cf., e.g. W. Benjamin Fite, Advanced Calculus (Macmillan: New York, 1938), chap. 12.

between functional dependence and consistency, consider the following systems:

(2.1) $$u_1 = x^2 + y^2 - 1 = 0$$
$$u_2 = x + y - 20 = 0$$
$$u_3 = x^2 + y^2 + x + y - 21 = 0$$

(2.2) $$w_1 = x + y - 2 = 0$$
$$w_2 = x + 2y - 2 = 0$$
$$w_3 = 3x + y - 3 = 0$$

(2.3) $$v_1 = x + y - 2 = 0$$
$$v_2 = x + 2y - 2 = 0$$
$$v_3 = 3x + y - 6 = 0$$

The functions of (2.1) and (2.2) are in each case dependent; for $u_3 = u_1 + u_2$, and $w_3 = 5w_1 - 2w_2 + 3$. But both systems are inconsistent in the real number system. It is interesting to note that these inconsistencies arise in two different ways. In system (2.1) we see from the functional relationship $u_3 = u_1 + u_2$ that any real point (x,y) which would satisfy $u_1 = u_2 = 0$, would also satisfy $u_3 = 0$; but the inconsistency flows from the fact that no such real point exists. In (2.2), on the other hand, there does exist a real point satisfying $w_1 = w_2 = 0$; but from the form of the functional dependence it is clear that this point cannot satisfy $w_3 = 0$; in fact, at this point $w_3 = 3$. In the case of (2.3) both of these difficulties are avoided: there does exist a real point satisfying $v_1 = v_2 = 0$; and the form of the functional dependence—$v_3 = 5v_1 - 2v_2$—assures that this point also satisfied $v_3 = 0$. Consequently system (2.3) is consistent.

Thus the presence of functional dependence implies nothing about consistency. Similarly, the reader can readily construct for himself systems of independent equations which are, respectively, consistent and inconsistent. In this way we see that <u>there is no relationship whatsoever between functional dependence and consistency</u>. Consequently, since we are interested in the consistency of the classical system, no further attention will be paid to its functional dependence.[10]

3. We shall now examine the Casselian system and show that it runs into difficulty at two different levels of analysis.

Using the same notation as that of "Indeterminacy," let us first consider the Casselian system of excess demand functions

$$(3.1) \qquad X_j = F_j \left(\frac{p_1}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}}\right) \qquad (j = 1, \ldots, n-1)$$

$$(3.2) \qquad X_n = F_n (p_1, \ldots, p_{n-1})$$

Classical economic theory made two statements about the form of the excess demand function for money, F_n—one explicit, the other implicit. On the one hand, it explicitly assumed this function to have the Cambridge form, so that

$$(3.3) \qquad X_n = M - K \sum_{j=1}^{n-1} p_j g_j \left(\frac{p_1}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}}\right)$$

where M is the given amount of money in circulation; K is the reciprocal of the velocity of circulation; and the g_j are the supply functions of the various commodities.[11] On the other hand, it assumed the commodity excess demand function to be homogeneous of degree zero in the prices; by virtue of Walras' law, we know that

$$(3.4) \qquad -F_n(p_1, \ldots, p_{n-1}) \equiv \sum_{j=1}^{n-1} p_j F_j \left(\frac{p_1}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}}\right)$$

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10. Hickman's basic error on this whole issue—one that vitiates most of his discussion in Part III, § 2—is his defining "a set of n equations [as] dependent if a solution of n-1 of them is necessarily a solution of the other." (Hickman, footnote 5). Hickman, of course, is free to define functional dependence as he likes; but he cannot attribute to his concept of "dependence" properties developed on the basis of the completely different definition given in the text above. In particular, neither the excess of functions over variables, nor the identical vanishing of the Jacobian implies "dependence" in Hickman's sense.

11. "Indeterminacy," (10.1) - (10.4).

identically in the p_j, so that the excess demand function for money must then be homogeneous of degree one in these variables.[12] But inspection of (3.3) shows that inasmuch as M and K are positive constants, this function cannot be homogeneous of degree one in the prices. Hence the classical assumption as to the form of the excess demand function for money is inconsistent with its assumption as to the form of the excess demand functions for commodities. For the sake of brevity this inconsistency of assumptions will be referred to as Invalidity I.[13,14] It holds in the classical system even under the assumption of Say's law.[15]

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12. Ibid., lemma on the top of p. 14.

13. This is the argument of "Indeterminacy," § 10(b). Use has also been made of the results of § 8 there. Justification for attributing the assumptions in the text to the classical economists is provided ibid., footnotes 5 and 7. Cf. also Oscar Lange, "Say's Law: A Restatement and Criticism," Studies in Mathematical Economics and Econometrics, ed. by O. Lange, et.al., Chicago, University of Chicago Press, 1942, footnote 2.

14. Hickman (p. 15) apparently misinterpreted Invalidity I as being based on the assumption that the Cambridge equation is an identity. He then goes on to point out (quite correctly) that it is only "an equation of condition and not an identity in the p's" (ibid.). It is difficult to find in the original article any possible basis for this misinterpretation; for nowhere in this article—neither in the mathematical equations nor in the literary discussion—is the Cambridge equation ever represented as an identity. But the main point here is that Hickman's entire discussion of Invalidity I is completely irrelevant. For, as has just been explained, this argument is concerned with the Cambridge function; not the Cambridge equation. Hence the question as to whether the Cambridge equation is a conditional equation or an identity can have no possible bearing on the issue in question.

15. This was shown by Lange, op. cit., p. 65. The only addition made by "Indeterminacy" § 10(b) is that classical price homogeneity alone—even without Say's law—is enough to cause Invalidity I. The converse also holds: for Lange's analysis shows that the assumption of Say's law for the commodity functions—even if they are not assumed to be homogeneous of degree zero in the prices—is inconsistent with the assumption of the Cambridge function.

4. Let us now drop the assumption that the excess demand for money has the Cambridge form, and instead assume that it is homogeneous of degree one in the prices. In this way Invalidity I is removed. Does this make the classical theory acceptable? As we shall see, the answer is in the negative. In order to show this it will be necessary to consider not the classical <u>functions</u>, but the classical <u>system of equations</u>

$$(4.1) \qquad F_j\left(\frac{p_1}{p_{n-1}},\ldots,\frac{p_{n-2}}{p_{n-1}}\right) = 0 \qquad\qquad (j = 1,\ldots,n-1)$$

$$(4.2) \qquad \sum_{j=1}^{n-1} p_j\, F_j\left(\frac{p_1}{p_{n-1}},\ldots,\frac{p_{n-2}}{p_{n-1}}\right) = 0$$

Here use has been made of (3.4) to replace the equation $F_n(p_1,\ldots,p_{n-1}) = 0$ by the one now appearing in (4.2). (It must be emphasized that at this stage of the argument—where Say's law is not assumed—(4.2) is a conditional equation and <u>not</u> an identity in the p_j.) Clearly, Invalidity I is not present in this system, for the homogeneity assumptions for (4.1) is consistent with the form of (4.2). Let us now examine the solution(s), if any, of this system.

First of all it should be noted that of necessity any solution satisfying (4.1) must also satisfy (4.2). Thus, on the one hand, equation (4.2) cannot be the source of any inconsistency in the system; on the other, if (4.1) has multiple solutions, (4.2) cannot be used to eliminate any of them. Consequently in examining (4.1) - (4.2) for solubility, we need only consider (4.1), and can completely disregard (4.2).

In (4.1) we have n-1 equation in the n-2 price ratios, $\frac{p_s}{p_{n-1}}$ $(s=1,\ldots,n-2)$. There are a total of three mutually exclusive possible alternatives: (a) There may be no subset of n-2 equations with a solution. This corresponds to (2.1) above, and in this case (4.1) is clearly inconsistent. (b) There may be a subset of n-2 equations with a solution, but one that does not satisfy the remaining equation. This corresponds to (2.2) above, and in this case (4.1) is again inconsistent. (c) There may be a subset of n-2 equations with a solution that does satisfy the remaining equation. This corresponds to (2.3) above. Even though (4.1) is, in this case, consistent, it is definitely indeterminate in the absolute prices. For by the classical homogeneity assumptions, the variables of the subset of n-2 equations are the n-2 price ratios $\frac{p_s}{p_{n-1}}$ $(s=1,\ldots,n-2)$. From the definition of alternative (c) any set of price ratios that satisfied these equations will also satisfy the (n-1)-th. Hence if a given set of absolute prices is a solution of (4.1), any multiple of this set is also a solution.

Thus we conclude that <u>even after the functions of the classical system are modified so as to be free of Invalidity I, the resulting system of classical excess demand equations must be either inconsistent or indeterminate.</u> That is, either there exists no set of absolute prices which will satisfy the classical system, or else there exists an infinite number of such sets. There are no other possibilities. For convenience, this will be referred to as Invalidity II.

It must be emphasized that the assumption of Say's law, so frequently made by classical economists, in no way affects this conclusion. For by virtue of Say's law we have that the form of $F_{n-1}\left(\frac{p_1}{p_{n-1}},\ldots,\frac{p_{n-2}}{p_{n-1}}\right)$ is

$$(4.3) \qquad -\sum_{s=1}^{n-2} p_s F_s \left(\frac{p_1}{p_{n-1}},\ldots,\frac{p_{n-2}}{p_{n-1}}\right).$$

Consequently, if the first n-2 equations have a solution, it must also satisfy the (n-1)-th. In other words, Say's law eliminates alternative (b) of the preceding paragraph. But from the remaining alternatives, (a) and (c), it is clear that Invalidity II still holds.[16,17]

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16. This section is a reformulation of §§10(a) and 13 of "Indeterminacy," and is intended, in particular, to replace the theorem on p. 14, <u>ibid</u>. For, as Hickman (p. 16) has shown, in the Casselian case the first $\overline{n-1}$ functions cannot be independent. However, as has been shown above, the whole issue of dependence is irrevelant here. Hence the basic conclusion of "Indeterminacy" is not affected.

(continued on next page).

Footnote 16 con't.

It should be pointed out that by resorting to microanalysis, alternative (b) above can be eliminated. For examination of the utility-maximizing conditions in "Demand for Money" showed that Say's law and the dependence of the commodity functions on relative prices were necessary and sufficient conditions for each other (ibid., Theorem XVI). Hence it is impossible to assume the latter without the former. However, it seems to me to be illegitimate to bring this fact in at the macroeconomic level of analysis, where such a relationship cannot be proved. Hence this alternative has been retained.

A question also arises with respect to alternative (a). Brunner (footnote 12) refuses to list this alternative on the grounds that it "is not really meaningful" since "if inconsistency is assumed, then there is nothing to analyse further." But on these same grounds Brunner should also refuse to list alternative (b)—which he does consider (ibid., §4). And he should also refuse to list alternative (c), since if indeterminacy is assumed, there is again nothing further to analyse. By this *reductio ad absurdum* we can see that this criticism misses the main point of the preceding argument, which is to list *all* the possible macroeconomic implications of the classical homogeneity assumption, and to show that *none* of them is acceptable.

17. The conclusions of this section can be compared to those of Hickman and Leontief in the following way:

Hickman, because of his incorrect definition of functional dependence, fails to recognize alternative (b). He then concentrates on alternative (c), and—in contrast with our conclusion here—claims that the resulting system is determinate. He achieves this result by introducing an additional equation which he calls a "constraint in the monetary sector" (Hickman, pp. 13 ff) that turns out to be the Cambridge equation. Unfortunately, Hickman fails to make clear the economic meaning of this "constraint" within his system. It cannot be a behavior equation, describing the community's behavior with reference to the holding of money; for Hickman has another (and distinct) equation in his system—the excess demand for money—which performs this function. Nor can it be a definition of the Cambridge K, inasmuch as Hickman (p. 13) considers this "an institutionally determined constant."

This leaves only the possibility that Hickman intends the "constraint" as some sort of technological relationship. But this does not correspond to the classical treatment of the Cambridge equation as a *behavior* equation. For example, Marshall's discussion of the Cambridge equation is concerned with "the amount of purchasing power which people...*elect to keep* in the form of currency." (*Money, Credit, and Commerce*, pp. 43 ff, italics added.) Finally, even if this question of correspondence with the classical theory were overlooked, and the "constraint" were introduced as some sort of technological relationship, Hickman's attempted defence of the classical dichotomy would still fail. This is proved in § 5 below; cf. especially footnote 28.

Leontief considers Say's law as an integral part of classical economic theory and—ignoring alternative (a)—argues that it is consistent. This, of course, is correct and was explicitly proved in the analysis of § 13 of "Indeterminacy." But what Leontief ignores—and what the analysis of § 13 goes on explicitly to point out—is that the system is then definitely indeterminate; that is, it is still subject to Invalidity II. In this § 13 was merely repeating Lange's well-known theorem that if "Say's law is assumed... money prices are indeterminate" (Lange, *op. cit.*, pp. 65-6). Leontief also overlooks Lange's proof that the assumption of Say's law is inconsistent with the classical assumption of the Cambridge function. Cf. above, footnote 15.

Finally, it is important to note that in the demonstration of Invalidity II there has been no need to resort to the method of counting equations and unknowns. Consequently, it is not necessary to make any assumption whatsoever about the relationship between the number of equations and variables in the system, on the one hand, and its consistency or determinacy, on the other.[18] It is even more obvious that no such assumption is involved in the proof of Invalidity I. Indeed, the excess demand equations are not even mentioned in that proof; only the excess demand functions are involved.

5. The "Indeterminacy" article discussed several other systems besides the Casselian one. Section 11 there describes a system whose distinctive property is that the commodity excess demand functions are homogeneous of degree zero only in a subset of prices. It is shown there that Invalidity I does not hold for this system. But, by an analysis completely analogous to that of the preceding section, the reader can readily establish that the system so described is guilty of Invalidity II.[19] Section 12 then discusses a system due to Lange. By dividing the excess demand equation for bonds by any one of the prices, it is easily seen that Invalidity II is again present.[20]

In § 14 of "Indeterminacy" two further systems were presented. The first is a "modified Lange system" (p. 22) which is identical with the Lange system except for the bond excess demand function, assumed now to be non-homogeneous. This system is shown to be free of both Invalidities I and II.[21]

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18. This is an improvement over the exposition of the "Indeterminacy," where such an assumption was made. Cf. ibid. §3.

19. In "Indeterminacy" alternative (c) was ignored so that the system was described as inconsistent.

20. Once again, the discussion there ignores alternative (c) and thus described the system as inconsistent. In "Indeterminacy," I attributed the Cambridge equation to Lange, and hence described his system as being guilty of Invalidity I as well (ibid., last paragraph of § 12). But, as Brunner points out, this is not correct: Lange rejects the Cambridge function and explicitly assumes that "the excess-demand function for money is homogeneous of first degree in the...prices." (Oscar Lange, Price Flexibility and Employment, Principia Press, Bloomington, Indiana, 1944, p. 100.)

21. Note that being free of these invalidities does not imply that the system must be consistent and determinate.

Nevertheless, this system was rejected there (see below). Instead, the major conclusion of the "Indeterminacy" paper was that the classical system must be modified in accordance with the system set out there in equations (14.1)-(14.4), and described at length in pp. 23-27. This modified classical system, as it is called there, can be written as follows:

$$(5.1) \qquad X_s\left(\frac{p_1}{p},\ldots,\frac{p_{n-2}}{p},\ r,\ \frac{p_A A + M}{p}\right) = 0 \qquad (s = 1,\ldots,n-2)$$

$$(5.2) \qquad X_{n-1}\left(\frac{p_1}{p},\ldots,\frac{p_{n-2}}{p},\ r,\ 1,\ \frac{p_A A + M}{p}\right) = 0$$

$$(5.3) \qquad \sum_{j=1}^{n-1} p_j X_j\left(\frac{p_1}{p},\ldots,\frac{p_{n-2}}{p},\ r,\ \frac{p_A A + M}{p}\right) = 0$$

where p is the general price level, p_A is the average price of assets held by the community, and M is the amount of money. Equations (5.1)-(5.3) are identical with (14.1) - (14.3) of "Indeterminacy" except for the following changes: <u>A</u> now represents the given physical amount of assets in the economy;[22] the functions are assumed to be dependent upon the total real value of assets, taken together, instead of the real values of non-monetary and monetary assets, respectively, considered separately; equation (14.2) of "Indeterminacy" has been divided through by p; equation (14.3) has been rewritten in the specific form dictated by Walras' law. It is readily seen that this system is free of Invalidities I and II.

As pointed out in the "Indeterminacy," the major modifications in this system is that "the excess demand for each commodity is affected by the value of all assets (monetary as well as non-monetary) in the economy" (p. 23). It was also pointed out that the "absolute price level appears everywhere in the system" (p. 25) and not only in the "monetary sector." Consequently the classical dichotomy does not hold. Nevertheless, it is shown that the system

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22. In "Indeterminacy" <u>A</u> represented the <u>money</u> value of these assets. I am indebted to Milton Friedman for pointing out the necessity for this change.

has other, distinctly classical, properties. In particular, a change in the amount of money changes all prices proportionately, but leaves the rate of interest invariant (ibid., pp. 25-7).

On what grounds did the analysis of "Indeterminacy" reject the modified Lange system in favor of the modified classical system? The explicit reason given was that the procedure of the former system in "singling out one particular equation and assuming it to be non-homogeneous" was unjustified and arbitrary.[23] But implicit in the description of the modified classical system are other, and more important reasons. Thus it is pointed out (ibid., p. 24) that the homogeneity properties assumed for (5.1) - (5.2) imply that the excess demand function for money must have the same homogeneity properties as the Cambridge function; the modified Lange system does not have this property. But the fundamental reason for rejecting this system can be seen only when we make use of microanalysis. As pointed out in the last paragraph of "Indeterminacy," "utility maximisation [implies] that the excess demand equations... must depend also on the amounts of assets in the economy." In the modified Lange system these assets do not appear; hence the system must be rejected.

Of still greater importance is the fact that utility maximization implies that the assets appear in the excess demand functions in such a way that these functions will not, in general, be homogeneous of degree zero in the prices alone. To prove this, consider an economy in which the individual is concerned with the real value of bonds and money that he holds at the end of the period in question.[24] Let $\bar{Z}_{ia}$ $(i = 1,\ldots,n)$ represent the initial stocks of the i-th

23. Ibid., p. 23. However, Brunner (§) has shown that under certain assumptions as to the operation of the banking system there may be good reasons for assuming non-homogeneity in the bond equation.

24. The following model follows that of Oscar Lange, Price Flexibility and Employment, p. 16, footnote 6. Several modifications and corrections have been made.

commodity held by the a-th individual; and let Z_{ia} be the stocks he plans to hold at the end of the period.[25] Let the (n-1)-th and the n-th commodities be bonds and money, respectively. Bonds are assumed to be all of one type: perpetuities paying $1 per period. Here it must be emphasized that, unlike other commodities, $Z_{n-1,a}$ and $\bar{Z}_{n-1,a}$ can take on negative values. That is, people with positive $Z_{n-1,a}$ are lenders; those with negative $Z_{n-1,a}$, borrowers. Furthermore, on the assumption of a closed economy, we must have

$$(5.4) \qquad \bar{Z}_{n-1} = \sum_{a=1}^{m} \bar{Z}_{n-1,a} = 0.$$

That is, every debt is owed by one member of the community to another. Finally, we should note the fact that the price of such a perpetuity, p_{n-1}, is the reciprocal of the rate of interest, r. We shall make this substitution throughout.

Assume that trading in bonds takes place only on the last day of the period, so that interest is received (paid) on all bonds held (outstanding) at the beginning of the period. Then the budget restraint of the a-th individual is

$$(5.5) \quad \sum_{s=1}^{n-2} p_s(Z_{sa} - \bar{Z}_{sa}) + \frac{(Z_{n-1,a} - \bar{Z}_{n-1,a})}{r} + (Z_{na} - \bar{Z}_{na}) - \bar{Z}_{n-1,a} = 0.$$

That is, the value of goods held at the end of the period must equal the value at the beginning plus (minus) interest receipts (payments) on bonds held (outstanding) at the beginning of the period. The individual is assumed to maximize his utility.

$$(5.6) \qquad V^a \left(Z_{1a},\ldots,Z_{n-2,a}, \frac{Z_{n-1,a}}{r\,p}, \frac{Z_{na}}{p}\right)$$

subject to the restraint (5.5) to yield

$$(5.7) \qquad \frac{V^a_s \left(Z_{1a},\ldots,Z_{n-2,a}, \frac{Z_{n-1,a}}{r\,p}, \frac{Z_{na}}{p}\right)}{V^a_t \left(Z_{1a},\ldots,Z_{n-2,a}, \frac{Z_{n-1,a}}{r\,p}, \frac{Z_{na}}{p}\right)} = \frac{p_s}{p} \qquad \begin{array}{l}(s = 1,\ldots,n-2)\\(t = n-1,n)\end{array}$$

where p now represents the price level of the first n-2 goods.

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25. This is the definition of Z_{ia} that should also be given in "Demand for Money" instead of the one now appearing there on p. 137.

Divide (5.5) through by p, and take as the dependent variables Z_{sa}, $\frac{Z_{n-1,a}}{p}$, and $\frac{Z_{na}}{p}$. It is then seen that the independent variables p_s, p, $\bar{Z}_{n-1,a}$, and $\bar{Z}_{na}$ always appear in (5.5), (5.7) in the form of ratios. Thus, when (5.5), (5.7) is solved out for the designated dependent variables, the Z_{sa} $(s=1,\dots,n-2)$ must be homogeneous of degree zero, in p_s, p, $\bar{Z}_{n-1,a}$, and $\bar{Z}_{na}$; and the Z_{ta} $(t=n-1,n)$, homogeneous of degree one in these same variables. Thus we can write the individual excess demand functions

$$(5.8)\qquad X_{sa} = Z_{sa}\left(\frac{p_1}{p},\dots,\frac{p_{n-2}}{p},r,\ \frac{\sum_{s=1}^{n-2} p_s\bar{Z}_{sa}}{p} + \frac{(1+r)\bar{Z}_{n-1,a}}{r\,p} + \frac{\bar{Z}_{na}}{p}\right) - \bar{Z}_{sa} \qquad (s = 1,\dots,n-2)$$

$$(5.9)\qquad X_{n-1,a} = pZ_{n-1,a}\left(\frac{p_1}{p},\dots,\frac{p_{n-2}}{p},r,\ \frac{\sum_{s=1}^{n-2} p_s\bar{Z}_{sa}}{p} + \frac{(1+r)\bar{Z}_{n-1,a}}{r\,p} + \frac{\bar{Z}_{na}}{p}\right) - \bar{Z}_{n-1,a}$$

$$(5.10)\qquad X_{na} = -\sum_{s=1}^{n-2} p_s\, X_{sa} - \frac{X_{n-1,a}}{r} + \bar{Z}_{n-1,a}$$

We have seen that the functions (5.8) are homogeneous of degree zero in the p_s, p, $\bar{Z}_{n-1,a}$, and $\bar{Z}_{na}$. Does this preclude their also being homogeneous in the prices alone? It should be emphasized that, for the general class of functions homogeneous of degree zero, homogeneity may simultaneously exist in a set of variables and in a proper subset of these variables. Thus the simple function $\frac{wx}{yz}$ is homogeneous of degree zero in w,x,y, and z, as well as in the four subsets w and y, w and z, x and y, x and z. In fact, by a simple application of Euler's theorem and its converse, it can be shown that if a function is homogeneous of degree zero in $y_1,\dots,y_n$, then a necessary and sufficient condition for it to be homogeneous of degree zero in the proper subset $y_1,\dots,y_m$ is that it also be homogeneous of degree zero in $y_{m+1},\dots,y_n$.

In our case this would mean that in order for (5.8) to be homogeneous of degree zero in the prices alone, it must be homogeneous of degree zero in $\bar{Z}_{n-1,a}$ and $\bar{Z}_{na}$. But the significant point here is that the way in which $\bar{Z}_{n-1,a}$ and

$\bar{Z}_{na}$ enter the demand functions is determined by the way they appear in the budget restraint (5.5); and here they appear as a sum, and not as a ratio. Consequently the only ways in which (5.8) can be homogeneous of degree zero in $\bar{Z}_{n-1,a}$ and $\bar{Z}_{na}$ (and hence in the prices alone) are the two trivial ones in which (a) the individual has no initial stocks of money or bonds, so that $\bar{Z}_{n-1,a} = \bar{Z}_{na} = 0$—an assumption that clearly cannot hold for every individual, since we are considering a money economy; or in which (b) the partial derivative of each of the functions in (5.8) with respect to its n-th argument—$\sum_{s=1}^{n-2} p_s \bar{Z}_{sa} + (1+\frac{1}{r})\bar{Z}_{n-1,a} + \bar{Z}_{na}/p$—is identically zero—an assumption that is clearly contradicted by the many empirical studies which show that the income effect is usually non-zero. Hence we conclude that the commodity excess demand functions (5.8) are not, in general, homogeneous of degree zero in prices alone.[26,27]

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26. In Theorem XIV of "Demand for Money"—a theorem due to Kenneth J. Arrow—it was shown that if the <u>nominal</u> amount of money entered the utility function, then utility maximization precluded the possibility of <u>all</u> the commodity demand functions of a given individual being homogeneous of degree zero in the prices. Looking back, it seems clear that an additional conclusion could have been drawn: since under the stated conditions prices do not enter the maximizing conditions as ratios, <u>none</u> of the functions can, in general, depend solely on relative prices. This is both stronger and weaker than the conclusion of theorem XIV: stronger, in that it refers to every function; weaker, in that it must add the qualifying phrase "in general" for reasons specified in the text above.

It should be noted that theorem XIV cannot be used in the case which interests us here; for we are now making the assumption that it is the <u>real</u> amount of money which enters the utility function. However, as will now be emphasized in the text, the assumption that money enters the utility function in any form is not necessary for the conclusions obtained here.

27. It should be noted that if the market excess demand equations are aggregated from (5.8) - (5.10)—making use of (5.4) and the assumption that it is only the total amount of assets in the economy, and not their distribution among individuals, which affects the market excess demand functions—we shall come out with the modified classical system of "Indeterminacy" § 14 rewritten in (5.1) - (5.3) above. Thus this system has a firm foundation in utility maximization.

In particular it should be emphasized that the market excess demand function for money resulting from such an aggregation—(5.3)—has the same homogeneity property as that of the classical Cambridge function. This, too, is homogeneous of degree one in the amount of money and prices. (Cf. "Indeterminacy," p. 24). <u>Thus the form of the classical money demand function is consistent with utility maximization.</u>

It is essential to note that the inability of the commodity demand functions to depend on relative prices alone is a direct result of the form the budget restraint must take in a money economy. To repeat, from the budget restraint (5.5) divided through by p we know that the commodity demand functions must depend upon the argument $\dfrac{\sum_{s=1}^{n-2} p_s \bar{Z}_{sa} + (1 + \frac{1}{r}) \bar{Z}_{n-1,a} + \bar{Z}_{na}}{p}$. In general, this argument is not homogeneous of degree zero in the prices; hence it is impossible for any functions dependent upon it to be so homogeneous.[28] This simple proof is the

28. Thus the emphasis in "Demand for Money" is misdirected; for the essential non-homogeneity flows from the budget restraint, and not from the entrance of money into the utility function.

This distinction raises the question as to whether a money economy can be described without assuming that money enters the utility function. In "Demand for Money" I thought the answer was no (cf. theorem on p. 142). Now I believe that this was probably too extreme a statement.

Consider first the transactions motive for holding money. Here Donald Fort (in an unpublished manuscript—cf. Brunner, footnote 3) has made the following comments: "The above argument [i.e. the theorem just cited in the preceding paragraph—that if money does not enter the utility function no one will want to hold any of it] seems to be a special case of the following: 'If individual utility depends on A and not on B, then, so long as B can be traded for a finite quantity of A, the individual will exchange B for A until he has no more B, thereby maximizing his utility.' This argument suffers from the obvious flaw of not allowing for side relations between A and B. If, for example, the individual is limited by a physical transformation function f(A, B) = 0, which states that in order to consume A it is necessary to consume (or hold) some B, then the individual equilibrium position will be determined by maximizing U(A) subject to the constraint f(A,B) = 0. This may well involve a substantial consumption (or holding) of B even though B does not enter directly into the utility function."

For goes on to explain that what he has in mind is the possibility of "certain constraints relating money stocks and consumption flows" arising, in the classical way, from differences in the timing of payments. This may well be what Hickman was thinking about—at the macroeconomic level—when he introduced his "constraint in the monetary sector" (cf. above, footnote 17). It is clearly explicit in Brunner's equations (9.12) - (9.31). Unfortunately, it does not seem to me that either Hickman or Brunner has succeeded in integrating such a restraint into the analysis he presents. Furthermore, there is the possibility that the convenience of holding money to take care of differences in the timing of receipts and payments can be represented by introducing money directly into the utility function, instead of adding a constraint. The fact that the former procedure (as shown in the preceding footnote) leads to an excess demand function for money consistent with the Cambridge function—a function derived from the consideration of such differences in timing—would seem to give some support to this procedure.

But whether the transaction motive is handled by putting money into the utility function, or by adding a constraint, this much is clear: the budget restraint of whichever model is adopted must have the form (5.5). Hence, as mathematical counterpart of the common-sense observation that, in a money economy, the individual's economic behavior is affected by the real value of the assets—including real cash balances—which he holds. It is through this dependence on real cash balances—$\frac{\bar{Z}_{na}}{p}$—that the absolute price level enters into the consideration of the individual. This is the simple argument on which the microeconomic refutation of the classical homogeneity assumptions is based.

6. Thus as soon as resort is had to microanalysis, the issue becomes unmistakeably clear: The commodity excess demand functions postulated by classical economics—and reaffirmed by Hickman, Leontief, and the modified Lange system—contradict the form of the budget restraint appropriate for a money economy; hence these functions can immediately be rejected as descriptions of a money economy without the necessity of examining them on the macroanalytic level for Invalidities I and II. Such functions are appropriate only for non-monetary economies: those in which money does not exist as a commodity, but serves merely as a formal unit of account. In such an economy, relative prices can, accordingly, be determined in the real sector alone. But it is impossible to complement this real sector by a money equation that will determine absolute

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Footnote 28 cont'd.

shown in the text above, the commodity excess demand functions cannot depend solely on relative prices. (For an explicit example of this in the case where money is left out of the utility function, but an additional constraint is used, cf. Brunner, equations (9.12) - (9.31).) Thus even if Hickman can justify the introduction of his "constraint," the classical dichotomy cannot be solved.

With reference to the speculative motive, Jacob Marschak ( ) has emphasised that if the individual expects some prices to go down, and none to go up, he will hold money even though it does not enter the utility function. Once again, the form of the budget restraint will insure that the demand functions cannot have the classical homogeneity properties.

This leaves the precautionary motive; and it is here that I believe that the introduction of money into the utility function is particularly appropriate. For this can then represent "the satisfaction derived by individuals from holding money as a means of dealing with uncertainty." (Demand for Money," p. 136).

This whole question of whether to develop monetary theory by introducing money into the utility function (the procedure followed here and in "Demand for Money") or to use approaches which leave money out of the utility function (the procedure of Brunner, Fort, Marschak) deserves much further investigation. In particular, attention should be paid to the economic significance of this difference in approach.

prices. For any money equation consistent with the homogeneity assumptions made for the commodity functions can again depend only upon relative prices. Thus, in a non-monetary economy, absolute prices cannot be determined. But, in such an economy, absolute prices are of no significance; only relative prices matter.[29]

Thus the fundamental conclusion of "Demand for Money" and "Indeterminacy" is strengthened and reaffirmed: The classical dichotomy is untenable: it is impossible to have an economy in which relative prices are determined in the real sector, and absolute prices in the money. For in a monetary economy absolute prices must appear in both sectors, and hence are determined in a truly general equilibrium fashion by the system as a whole;[30,31] and in a non-monetary economy--where relative prices are determined in the real sector alone--absolute prices cannot be determined at all.

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29. The latter part of this paragraph is based on Theorems VI, VIII, XI, and paragraph four, p. 136 of "Demand for Money." Cf. also "Indeterminacy" §13 and first paragraph of §14.

30. Another characteristic of the money economy described above is that Say's law cannot hold. This was shown to follow as a direct implication of utility maximization in Theorem XIII and Corollary I of "Demand for Money." The proof there is carried out on the assumption that nominal money balances enter the utility function; the reader can readily establish the fact that the same proof holds even if real money balances enter. Similarly, Theorem XVI--that Say's law and dependence of the commodity demand functions on relative prices alone are necessary and sufficient conditions for each other--is unchanged by the substitution of real, for nominal, money balances in the utility function.

However, this proof cannot be used if we follow the approach to monetary analysis which does not put money in the utility function (cf. above, footnote 28). Hence the microeconomic role of Say's law in such an economy requires further investigation.

On the macroanalytic level, the impossibility of the existence of Say's law in any money economy was, of course, demonstrated by Lange, op. cit.

31. As an immediate result of this proposition we must also reject Leontief's attempt to identify the fundamental distinction between classical and Keynesian economics in the latter's denial of the "homogeneity postulate." For, as the proposition in the text shows, any system describing a money economy must deny this postulate. Leontief's interpretation is presented in "The Fundamental Assumption of Mr. Keynes' Monetary Theory of Unemployment," Quarterly Journal of Economics, v. 51 (1936-7), pp. 192-7; and "Postulates: Keynes' General Theory and the Classicists," in Seymour Harris (editor), The New Economics, New York, 1948, pp. 232-45.

Footnote 31 cont'd.

In "Involuntary Unemployment and the Keynesian Supply Function," Economic Journal, v. 59 (1949), pp. 360-83, I have attempted to present an alternative interpretation of the distinction between Keynesian and classical economics, assuming that both approaches reject the "homogeneity postulate."

It should, however, be pointed out that by redefining "homogeneity postulate" to mean "homogeneous in the prices and in the amount of money" Leontief's distinction can be made logically tenable. But I do not believe that the rejection of the postulate in this sense is an essential assumption of Keynes' theory of employment. Thus the article referred to in the preceding paragraph explains the distinction between the classical and Keynesian positions on the assumption that both accept homogeneity in this sense.