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The Coefficient of Resource-Utilization

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I

ℓ being the total number of goods, the economic activity of the ith individual ($i=1,\dots,m$) is defined by $C_{1i},\dots,C_{hi},\dots,C_{\ell i}$, the quantities of the different goods he consumes, or the negative of the quantities of the different kinds of labor he produces. This can be better represented by a consumption-vector $\vec{C}_i$ of the space R_ℓ. His satisfaction is thus $S_i\ (\vec{C}_i)$. The m satisfactions S_i define a vector $\vec{S}$ of R_m $\vec{S}\ (\vec{C}_1,\dots,\vec{C}_m)$.

The economic activity of the jth industry ($j=1,\dots,n$) is characterized by $I_{1J},\dots, I_{hJ},\dots,I_{\ell J}$ the quantities consumed (inputs), or the negative of the quantities produced (outputs). This can be better represented by an input-vector $\vec{I}_J$ of R_ℓ. The limited technological knowledge of the economic system considered imposes on I_J the condition

$$P_J\ (\vec{I}_J) \leqslant P_J^o \tag{1}$$

(technological progress in this industry will be represented by an increase of P_J^o). The n P_J define a vector $\vec{P}$ of R_n, $\vec{P}\ (\vec{I}_1,\dots,\vec{I}_n)$.

Taking $\vec{C}=\sum_i \vec{C}_i$, $\vec{I}=\sum_J \vec{I}_J$, we finally define $\vec{A} = \vec{C} + \vec{I}$ utilized-physical-resources-vector.

In each one of the three vector spaces R_ℓ, R_m, R_n we introduce the order relation for vectors $\vec{u} \leqslant \vec{v}$ if this inequality holds true for every pair of corresponding components

($\vec{u} < \vec{v}$ if the strict inequality holds true for at least one such pair). And according to the theory of sets terminology, a vector $\vec{u}$ is maximal if there is no $\vec{v} > \vec{u}$, minimal if there is no $\vec{v} < \vec{u}$.

If $\vec{A}^{o}$ is the utilisable-physical-resources-vector one has necessarily $\vec{A} \leq \vec{A}^{o}$. The relations (1), on the other hand, can be condensed in $\vec{P} \leq \vec{P}^{o}$.

II

According to the Paretian definition, $\vec{S}$ is optimal if it is maximal under the constraints $\vec{A} \leq \vec{A}_o$, $\vec{P} \leq \vec{P}_o$. The set of the extremities of the maximal elements is an hypersurface (Σ) of R_m.

If conditions of continuity and derivability are fulfilled, one can apply to this problem the generalized theory of Lagrangean multipliers and express that

$$\frac{\vec{1}}{\sigma} \cdot \vec{S} + \frac{\vec{1}}{\pi} \cdot (\vec{P} - \vec{P}_o) - \vec{p} \cdot (\vec{A} - \vec{A}_o)$$

has a free extremum.

We take the derivative with respect to $\vec{C_i}$ $\frac{1}{\sigma_i} \overrightarrow{\text{grad}}\, S_i - \vec{p} = 0$, and to

$\vec{I_j}$ $\frac{1}{\pi_j}$ $\overrightarrow{\text{grad}}\, P_j - \vec{p} = 0$ and therefore: for all i $\frac{\text{grad } S_i}{\vec{p}} = \sigma_i$, for all j $\frac{\text{grad } P_j}{\vec{p}} = \pi_j$. This contains the complete theorem about optimal situations: a price-vector $\vec{p}$ perfectly determined (but for a scalar factor) implicitly exists.

The definition of the loss associated with a non-optimal $\vec{S_o}$ implies a measure of the distance from a point inside (Σ) to the surface (Σ). Each satisfaction involving an arbitrary increasing function, the problem seems hopeless.

III

Let us therefore consider the following dual problem: we take as a datum a certain set of satisfactions $\vec{S}^{o}$ and imposing the conditions $\vec{P} \leqslant \vec{P}_o$, $\vec{S} = \vec{S}^{o}$, we look for the minimal elements $\vec{A}$. The set of their extremities is an hypersurface (Γ) of R_γ. If $\vec{S}_o$ is on the left side of (Σ), $\vec{A}_o$ is on the right side of (Γ). We are confronted with the new problem of measuring the distance from $\vec{A}_o$ to (Γ) but the coordinates are now <u>quantities of goods</u>.

The usual conditions of continuity and derivability being fulfilled, one proves in the same way as above that with each point $\vec{A}$ of (Γ) is associated a price-vector $\vec{p}$ (determined but for a scalar factor) normal to (Γ).

$\vec{A}$ being an arbitrary point of (Γ), $\vec{A}^{o} - \vec{A}$ is the non-utilized part of the available resources, it can be valuated by $\vec{p}.(\vec{A}^{o}-\vec{A})$; we divide this value by $\vec{p}.\vec{A}$ to eliminate the influence of the undetermined scalar factor included in $\vec{p}$, and we take

$\mathrm{Min}_{A \in (\Gamma)} \dfrac{\vec{p}.(\vec{A}^{o}-\vec{A})}{\vec{p}.\vec{A}}$ as an index of the loss involved. This amounts to finding

$$\mathrm{Max}_{A \in (\Gamma)} \frac{\vec{p}.\vec{A}}{\vec{p}.\vec{A}^{o}}$$

We will then prove under wide assumptions that (Γ) is convex.

It will result that the Maximum is obtained for <u>$\vec{A}_M$ collinear with</u> $\vec{A}_o$, $\vec{A}_M = \rho\,\vec{A}_o$.

The maximum itself is ρ perfectly defined as a function $\rho(\vec{S}_o, \vec{P}_o, \vec{A}_o)$, coefficient of resource-utilization of the economic system.

If a price-vector $\vec{p}_o$ exists in the concrete economic situation observed, one can deduce the value of the loss: $\vec{p}_o . \vec{A}_o$ $(1-\rho)$

$(1-\rho)$

Problems of valuation of the two factors $\vec{p}_o.\vec{A}_o$ and are considered.

IV

The properties of ρ (which is equal to 1 if $\vec{S}_0$ is optimal, smaller than 1 if $\vec{S}_0$ is not optimal, increases when $\vec{S}_0$ increases, decreases when either $\vec{P}_0$ or $\vec{A}_0$ increases) are worked out. Its differentials of the first two orders are computed by an extensive use of matrix calculus.

One particular case is then given special attention, the one where the initial situation is optimal and where $\vec{A}_0$ and $\vec{P}_0$ are constant. This is a comparative study of different types of economic organizations, and problems of taxation, subsidies, railroad rates,... are included here, as well as problems of measurement of the degree of monopoly.

ρ is finally shown to yield a measure of technological progress.