Inconsistency and Indeterminacy in Classical Economics

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Patinkin's two articles on the properties of classical systems proved to be highly stimulating. The second article, "The Indeterminacy of Absolute Prices in Classical Economic Theory" gave rise to a lively controversy. Patinkin's analysis was honored by the publication of 3 critical contributions to Econometrica in January, 1950, on the problem developed mainly in his second paper. Five additional manuscripts were submitted to Econometrica, including a paper by Patinkin answering the criticisms formulated. This development of the discussion shows the need for a survey of the problem. The present paper attempts to take stock of the discussion to see where we are in the evaluation of the "Patinkin problem." A careful study of the manuscripts submitted to Econometrica helped considerably to clarify the issues involved, and they have to be regarded as an important basis of the present paper. Positive or critical acknowledgment

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Cecil G. Phipps: A Note on Patinkin's "Relative Prices."

3. These manuscripts were submitted by Kenneth Arrow, Gary S. Becker, Donald Fort, and Cecil G. Phipps. Patinkin's reply paper will henceforth be referred to as Patinkin III.
will be given them at appropriate points of the discussion.¹

I. A Restatement of Patinkin's Conclusions and Analysis

1. Patinkin is concerned with the determinacy and consistency of the "Cassian system." Unlike the Walrasian variant of Classical economics the Cassian system of market relations is not explicitly derived from first principles — a maximizing hypothesis with respect to individual behavior. But, the postulational structure of the system implies the existence of some such notion — even if Cassel explicitly denied it.

Consider the following system:

(1.1) \[ D_i = D_i (p_1, \ldots, p_{n-1}) \quad i = 1, \ldots, n-1 \]

(1.2) \[ S_i = S_i (p_1, \ldots, p_{n-1}) \]

(1.3) \[ D_i = S_i \]

with the usual notation. It is postulated that these functions are homogenous of degree zero in the prices. This property allows an effective reduction in the number of variables by one. Expressing the system in terms of real prices and treating the (n-1)th commodity as a provisional numéraire

(1.4) \[ D_i = D_i \left( \frac{p_1}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}} \right) \]

(1.5) \[ S_i = S_i \left( \frac{p_1}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}} \right) \quad i = 1, \ldots, n-1 \]

(1.6) \[ D_i = S_i \]

These functions are not homogenous in the real prices. In terms of these new variables homogeneity has been eliminated. The system contains 3n-3 equations in 3n-4 unknown variables. But the equilibrium condition of the numéraire commodity \( D_{n-1} - S_{n-1} = 0 \) is not independent from the other equations in (1.6). Thus we have actually an equal number of variables and independent equations, insuring in general the consistency of

¹. I feel very much indebted to Carl Christ of the Cowles Commission for a patient and detailed discussion of the whole Patinkin problem.
the system of real variables. The quantities of commodities bought and sold together with their real prices is in this way uniquely determined, without taking account of money variables. In order to determine money prices an additional equation is added to (1.4)-(1.6)

\[(1.7) \quad \sum_{r=1}^{n-1} \frac{p_r}{P_{n-1}} \cdot S_r - M^s = 0\]

which expresses the equilibrium condition of the monetary system. \(M^s\) describes the (institutionally determined) money supply and \(K\) the institutionally imposed money holding habits of the public. The monetary system contains only one variable — \(p_{n-1}\) — the money price of the numéraire commodity. This variable is determined by (1.7).

In this way the whole system is built up by two closed subsystems: the real and the monetary subsystem. The simultaneous existence of these two subsystems gives rise to what has been termed the dichotomization of the pricing process. Behavior in the real part depends solely on the real variables, which are uniquely determined by these behavior patterns. In the monetary sector behavior is linked to money values, which result thus from behavior patterns relevant only for this sector of the economy.

This theory seems logically watertight, but according to Patinkin the "additivity property" of real and monetary subsystems does not hold. The dichotomization of the pricing mechanism is based on serious misconceptions about the structure of economic processes. This statement follows immediately from Patinkin's central conclusion: namely, that the Casselian variant of classical economics is either inconsistent or indeterminate. If the real subsystem is inconsistent, then — by definition — no solution exists for the whole system, implying that no "image" of it could exist in reality. If the system is indeterminate this means that we can only solve it for the real variables. There is no escape from this dilemma within the Casselian system, and this fact dissolves the dichotomization of the pricing process. If we choose in
favor of a consistent real system, there exists no meaningful monetary equation consistent with the non-monetary equations, which will remove the indeterminacy of prices. In this respect the Casselian system meets an additional dilemma: Either the monetary equation is useful (in determining absolute prices), but then it is inconsistent with the rest of the system, or it is consistent with the non-monetary equations, but useless.

2. Patinkin's analysis of the logic of the Casselian system is based on the following postulates:

   a) All commodity demand and supply functions are homogenous of degree zero in \( 2 \leq m \leq n-1 \) prices, when \( n \) denotes money.

   b) A system of \( n \) excess demand equations contains at least \( n-2 \) independent and consistent equations.\(^5\)

   c) A necessary and sufficient condition for the consistency of a system is the equality in the number of variables and the number of independent equations.\(^5\)

   d) As a corollary to c) above it follows that a system with more equations than variables is inconsistent, i.e., possesses no solution. Further, it is assumed that a system with less equations than variables has no unique solution.

3. Patinkin's criticisms of the logic of the Casselian system proceed on two levels. He finds this system deficient on two accounts: first, the functions of the system (1.4)-(1.7) are not consistent, and second, if the functions have been changed in such a way as to make them consistent, the equations are either inconsistent or

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\(^5\) Patinkin II contains somewhat contradictory statements on this point, but the practice conforms to the postulate. On p. 4 it is stated: "...we shall assume throughout that for the system under study equality in numbers of equations and variables is a necessary, but not a sufficient condition for the existence of a unique solution." On p. 11 a corollary of the homogeneity theorem is developed, which clearly implies that equality in number of equations and variables is a sufficient condition for consistency.
indeterminate in money prices. In the following paragraph the inconsistency of the functions will be demonstrated.

Let us designate by \( M^D \), the demand for money in terms of stocks; \( M^S \) supply of money in terms of stocks; \( D_n \) the demand for money in terms of flows (inflow of money); \( S_n \) supply of money in terms of flows (outflow of money). The economy is divided in two groups: the private sector \( P \), and the banking-government sector \( B \). With \( M_o^S \) we designate the money supply existing before the marketing process begins. Then the planned holdings of money are

\[
(3.1) \quad M^D = M_o^S + D_n^P - S_n^P
\]

which expresses that the sum of original quantity plus planned net inflow gives us the holdings desired. On the supply side we have

\[
(3.2) \quad M^S = M_o^S + S_n^B + D_n^B
\]

money supply is equal to original supply plus planned net injection on the part of the banking system. If

\[
(3.3) \quad S_n^B - D_n^B = 0 \quad \text{in} \quad F's
\]

vanishes identically in the prices, so that the money stock remains unchanged whatever happens in the system, then the excess demand for money can be written

\[
(3.4) \quad M^D - M^S = D_n^P - S_n^P
\]

or

\[
(3.5) \quad M^D = S_n^P
\]

(3.1) and (3.2) are definitions, so (3.5) expresses that on the basis of the definition- al relations accepted excess demand functions for money in terms of stocks and flows are equivalent. In this way the two quantities \( X_n^P \) and \( M^D \) can be substituted for each other.\(^6\)

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6. This result has been derived in Patinkin II, p. 8. The restriction (3.3) was not mentioned by Patinkin, but expressions like (10.12) and (11.4), for example, (both in Patinkin II) clearly imply it. It will be shown that the elimination of this restriction plays queer tricks on Patinkin's analysis.
The money equation (1.7), describing the structure of the monetary system, consists of two parts, one homogeneous of degree 0 and the other homogeneous of degree 1 in prices. The net result is non-homogeneity of the excess demand function with respect to prices. (1.7) is clearly of the general form \( M^x \). But if \( M^x \) is non-homogeneous in prices, then it follows by (3.5) that \( X^P_n \) must be non-homogeneous in prices as well. The inflow and outflow component of \( X^P_n \) is linked up with the demand and supply decisions on the commodity market. \( D^P_n \), the inflow of money is generated by the supply of commodities, and \( S^P_n \) the outflow by the demand for commodities. Thus we have another pair of definitional relations:

\[
D^P_n = \sum_{i=1}^{n-1} p_i \cdot S_i (p_1, \ldots, p_{n-1})
\]

\[
S^P_n = \sum_{i=1}^{n-1} p_i \cdot D_i (p_1, \ldots, p_{n-1})
\]

By postulate a) all \( S_i \) and \( D_i \) functions \((i = 1, \ldots, n-1)\) are homogeneous of degree 0 in \( p \)'s. This fact makes the expressions on the right of the definition homogeneous of degree 1 in \( p \)'s. Thus the homogeneity property of the commodity functions implies that \( D^P_n \) and \( S^P_n \) be homogeneous of degree 1 in \( p \)'s. This result makes \( X^P_n \) homogeneous of degree 1 in the \( p \)'s as well. But this contradicts \( M^x \) being non-homogeneous in \( p \)'s. So, either \( M^x \) is non-homogeneous and the same has then to hold for \( X^P_n \), which is inconsistent with postulate a) when \( m = n-1 \) or \( M^x \) is homogeneous and thus compatible with the \( S_i \) and \( D_i \) functions, but then the excess demand function (1.7) has to be replaced by some other expression. As the Casselian system stands, there exists an inconsistency (we might term of the "first kind") between the properties of the commodity functions and the monetary constraint imposed. So Patinkin is able to point out that "the classical homogeneity assumption is logically inconsistent with the classical monetary equation."^7

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4. This is a refutation of the Casselian variant at one level. If we improve on this deficiency by amending the money function to make it consistent with the non-monetary functions we meet a second difficulty: the inconsistency of the "second kind." The system to be discussed is

\[ X_i = X^p_i(p_1, \ldots, p_{n-1}) = 0 \quad i = 1, \ldots, n \]

The money function (1.7) has been substituted by \( X^p_i \), the components of which are defined by (3.6) and (3.7). All the functions are now consistent. Are now the equations determinate in p's and consistent? The system is built up out of (n-1) variables, which can be effectively reduced to (n-2) because of the homogeneity of all functions, and n equations. On the basis of our postulates we must look around for some degrees of dependence, before we can decide in favor of consistency. By the definition of \( X^p_i \) in (4.1) the following Lemma can be formulated:

**Lemma 1:** \( X^p_i \) is symmetrically dependent upon \( X^p_j \) \((j = 1, \ldots, n-1)\).

Hence the consistency of the system (4.1) depends only on the consistency of the (n-1) commodity market equations, and henceforth, the money equation may be neglected.\(^8\),\(^9\)

The symmetrical dependence of the money function on the commodity functions assures that any set of values which satisfies the commodity equations will be a solution for the monetary equation. Thus Lemma 1 leaves us with (n-1) equations and (n-2) real variables. An additional dependence is introduced into the system because of the homogeneity postulate. The Jacobian of a set of n functions in n variables, each of which is homogeneous of degree zero in the same set of 2 \( \leq m \leq n \) variables vanishes.

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8. The development in the text follows closely Patinkin's modified version of his overdeterminacy theorem as presented in the last manuscript, after he took account of Hickmann's critical points.

9. Patinkin III elaborated at length on the relationship between functional dependence and consistency of a system. Patinkin distinguished there two forms of functional dependence:

a) **symmetrical dependence:** in this case the functional dependence
   \[ F(X_1, \ldots, X_{n-1}) = 0 \] is of the special form \[ F(0,0,\ldots,0) = 0. \]

b) **asymmetrical dependence:** functional dependence is not of the special form
   \[ F(0,0,\ldots,0) = 0. \] Symmetrical dependence is considered a necessary, but not a sufficient condition for consistency and asymmetrical dependence a sufficient condition for inconsistency.
identically. This fact implies that such a set of functions is functionally dependent. This provides a second lemma:

Lemma 2: If the \((n-1)\) commodity functions are homogeneous of degree zero in \(2 \leq m \leq (n-1)\) money prices, then the set of \((n-1)\) commodity functions is functionally dependent.

According to Patinkin III there is an essential difference between Lemma 1 and Lemma 2. Lemma 1 is related to the budget restriction, within which individual units have to operate. So we know from a priori economic reasons that the first dependence is of a symmetrical form. But, in Lemma 2 there is no "economic evidence to specify the form of dependence involved," the commodity functions may be symmetrically or asymmetrically dependent. Say's law is considered by Patinkin as an additional postulate, logically independent from Lemma 2.\(^{11}\)

The results established can be summarized in the following way: The \((n-1)\) non-monetary equations of the system \((h.1)\) are dependent. They may be either

a) symmetrically dependent and consistent

b) asymmetrically dependent and hence inconsistent.\(^{12}\)

Thus Patinkin is able to conclude that the Casselian system is either inconsistent in the real variables, so that no solution exists for the system as a whole or then consistent in real variables but indeterminate in money variables. Any set of values in \(p's\) satisfying \(X_p(i = 1, \ldots, n-1)\) in \((h.1)\) will also satisfy \(X_n\). Thus there will be an infinite number of sets satisfying \(X_n\) and the money equation cannot determine money prices.

\(^{10}\) The proof for the case \(m = n\) is given in Patinkin II, p. 15.

\(^{11}\) This is the position taken within macro-analysis. Within micro-analysis Patinkin I states that homogeneity of degree 0 in prices and Say's law are equivalent properties, see Theorem XVI in Patinkin I, p. 153.

\(^{12}\) It is conceivable to list a third category: symmetrically dependent and inconsistent, but this is not really meaningful. If inconsistency is assumed then there is nothing to analyze further (if we keep to statics and exclude mutual adjustments of behavior patterns).
II. Meaningful Monetary Equations and Homogeneity of Non-Monetary Equations.

Patinkin's argument seems at first sight to be definitely valid. But there are a number of analytical inaccuracies which change the nature of his conclusions in a definite way. This part will be devoted to a critical analysis of Patinkin's and his critics' arguments. The first sections will be concerned with the inconsistency of the first kind and later sections will then take up the inconsistency of the second kind.

5. Patinkin pointed out that the classical homogeneity assumption is not compatible with the classical monetary equation. This statement must be considerably qualified, before it holds. It does not hold in the generality formulated by Patinkin.\textsuperscript{13} Let us consider the following equational system: \textsuperscript{14} n-1 commodity excess demand functions are homogeneous of degree zero with respect to \( p_2, p_3, \ldots, p_{n-1} \), but not with respect to \( p_1 \).

Then we have

\begin{align*}
(5.1) & \quad x_i = x_i(p_2, \ldots, p_{n-1}) = 0 \quad i = 1, \ldots, n-1 \\
(5.2) & \quad x_n = x_n(p_1, \ldots, p_{n-1}) = 0
\end{align*}

By a simple proof\textsuperscript{15} it can be shown that \( x_n \) is not homogeneous of any degree in the \( p \)'s. The fact that homogeneity is limited to a proper subset of the \( p \)'s is a sufficient condition to eliminate any transmission of homogeneity properties to the money function.

The subsystem (5.1) contains (n-1) variables, which can be reduced to (n-3) real prices and one money price. Then (n-2) unknowns are balanced by (n-1) equations. But according to Lemma 2 the set of \( x_i(1 = 1, \ldots, n-1) \) functions is functionally dependent, which leaves us the choice between symmetrical or asymmetrical dependence. If we have reason to decide in favor of symmetrical dependence, then we can solve (because of postulate b) and c) the system (5.1) for \( p_1 \) and \( \frac{p_r}{p_{n-1}} (r = 1, \ldots, n-2) \). Inserting \( p_1 \) and

\textsuperscript{13} Patinkin II, p. 16

\textsuperscript{14} This system has been analyzed in Patinkin II, p. 16 & 17.

\textsuperscript{15} Patinkin II, p. 16, equation (10.6).
s_r - \frac{p_r}{p_{n-1}} \) into (5.2) the money function seems to give us a solution for the price level. The system (5.1)-(5.2) does not break down, because of any inconsistency in the functions as defined by Patinkin. No adjustment is needed here. We are evidently left with the relatively uninteresting conclusion that the system is either inconsistent in the subsystem (5.1) thus forbidding any solution for the system as a whole, or then it seems consistent in (5.1) and determinate in money prices.

6. There exists a more serious type of limitation to the general proposition of incompatibility between the homogeneity property and a non-homogeneous money function. Patinkin developed his argument on the implicit hypothesis

\[(6.1) \quad S_n^B - D_n^B = 0 \text{ identical in } p's\]

that the net injection of money by the banking-government sector vanishes identically in the p's.

Let us suppose that this restriction does not hold and let us assume that all money is created and destructed by fiat; there is no counter-flow to the components of the net injection. In this case the definition of the demand for money in terms of stocks will be changed in the following way

\[(6.2) \quad M^D = M_0^S + [D_n^P + S_n^B] - [S_n^P + D_n^B]\]

Because of the fiat components the inflow and outflow expressions of private cash-balances have to be adjusted. The inflow consists in this case not only of the sales value of commodities but also of receipts from transfer payments from the B sector, and an analogous statement holds for the outflow. Forming the excess demand we find

\[(6.3) \quad M^D - M^S = D_n^P - S_n^P\]

a result which establishes the same result, as if the restriction (6.1) holds.

Let us now suppose that (6.1) does not hold and that the components of the net injection are related to transactions on the securities market: \(S_n^B\) is the supply of money resulting from the demand for securities by the banking system, and \(D_n^B\) is the demand for money emanating from the supply of securities by the banking system.
So $D_n^B$ and $S_n^B$ are determined by the decisions about securities holding of the various parts of the banking system. The various units of the banking system enter the marketing period with a given balance-sheet, which describes the initial quantities. The price-system and some maximizing behavior, subject to the balance-sheet restriction determine the desired quantities in the balance-sheet. By taking the difference between the initial and the desired quantities and adding up all positive differences and negative differences separately, we get finally their sums $D_n^B$ and $S_n^B$, the value of the initial stocks which are planned to be transacted. Under these circumstances we have the following relationships

\[(6.4) \quad p_{n-1} \cdot S_{n-1}^B = D_n^B \]
\[(6.5) \quad p_{n-1} \cdot D_{n-1}^B = S_n^B \]

$D_{n-1}^B$ and $S_{n-1}^B$ are clearly special components of a general securities equation of the system. If we specify demand and supply of securities by units of/private (P) sector, $D_{n-1}^P$ and $S_{n-1}^P$, we can formulate total demand and supply of securities

\[(6.6) \quad D_{n-1}^P + D_{n-1}^B = D_{n-1} \]
\[(6.7) \quad S_{n-1}^P + S_{n-1}^B = S_{n-1} \]

With respect to the P components it has been deemed plausible to suppose that they are homogeneous of degree 1 in the prices. Such a hypothesis with respect to the B components would hold if banks not only professed but actually followed the commercial loan theory. In case of an ultimate reserve system, where the reserve (or at least a part of it) has to be treated analytically as an exogeneous element, or in case of a purely "managed currency" the B components will not be homogeneous of any degree in the p's. This property will then be transmitted [because of (6.4)-(6.7)] to the securities demand and supply function. Our system, after eliminating the special restriction imposed on the behavior of the banking system, will now look like:

\[(6.8) \quad D_i(p_1,\ldots,p_{n-1}) - S_i(p_1,\ldots,p_{n-1}) = 0 \quad 1 - l, \ldots, n-2 \]
\[(6.9) \quad D_{n-1}(p_1,\ldots,p_{n-1}) - S_{n-1}(p_1,\ldots,p_{n-1}) = 0 \]
\[(6.10) \quad D_n(p_1,\ldots,p_{n-1}) - S_n(p_1,\ldots,p_{n-1}) = 0 \]
All the postulates mentioned earlier are still applicable to this system. Functions in equations (6.8) are homogeneous of degree 0 in $p_1, \ldots, p_{n-2}$. $D_{n-1}$ and $S_{n-1}$ are definitely non-homogeneous, and the same holds for $D_n$ and $S_n$. These latter expressions are defined by

\begin{align}
(6.11) & \quad D_n = D_n^P + D_n^B \\
(6.12) & \quad S_n = S_n^P + S_n^B
\end{align}

both containing a non-vanishing, non-homogeneous component. This fact makes $x_n^p$ or $x_n^s$ as well, non-homogeneous in $p$'s. Thus our system contains two excess demand functions for which the homogeneity property does not apply. Clearly, the inconsistency of the first kind -- as has been defined by Patinkin -- does not exist in this system. The linear combination of $D_i$'s and $S_i$'s ($i = 1, \ldots, n-1$) gives a money function compatible with classical notions. So the first hurdle has been taken. With respect to inconsistency of the second kind, we observe the following fact: The $(n-2)$ commodity equations contain $(n-3)$ real prices $\frac{p_r}{p_{n-1}}$ ($r = 1, \ldots, n-3$) and securities prices $p_{n-1}$ -- altogether $(n-2)$ unknown variables. On basis of postulate b) and c) we can solve in the commodity sector for these $(n-2)$ variables and use either the securities or the money equation, after inserting the solutions of (6.8), for the determination of money prices. Because of the interdependence formulated by Lemma 1, which is based on Walras law, either the securities or the money equation is redundant.

Collecting the results of this section we observe that after the elimination of the special restriction imposed on the behavior of the B sector the dichotomization of the pricing process seems to hold again. Real prices and money prices are determined in co-existing closed sub-systems and inconsistencies of both kinds do not appear. So Patinkin's main result does not hold for this case.

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16. In fairness to Patinkin it must be mentioned that he was fully aware of the existence of a system with the properties described. In Patinkin II we read on p. 22 "... assume, that for some unspecified reason, the excess demand equation for the bond market, $x_{n-1}$, is not homogeneous of degree 1 in $p_1, \ldots, p_{n-2}$. Then $x_n$ is not homogeneous of degree 1 in $p_1, \ldots, p_{n-2}$, and the overdeterminacy theorem... cannot be applied." "Then the classical... (continued on p. 13)
7. There exists an important relationship between the homogeneity property, functional dependence and Say's law (of the Lange kind). Patinkin's proof of the inconsistency of the second kind was based on his fundamental homogeneity theorem.

Applying this theorem on the system of \( n \) excess demand functions
\[ X_i (i = 1, \ldots, n), \]
each one put equal to zero, Patinkin finds a situation to which the homogeneity theorem seems to apply. \( X_n = 0 \) drops out because of Walras law. So there remain \( (n-1) \) equations in \( (n-2) \) effective variables. This fact provides the basis for Patinkin's overdeterminacy theorem. At a later stage of his argument Patinkin introduced Say's law 1 as an additional and independent postulate. Patinkin made it clear that this postulate had no logical relation to any of the other postulates of the system. In this way the \( (n-1) \) commodity equations were made linearly dependent, one became redundant, and the system contained \( (n-2) \) independent equations in \( (n-2) \) real prices. Thus because of postulate c) the system has a solution in real variables, but the identity in the p's, termed Say's law 1 (or the money function consistent with the homogeneous commodity functions) makes the system indeterminate in money prices.

Hickman's paper attacked squarely Patinkin's fundamental theorem: the homogeneity and the overdeterminacy theorem. He pointed out correctly that the assumption of independence in the set of functions considered is not compatible with the assumption of homogeneity of any degree in the same set of variables. A set of functions home-

... would hold -- with relative prices determined in the real part...and money prices through the monetary equation..." But Patinkin disposes of the case by stating: "What justification is there for singling out one particular equation and assuming it to be non-homogeneous? A satisfying solution...cannot be achieved by such ad hoc and arbitrary assumptions." After reading this statement, it is necessary to remember that the result of the above section followed by dropping Patinkin's restriction \( S^B - D^B = 0 \), and supposing that the components of the not injection are linked to the decisions of the banking system with respect to securities holding.

17. I feel considerable sympathy for Becker's distinction of two types of Say's law. First, we have Lange's definition of it
\[ \sum_{i=1}^{n} p_i (D_i - S_i) = 0 \]
an identity in p's
and second Say's law may be defined as the coincidence of supply and demand values as a function of a set of finite and positive prices.
\[ \sum_{i=1}^{n} p_i (D_i - S_i) = 0 \]
a conditional statement in p's.
The realm of these two definitions differs essentially.

18. Patinkin II, p. 10: "Theorem: If every equation of a system of \( K \) independent equations in \( K \) variables is homogeneous of some degree \( t \) in the same set of variables, then the system possesses no solution..."
geneous of degree 0 in the same set of variables has an identically vanishing Jacobian. This is a necessary and sufficient condition for functional dependence.\textsuperscript{20} So the homogeneity property postulated for the Casselian system implies functional dependence, and a combination of independence and homogeneity of degree 0 cannot hold. Patinkin’s theorems must be qualified in order to be correct. If we add to each one: "provided $t \neq 0$ for at least one of the non-money functions" then his two theorems will hold. Formulated in this way, the overdeterminacy theorem is not relevant for the simple Casselian system discussed, which by hypothesis has only non-money functions homogeneous of degree 0.

It is interesting to observe that it still remains relevant for the evaluation of the Lange system.\textsuperscript{21} This system is of the same general type as the Casselian variant, but besides commodities and money it contains as well bonds. The $(n-2)$ commodity excess functions are homogeneous of degree 0 the the bonds excess function is homogeneous of degree 1 in $p_1, \ldots, p_{n-2}$. A simple proof developed by Patinkin shows that the money excess function, $X_n$, is then necessarily homogeneous of degree 1 in $p_1, \ldots, p_{n-2}$. One of the $n$ excess demand equations can be deleted because of Walras law. So there remain $(n-1)$ equations, out of which at least one is homogeneous of degree 1 in $p_1, \ldots, p_{n-2}$. This fact is sufficient to prevent the Jacobian of the functions involved to be of an identically vanishing type. The equations of the Lange system are thus independent, a property which renders the system overdetermined, because "we are left with $(n-1)$ equations homogeneous of degree 0 or 1 in the same subset of variables, so that the effective number of variables can be reduced to $(n-2)$." We see then, that Patinkin’s con-

\begin{itemize}
\item \textit{19.} Patinkin II, p. 14: "Theorem: If the $f_i$ and $g_i$ ($i = 1, \ldots, n-1$) are independent and homogeneous of degree $t'$ in all the variables, then the system...is overdetermined."
\item \textit{21.} Patinkin II, p. 18 and 19.
\end{itemize}
clusions with respect to the Lange system remain valid in spite of Hickmann's well taken criticisms against the homogeneity and overdeterminacy theorem.

Hickmann's thesis about the relationship between homogeneity of degree 0 and functional dependence was accepted in Patinkin III. This led Patinkin to modify his theorems, but not in the way suggested above. This difference in the qualifications is of some importance. Patinkin contends, that because of the special homogeneity property we know that the set of functions involved is dependent, but that we do not know if this dependence is asymmetrical or symmetrical. If it is symmetrical then the overdeterminacy and the consequent inconsistency does not hold, but if it is asymmetrical then the system remains inconsistent. Thus Patinkin modifies his overdeterminacy theorem by substituting "asymetrically dependent" for "independent."

This qualification makes the theorem logically correct, but the question of relevancy remains. By postulate b) it was assumed that there exist at least (n-2) independent and consistent commodity equations. To introduce later on asymmetrical dependence of (n-1) functions is equivalent to the assumption that the (n-1)th equation is inconsistent with the other (n-2) ones. From the point of view of economics the modified theorem does not seem to be meaningful. We are of course at liberty to set up inconsistent static equational system, but in such cases it is not the conclusion of inconsistency which can interest us, but the dynamic implications of this construction.

There is another aspect which renders the modified version of the overdeterminacy theorem of doubtful value: the connection between Say's law 1 and the classical homogeneity property. The latter implies that decisions to buy and sell are taken independent of money prices, dependent only on real prices. For any excess demand function of a "homogeneous" system only real prices will be relevant. It might be termed an identity in money prices. The same property holds for Say's law 1. As it is an identity in p's the relation postulated holds for any level of money prices whatever, but not for any set of real prices. Thus, if homogeneity is postulated, the relation
formulated by Say's law 1 is already implied in the postulate: Whatever the level of money prices, there exists a set of real prices which will equalize demand and supply values. Say's law is contrary to Patinkin's presentation, even on the level of macro-analysis, not an additional, independent postulate to the homogeneity postulate. In such a situation the functional dependence expressed by Lemma 2 is to be understood simply as the mathematical image of the economics involved in Say's law 1. As soon as this fact is accepted, the original and the modified overdeterminacy theorem lose all their relevance for the evaluation of classical systems.

There is an additional moment pointing in the same direction. Economists have to move cautiously in macro-analysis. It is important that the fundamental properties of the macro-functions should be derived from micro-analysis. In Patinkin III there is an unambiguous statement that "by resorting to micro-analysis the issue is made quite clear." This statement applies immediately to the question which Patinkin left unsolved after having formulated Lemma 2: is the functional dependence implied by the homogeneity property of symmetrical or asymmetrical form. By relating this problem to micro-economic analysis Patinkin would have found an unambiguous answer. When we look up Theorem XVI in Patinkin I, we observe that Say's law 1 and homogeneity of degree 0 are logically equivalent properties of a system.

Leontief's paper formulates essentially the same criticism against Patinkin's original (and — implicitly — modified) overdeterminacy theorem as Hickmann's. But whereas Hickmann concentrated on the purely mathematical property of the homogeneous system, Leontief was immediately concerned with the economics of the interdependence between the commodity functions. Say's law 1 is introduced by reference to the

22. Op. cit., p. 153, Theorem XVI: "A necessary and sufficient condition for homogeneity of degree zero for the individual in the p's for the individual demand function is that Say's law 1 be true." — A proof is given there.
classical hypothesis, that money does not enter the utility function. Such a postulate is a necessary but not a sufficient condition for the existence of Say's law. If the utility hypothesis holds and there exists no institutionally determined finite upper boundary to the rate of utilisation of money stocks, then Say's law will follow together with the homogeneity property. A proof of this statement will be given in a later section. So Leontief disposes correctly of the inconsistency in the real relations, by stating the incompatibility of independence and homogeneity of degree 0 in the functions, with respect to the original overdeterminacy theorem, and by pointing out implicitly the irrelevance of asymmetrical dependence, with respect to the modified form of the theorem. But Patinkin concluded that the Casselian system is either inconsistent (in the real relations) or indeterminate in money prices. Leontief counters by stating categorically and correctly, that the real system is determinate and consistent, but he says nothing about determinacy in money prices, in spite of the fact that his paper is entitled "Consistency of Classical Theory of Money and Prices." If this title implies that Leontief thinks an addition of a (non-homogeneous) money equation to the determinate real system will create a determinate and consistent monetary system, then he is definitely wrong. We remember Patinkin's inconsistency of the first kind, on which such a construction would break down. This breakdown reflects a more fundamental incompatibility in certain constructions of classical monetary theories, than Patinkin's arguments imply.

23. Op. cit., p. 23: "Since money does not enter in his utility function, an individual according to the classical theory of economic behavior offers real commodities and services for sale only in order to be able to purchase other real goods and services. This means that the shapes of the 2n-2 functions describing his demand and supply for each of the real goods are necessarily interrelated in such a way that the unknown form of any one of them can be directly derived from the given shapes of the other 2n-3.
8. Before we treat this problem of dominant importance it might be interesting to collect the provisional interim results of the discussion. How does Patinkin's conclusion stand up? We may summarize perhaps in the following way:

a) Insofar as the inconsistency of the first kind is concerned, it only holds under the condition that

1) \( S^n_B - D^n_B \equiv 0 \) identically in p's, or money is created or destructed purely by fiat, without connection to market transactions of any sort.

2) all non-monetary functions are homogeneous of degree 0 in all the variables.

b) Insofar as the inconsistency of the second kind is concerned it only holds under the condition that at least one function besides the money function is homogeneous of some other degree than zero.

c) Patinkin's main result -- the breakdown of the classical dichotomy does not hold:

1) if \( S^n_B - D^n_B \) does not vanish identically and they do not reflect fiat decisions.

2) if the commodity functions are not homogeneous in all the prices.

9. These interim results were established by keeping strictly to the lines of reasoning of Patinkin and some of his critics. It has to be shown that they are partly irrelevant and partly wrong. We shall have to point out that any version of classical economics which implies the dichotomy of the pricing process will break down because of its internal inconsistency. Thus Patinkin was actually correct in stating that dichotomization of the pricing process cannot hold, and his critics wrong in disputing it. But the argument developed by Patinkin in order to establish his thesis is fallacious: Dichotomization is impossible, not because the classical system with its characteristic "additive property" is either inconsistent (in the real relation) or indeterminate. The main deficiency of such a classical system is to be
found in the incompatibility of the functions involved: This incompatibility permeates a good part of the whole discussion. The central thesis, on which the following development is based states: that a meaningful monetary equation (or function) is inconsistent with non-monetary equations (or functions) homogeneous of any degree in the prices. The existence of a meaningful monetary equation implies "money illusion" for the system. The creation of "money illusion," by the existence of money, disrupts the dichotomization process, so that all money and real prices are determined in a truly general equilibrium manner.

a) The inconsistency in the functions of the system which makes the system envisaged economically meaningless is of a different type than Patinkin's inconsistency of the first kind. Patinkin considers the following as a sufficient criterion for establishing or evaluating consistency in the functions: Given an excess demand function \( f(x_1, \ldots, x_{n-1}) \). This is equivalent to the excess demand function \( g_1(p_1, \ldots, p_{n-1}) \). But this latter can be expressed as a linear combination of the excess demand functions of commodities. The properties of the \( g_1 \) function are by definition the properties of the linear combination of commodity functions. A comparison of these latter properties with the properties of the \( M_x \) functions establishes for Patinkin consistency or inconsistency of the first kind.

The inconsistency considered in this section differs from Patinkin's on the following essential point: Patinkin treats the coincidence of the homogeneity properties in the \( M_x \) function and of the above mentioned linear combination as a necessary and sufficient condition of consistency whereas we shall regard it as a necessary condition, but far from sufficient.

\[ \text{In case a) money enters the utility function or} \]
\[ \text{b) money does not enter the utility function, but there exists a finite upper boundary to the rate of utilisation of money.} \]
An example developed in some earlier section shows clearly that the "Patinkin condition" does not necessarily assure solubility in terms of money prices. Let us revert to the system (5.1) and (5.2). Patinkin proceeds to solve the system in the following way: (n-2) commodity equations will determine (n-3) real prices and the money price \( p_1 \). Inserting \( z_r \cdot p_{n-1} \) (when \( z_r = \frac{P_r}{P_{n-1}} \), \( r = 1, \ldots, n-2 \)) into (5.2) evidently determines \( p_{n-1} \), and so money prices. But we have to remember that (5.2) is simply the linear combination of (5.1). The system looks then really like

\[
(9.1) \quad X_i = X_i(p_1, \frac{P_2}{P_{n-1}}, \ldots, \frac{P_{n-2}}{P_{n-1}}) = 0 \quad i = 1, \ldots, n-2
\]

\[
(9.2) \quad X_n = \sum_{i=1}^{n-1} p_i X_i(p_1, \frac{P_2}{P_{n-1}}, \ldots, \frac{P_{n-2}}{P_{n-1}}) = 0
\]

Dropping \( X_{n-1} = 0 \) because of Walras law, we are left with \( X_1, \ldots, X_{n-2} \) and \( X_n \). Suppose now \( X_1 = 0, \ldots, X_{n-2} = 0 \) and solved in \( p_1 \) and \( z_r \). But this solution eliminates simultaneously \( X_1, \ldots, X_{n-2} \) in the linear combination (9.2). Thus we are actually left with

\[
(9.3) \quad p_{n-1} \cdot X_{n-1}(p_1, \frac{P_2}{P_{n-1}}, \ldots, \frac{P_{n-2}}{P_{n-1}}) = 0.
\]

Supposing \( p_{n-1} \neq 0 \) (9.3) shows that the solutions of the (n-2) equations satisfies \( X_{n-1} = 0 \) as well. But that is all. (9.3) and thus (9.2) cannot determine \( p_{n-1} \), as it is supposed to do. As the system stands, it is indeterminate in money prices in spite of the non-homogeneous money equation. 25

b) The deficiency in Patinkin's criterion is to be found in the fact that some homogeneity in a proper subset of variables and functions only, already assures a... 

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25. In Patinkin II, p.16 &17, it is stated that "the dropping of the extra equation is not as symmetrical a process as Modigliani implies." It seems to me that the case discussed implies symmetry, whatever equation is dropped. The eliminated commodity equation always enters again via the linear combination. So ultimately we have only to deal with (9.1). But because of the interrelation between homogeneity and Say's law 1, we have then (n-2) equations in the same number of variables. Patinkin's conclusion ought then be: the system has a meaningless money equation, which does not determine money prices, but it is determinate in real prices.
non-homogeneous money function; but such a state of affairs leaves still the basic inconsistency intact, which says that non-zero liquid assets are incompatible with any sort of homogeneity in prices. We shall prove this thesis for the two situations: that liquid assets enter the utility function, and when they are excluded from it. Let us designate the utility function by

\[ u^\alpha = u^\alpha (Z_1^\alpha, \ldots, Z_{n-2}^\alpha, \frac{Z_{n-1}^\alpha}{r \cdot p}, \frac{Z_n^\alpha}{p}) \]

when \[ p = \sum_{i=1}^{n-2} w_i \cdot p_i \]

\( p \) = price index, \( p_i \) = price of commodity \( i \), \( w_i \) = weight of commodity \( i \), \( Z_i^\alpha (i = 1, \ldots, n-2) \) = amounts of commodities (for the static case considered, it is actually irrelevant if we deal in stocks or flows; in strict logic, for the uni-period model the relevant quantities are stocks). \( Z_{n-1}^\alpha \) = securities, \( r \) = rate of interest, \( Z_n^\alpha \) = money and \( \alpha = 1, \ldots, m \) designates the individual. This function has to be maximized subject to the restriction expressing the limitation of resources for each individual.

\[ \sum_{i=1}^{n-2} p_i (Z_{i+1}^\alpha - \bar{Z}_i^\alpha) + \frac{(Z_{n-1}^\alpha - \bar{Z}_{n-1}^\alpha)}{r} + (Z_n^\alpha - \bar{Z}_n^\alpha) - r \cdot \bar{Z}_{n-1}^\alpha = 0 \]

where \( \bar{Z}_i^\alpha \), \( i = 1, \ldots, n \), designates the quantities brought to the market. (9.5) states that the value of disposable resources (including interest receipts) before and after marketing are equal. By maximizing (9.4) subject to (9.5) we find the first order conditions:

\[ u_j^\alpha (Z_1^\alpha, \ldots, Z_{n-2}^\alpha, \frac{Z_{n-1}^\alpha}{r \cdot p}, \frac{Z_n^\alpha}{p}) - \lambda \cdot p_i = 0 \]

when \( j = 1, \ldots, n-2 \)

\[ u_r^\alpha (Z_1^\alpha, \ldots, Z_{n-2}^\alpha, \frac{Z_{n-1}^\alpha}{r \cdot p}, \frac{Z_n^\alpha}{p}) - \lambda \cdot p = 0 \]

where \( r = n-1,n \)

and further

\[ u_{n-1}^\alpha \frac{\partial u^\alpha}{\partial (\frac{Z_{n-1}^\alpha}{r \cdot p})} \quad u_n^\alpha = \frac{\partial u^\alpha}{\partial (\frac{Z_n^\alpha}{p})} \]

26. It must be observed that according to a short note in Patinkin III, he was aware of this fact, but strangely he did not adjust his analysis to it. Otherwise, he could not have granted the correctness of the system on p. 22 in his II.
If we divide (9.5) by p we have the restriction

\[ (9.8) \quad \sum_{s=1}^{n-2} \frac{p_s}{p} \left( Z_{s\alpha} - \frac{Z_{s-1,\alpha}}{r} \right) + \frac{Z_{n-1,\alpha} - \frac{Z_{n-1,\alpha}}{r}}{p} + \frac{Z_{n,\alpha}}{p} - r \cdot \frac{Z_{n-1,\alpha}}{p} = 0 \]

(9.6), (9.7) and (9.8) contain (r+1) equations for \( \alpha \), in the r+1 dependent variables \( Z_{s\alpha} (s = 1, \ldots, n-2), \frac{Z_{n-1,\alpha}}{p}, \frac{Z_{n,\alpha}}{p} \) and \( \lambda \). Thus we can solve for them in terms of the independent variables \( p_s, p, \frac{Z_{s\alpha}}{p}, \frac{Z_{n-1,\alpha}}{p}, \frac{Z_{n,\alpha}}{p} \) and \( r \). By inspection of the system of equations we can see that the independent variables \( p_s, p, Z_{n-1,\alpha} \) and \( \frac{Z_{n,\alpha}}{p} \) always appear in ratios of the form \( \frac{p_s}{p}, \frac{Z_{n-1,\alpha}}{p}, \frac{Z_{n,\alpha}}{p} \). Because of this feature the solutions of the system with respect to \( Z_{s\alpha} (s = 1, \ldots, n-2) \) must be homogeneous of degree 0 in \( p_s, p, \frac{Z_{n-1,\alpha}}{p} \), and \( \frac{Z_{n,\alpha}}{p} \) whereas \( Z_{n-1,\alpha} \) and \( Z_{n,\alpha} \) must be homogeneous of degree 1 in these same variables. The various demand functions are then of the form:

\[ (9.9) \quad Z_{s\alpha} = \frac{p_l}{p}, \ldots, \frac{p_{n-2}}{p}, r, \frac{Z_{n-1,\alpha}}{p}, \frac{Z_{n,\alpha}}{p} \] \( s = 1, \ldots, n-2 \)

\[ (9.10) \quad Z_{r\alpha} = \frac{p_l}{p}, \ldots, \frac{p_{n-2}}{p}, r, \frac{Z_{n-1,\alpha}}{p}, \frac{Z_{n,\alpha}}{p} \]

where \( r = n - 1, n \).

While these functions are actually homogeneous in all the variables except \( r \), they are non-homogeneous in the prices \( p_s \) and \( p \).

The thesis has to be proved for the case that liquid assets do not enter the utility function. The utility function is in this case

\[ (9.11) \quad U^* = U^* (Z_{1\alpha}, \ldots, Z_{n-1,\alpha}) \] where \( 1, \ldots, n-1 \) designate commodities.

The budget restriction is according to Patinkin I

\[ (9.12) \quad \sum_{j=1}^{n} p_j \cdot Z_{j\alpha} = \sum_{i=1}^{n} p_i \cdot Z_{i\alpha} \]

Maximizing (9.11) subject to (9.12) gives us

\[ (9.13) \quad \frac{U^*}{U^*_{n-1}} = \frac{p_r}{p_{n-1}} \quad r = 1, \ldots, n-2. \]

27. This fits in with Theorem XIV in Patinkin I, p. 152: "A necessary condition for \( A \) to be true is that not all the \( Z_{j\alpha} (j = 1, \ldots, n-1; \alpha = 1, \ldots, m) \) are h.d.i.o in the \( p_j \)."
We have further

\[(9.14) \quad z_{n\alpha} = 0\]

Rewriting the budget restriction

\[(9.15) \quad \sum_{j=1}^{n-2} \frac{p_j}{p_{n-1}} z_{j\alpha} + z_{n-1} - \sum_{j=1}^{n-2} \frac{p_{n-1}}{p_{n-1}} \tilde{z}_{j\alpha} - \tilde{z}_{n-1} - \frac{\tilde{z}}{p_{n-1}} = 0\]

By inspection of the system (9.13), (9.14) and (9.15) we observe that we have \(n\) equations for \(\alpha\) in \(n\) independent variables \(z_{i\alpha} \quad (i = 1, \ldots, n)\) in terms of \(p_j \quad (j = 1, \ldots, n-2)\), \(p_{n-1}\), \(\tilde{z}_{i\alpha} \quad (i = 1, \ldots, n-1)\) and \(\tilde{z}_{n\alpha}\). We can see that the independent variables appear in ratios, in such ways, as to make \(z_{j\alpha} \quad (j = 1, \ldots, n-1)\) homogeneous of degree \(0\) in \(p_j\), \(p_{n-1}\) and \(\tilde{z}_{n\alpha}\), but this fact implies that the functions are non-homogeneous in \(p_j\) and \(p_{n-1}\).

If we form for all \(\alpha \quad (\alpha = 1, \ldots, m)\) the resulting system of market excess demand functions

\[(9.16) \quad z_1 = \sum_{\alpha=1}^{m} z_{1\alpha} = \sum_{\alpha=1}^{m} \alpha (\frac{p_1}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}}, \frac{\tilde{z}}{p_{n-1}})\]

\[(9.17) \quad \tilde{z}_1 = \sum_{\alpha=1}^{m} \tilde{z}_{1\alpha}\]

we have

\[(9.18) \quad x_1 \left(\frac{p_1}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}}, \frac{\tilde{z}}{p_{n-1}}\right) = z_1 - \tilde{z}_1 = 0 \quad i = 1, \ldots, n-1\]

\[(9.19) \quad x = z_1 - \tilde{z}_1 = \sum_{\alpha=1}^{m} z_{1\alpha} - \alpha \sum_{\alpha=1}^{m} \tilde{z}_{1\alpha} = 0 - \tilde{z}_n = -\tilde{z}_n\]

It is evident, that for the system (9.18) and (9.19) to have a solution the last term in the commodity excess functions must vanish

\[(9.20) \quad \frac{\tilde{z}}{p_{n-1}} = 0\]

This condition can be fulfilled in two ways:

1. if money stocks are zero: \(\tilde{z}_{n\alpha} \neq 0\)
2. if money prices are infinite: \(p_{n-1} = \infty\)

Both results are in a way equivalent properties of a "money rejecting" system: All money brought into the system is rejected again. There are no money prices, but real prices are determinate. But if (9.20) is satisfied then (9.18) is homogeneous in the prices.
In order to establish non-homogeneous functions we have to eliminate (9.20) without rendering the system inconsistent. In case \( \tilde{\mathcal{Z}}_{n} > 0 \), (9.20) implies, that velocity of circulation is driven up to infinity with prices. This follows by the definition of velocity,

\[
V = \frac{\sum_{i=1}^{n-1} \frac{P_i}{P_{n-1}} \cdot \frac{\tilde{S}_i}{P_{n-1}} \cdot \frac{\tilde{Z}_n}{P_{n-1}}}{\tilde{Z}_n}
\]

where \( S_i \) is the sum of all net supplies, that is \( S_i = \sum_{\alpha = 1}^{r} (\tilde{Z}_{i\alpha} - \tilde{X}_{i\alpha}) \)

when \( r < m \) and \( \tilde{Z}_{1\alpha} - \tilde{Z}_{1\alpha} > 0 \)

What we have to do then is to introduce an additional restriction, superimposed on the maximizing behavior of individuals, which assures that \( V \) can only be driven up to some finite upper boundary.

We proceed again with the utility function (9.11), but substitute for the budget restriction (9.12) the following one:

\[
\sum_{i=1}^{n} p_i (\tilde{Z}_{i\alpha} - \tilde{Z}_{i\alpha}) = 0 \quad \text{where} \quad p_n \neq 1
\]

Further (9.11) has to be maximized not only subject to (9.22), but as well subject to some restriction with respect to the rate of utilization of given stocks of money.

\[
Z_{n\alpha} = k_{\alpha} \sum_{i=1}^{r} p_i (\tilde{Z}_{i\alpha} - \tilde{Z}_{i\alpha}), \quad \text{where} \quad r < n - 1, \quad \text{and} \quad \tilde{Z}_{i\alpha} - \tilde{Z}_{i\alpha} > 0
\]

The summation on the right defines the total value of the individuals net purchases. \( k_{\alpha} \) is a given constant, determined outside the economic nexus. Solving (9.22) for \( Z_{n\alpha} \), we have

\[
Z_{n\alpha} = \tilde{Z}_{n\alpha} - \sum_{j=1}^{n-1} p_j (\tilde{Z}_{j\alpha} - \tilde{Z}_{j\alpha})
\]

Thus we have formulated in (9.23) and (9.24) two expressions defining \( Z_{n\alpha} \). Putting them together we may establish

\[
\sum_{j=1}^{n-1} p_j (\tilde{Z}_{j\alpha} - \tilde{Z}_{j\alpha}) = k_{\alpha} \sum_{r=1}^{s} p_r (\tilde{Z}_{r\alpha} - \tilde{Z}_{r\alpha}) \quad \text{for} \quad s < n - 1
\]

Subject to (9.25) the utility function has to be maximized. The first order conditions are:
(9.26) \[ \sum_{r} u_r^\alpha + \lambda \cdot p_r (1 + k_{r\alpha}) = 0 \quad r \leq s < n-1 \]

(9.27) \[ u_k + \lambda \cdot p_k = 0 \quad k = n-s, n-s+1, \ldots, n-1 \]

Let us order the commodities in such a way that all cases \( Z_r \approx -\bar{Z}_r > 0 \) comprise the first \( s \) of \( n-1 \). Then let us choose \( (n-1) \) as numéraire. Dividing through (9.25) by \( p_{n-1} \) we have

\[ \frac{\bar{Z}_{n\alpha}}{p_{n-1}} = \sum_{j=1}^{n-1} \frac{p_j}{p_{n-1}} (Z_j \approx -\bar{Z}_j) = k_{\alpha} \sum_{r=1}^{s} \frac{p_r}{p_{n-1}} (Z_r \approx -\bar{Z}_r) \]

and from (9.26) and (9.27) we derive

\[ \frac{u_s^\alpha}{p_{n-1}} = \frac{p_s}{p_{n-1}} \cdot (1 + k_{\alpha}) \quad \text{when} \quad r = 1, \ldots, s \]

and

\[ \frac{u_k^\alpha}{p_{n-1}} = \frac{p_k}{p_{n-1}} \quad k = s + 1, \ldots, n-2 \]

(9.28), (9.29) and (9.30) are together \((n-1)\) equations for \( \alpha \) in \((n-1)\) unknown \( Z_{1\alpha} \) \((i = 1, \ldots, n-1)\). By inspection of the system it follows that \( p_j \) \((j = 1, \ldots, n-2)\), \( p_{n-1} \) and \( \bar{Z}_{n\alpha} \) always appear in ratios. Clearly, the solutions of \( Z_{1\alpha} \) in terms of \( p_j, p_{n-1}, \bar{Z}_{j\alpha} \) and \( \bar{Z}_{n\alpha} \) are homogeneous of degree 0 in the prices \( p_j, p_{n-1} \) and the money stock \( Z_{n\alpha} \), but non-homogeneous in the prices alone.

Net demand and supply are then defined in the following way:

\[ D_{r\alpha} = Z_{r\alpha} - \bar{Z}_{r\alpha} \quad r = 1, \ldots, s < n-1, \text{ and } Z_{r\alpha} - \bar{Z}_{r\alpha} > 0 \]

\[ S_{k\alpha} = Z_{k\alpha} - Z_{k\alpha} \quad k = s + 1, \ldots, n-1, \text{ and } \bar{Z}_{s\alpha} - Z_{s\alpha} > 0 \]

and by adding over all \( \alpha \) \((-1, \ldots, m)\) we find the market (net) demand and (net) supply functions:

\[ D_i = \sum_{\alpha=1}^{m} (Z_{1\alpha} - \bar{Z}_{1\alpha}) \quad i = 1, \ldots, n-1, \text{ and } Z_{1\alpha} - \bar{Z}_{1\alpha} > 0 \]

\[ S_i = \sum_{\alpha=1}^{m} (\bar{Z}_{1\alpha} - Z_{1\alpha}) \quad i = 1, \ldots, n-1, \text{ and } \bar{Z}_{1\alpha} - Z_{1\alpha} > 0 \]

And by adding over all money restrictions, the market function of money is found.

These results are organized in the following system:

\[ D_i = D_i \left( \frac{p_1}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}}, \frac{\bar{Z}_n}{p_{n-1}} \right) \]
(9.36) \[ S_1 = S_1 \left( \frac{p_1}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}}, \frac{\tilde{z}_n}{p_{n-1}} \right) \]

(9.37) \[ D_1 = \tilde{S}_1 \]

(9.38) \[ \tilde{z}_n = K, \quad p_{n-1} \sum_{j=1}^{n-1} \frac{y_j}{p_{n-1}} - D_1 \]

where \[ \tilde{z}_n = \sum_{i=1}^{m} \tilde{z}_n \quad \tilde{z}_n = \sum_{i=1}^{m} k_i \]

which seems at first to be overdetermined: It contains \((n-1)\) money prices, \((n-1)\) demand quantities and \((n-1)\) supply quantities — altogether \(3n-3\) unknown variables. Equations on the other hand we count \(3n-2\). But there is no need for worry. The integrated restriction formulated by (9.22) implies a linear dependence of the commodity and money functions. If we sum (9.25) over the \(a_i's\) the result is

\[
\sum_{i=1}^{m} \tilde{z}_a - \sum_{j=1}^{n-1} p_j (\tilde{z}_a - \tilde{z}_a) = \sum_{i=1}^{m} k_i \sum_{i=1}^{n-1} r_i p_r (\tilde{z}_a - \tilde{z}_a)
\]

an expression resulting in:

\[
\tilde{z}_n = \sum_{i=1}^{n-1} p_i (D_i - S_i) = K \cdot \sum_{i=1}^{n-1} p_i \cdot D_i
\]

by further rearrangement:

\[
P_{n-1} (D_{n-1} - S_{n-1}) = \left[ \tilde{z}_n - K \cdot \sum_{i=1}^{n-1} p_i \cdot D_i \right] + \sum_{j=1}^{n-2} p_j \cdot (D_j - S_j)
\]

that is, we can express the excess demand function for the \((n-1)\)th commodity, for example, as a linear combination of the excess demand functions for money and the other

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28. It has become fashionable of late to decry the futility of counting equations and variables. I definitely disagree with this attitude. Several remarks may be appropriate about "equation counting:"

a) The fact that an equal number of equations and variables is neither a sufficient nor a necessary condition for the determinacy and consistency of a system, does not prevent us to accept the view, expressed in standard books on mathematics, that as a rule equality in the numbers mentioned is a necessary and sufficient condition. Vide Courant, Differential- und Integralrechnung II, Berlin, 1929, p. 136, for example.

b) So long as an analysis of parameters is not possible, we have to formulate postulate c) of I.1 if our talk has to be meaningful at all.

c) This fact does not imply that we have to stop at counting equations and variables, a habit correctly criticized by Samuelson. But I still think that Tinbergen was right when he stated that the happy sigh of an economist, having just established the equality in numbers of equations and variables may seem ridiculous. But to forget about bothering with "equation counting" may not be ridiculous, but rather serious from the point of view of logic.
(n-2) commodities. The dependence is clearly symmetrical. In order for it to be meaningful, we suppose $p_{n-1} > 0$. This is a postulate of the system. (9.61) indicates that a set of prices, equilibrating demand and supply for money and for (n-2) commodities, will also equilibrate $D_{n-1}$ and $S_{n-1}$. So one of the commodity equations in (9.37) is actually redundant. Thus we have 3n-3 independent equations [or (n-1) excess equations] in 3n-3 variables [(n-1) variables]. According to postulate c) the system can be solved in all the variables.

d) Various aspects of the system developed must be discussed:

1. Hickmann insisted\(^{29}\) that it is possible to build up a consistent classical system containing a closed real and a closed monetary subsystem. The analysis developed above shows that Hickmann's thesis is fundamentally wrong. If a monetary equation is at all meaningful, by which we mean that the money stocks are non-zero with finite money prices, then at least some of the non-monetary functions must be non-homogeneous in prices. Independent of the inclusion or exclusion of money into the utility function, if a system contains non-zero money stocks and finite prices, then any general assumption of homogeneity in money prices is logically excluded. It is here that Hickmann's argument goes wrong. It is an interesting example to the thesis that the existence of a mathematical solution of a given system of relationships does not assure any economic meaning for the system under consideration. The Casselian system presented by Hickmann possesses a mathematical solution, but its economics is extremely doubtful. The solution of the real part implies that a simultaneously existing positive excess demand for all commodities is rigidly excluded, and that these excess demands are independent of money prices. The monetary part implies that a positive excess demand for all commodities is perfectly conceivable and it depends essentially on money prices. These two statements are evidently contradictory — but Hickmann thinks it possible to combine them in one total system. But such a

\(^{29}\) Becker and Fort agree in this respect.
system is internally inconsistent and cannot hold. Thus he has either to drop the homogeneity assumption or the monetary constraint. If he chooses the first then he may formulate what was shown above; a consistent system determinate in money and real prices, but without any "additivity property" linking real and money relations. Money prices are not determined by the money equation, but all equations together. If he chooses to drop the monetary constraint, then we have a consistent system determinate in real variables. But the dichotomy of the pricing process within one system is definitely ruled out.

2. In a footnote, two types of Say's law were distinguished: Say's law as an identity and as a conditional statement. In Hickmann's presentation of the Casselian system Say's law is clearly implied. First, Hickmann defines functional dependence to make it equivalent to symmetrical dependence.\(^{30}\) Secondly, the real part of his system implies that any linear combination of commodity excess demand functions vanishes, whatever the money price level may be. But this actually is the content of Say's law 1. If the Casselian system is adequately adjusted as mentioned above, this will mean the following for Say's law 1: In case the monetary constraint is eliminated, Say's law 1 holds in a consistent system and coincides with Say's law 2. In case homogeneity goes overboard, the system may have the property defined by Say's law 2, but is definitely at variance with Say's law 1. The excess demand of the various commodities is not independent of money prices — so the identity in p cannot hold.

In Patinkin I it has been shown that Say's law 1 cannot hold in a system, where money is included in the utility function.\(^{31}\) Now we have shown as well that Say's

\(^{30}\) This fact was correctly emphasised in Patinkin III.

\(^{31}\) Patinkin I, Theorem XIII, p. 151, and Corollary. It must be added that according to the proof given by Patinkin, Say's law must hold in system A (money included in the utility function) if there is a neighborhood of the maximum position \(X^0\) for which the condition holds

\[ f(x_1, \ldots, x_n) = f(x_1^0, \ldots, x_n^0) \quad \text{when} \quad \left| x_i - x_i^0 \right| < \alpha_i \quad i = 1, \ldots, n \]

where \(\alpha_i\) are arbitrarily small positive numbers.
law 1 is excluded in the same way from a monetary system where money is not included in the utility function. Thus Say's law 1 and any monetary system are incompatible with each other. Say's law 2 is different, it may hold in a monetary system but it must not. By postulating it, we imply forms of the functions which will assure an effective clearing of all markets at a given set of equilibrium prices. To postulate Say's law 2 means, to attribute to the monetary system such properties that any excess supply (for instance: unemployment) can only exist in disequilibrium. Within a homogeneous system the two types of Say's law merge into one expression, as money prices become necessarily irrelevant.

3. It was seen that there are two ways to construct a monetary theory. One may assume money and other assets to be an integral part of the utility function. Here again, there are various ways of handling the matter. We may include nominal or real cash-balances. The system (A of Patinkin I) derived on such a hypothesis will in general be determinate in money and real variables, amongst which will be the velocity of circulation of money. This quantity will not be an explicit variable, but the components of its definition will be uniquely determined by the system. Or we may assume money to be completely outside the utility function, but its utilization subject to an institutional constraint.32 Both approaches give us a monetary theory. They may differ essentially in their respective relevancy, or in their comparative meaningfulness as pieces of economic theory. This difference actually exists. The first type of monetary theory pushes the frontiers of investigations much further than the second. The second one treats methods of payments, the relative dates of receipts and payments for an individual cash-balance and between individual cash-balances like a technological relationship imposed on the maximizing individual. These same phenomena are among problems of

32. Because of this possibility Theorems IV-IX and XIII in Patinkin hold only subject to the proviso: "provided there exists no finite upper boundary to velocity of money."
the first approach. This theory is able to show that maximizing behavior leads up to
a definite pattern of observable phenomena which immediately determine velocity. There
can be no question about the difference in the fruitfulness of the two types of theories,
but it may be well to remember that the second approach is not meaningless, if properly
handled. It still is a definitely better monetary theory, than any of the monetary
theories propagated by various European Central Banks in the interwar period.33

4. Apart from the fundamental internal inconsistency of the Casselian system,
there is a distinct difference in the implications of the Casselian system and the
monetary system with an institutional constraint. An increase in the supply of money
involves in both systems a proportional increase in money prices, and the real variables
remain unchanged. But an increased velocity of circulation (reduction in V) increases
prices proportionately in the Casselian system, while leaving real variables the same,
but in the non-homogeneous monetary system real variables will be affected and money
prices will not move proportionately.

There exists a further difference: In the Casselian system changes in the
distribution of the money stock were of no importance, money prices and real variables
remained necessarily completely unaffected. This may be so for the non-homogeneous
money system, but it is not necessarily so. In case of a linear approximation to the
individual demand functions, the parameters attached to might differ rather
sharply. The parameters of the market demand functions aggregated from the individual
functions will then be:

where \( a_{\alpha i} \) is the parameter of the \( \alpha_i \)th
individuals demand function.

33. There is no justification for F.oth categorical statement in his paper: "...an
error is involved in attempting to construct a consistent and determinate system by
including money stocks in utility functions instead of taking into account the effect
of the time pattern of money flows on the transactions demand for money." These time
patterns are variables from the point of view of utility analysis.
In case the $a_\alpha$ differ considerably and there is a redistribution of $\tilde{h}$ among the $\alpha$, then the form of the market functions will be changed. Thus there seems to be a definite relation between the variance of the $a_\alpha$ and the importance of redistributive effects of the money stock on real variables and money prices.

III. Conclusions

Patinkin thought that the classical dichotomy of money and prices meets two obstacles: the inconsistency in the functions and the inconsistency or indeterminacy of the equations of the system. With the exception of one of the classical systems discussed all were thought of as breaking down on these inconsistencies. The exception is the system with the homogeneous commodity functions and the non-homogeneous bond and money function. Patinkin had no logical argument against this system, he criticized simply the arbitrariness and ad hoc character of introducing non-homogeneity into the system.

Collecting our main results we observe that Patinkin's statement about the inconsistency of a classical system, which is built up out of two closed subsystems, one in the real and the other in money variables, holds true. In this respect Hickmann's thesis and others' cannot be upheld, but the reason differs eventually from Patinkin's argumentations. This type of classical theory breaks down, because of the following inconsistency in the functions of the system. If the monetary equation is meaningful, then homogeneity throughout the market functions is eliminated. If the market functions are non-homogeneous, then there exists no solution of real prices in a subset of the system's equation, or of money prices in the monetary equation. This type of inconsistency in the functions differs essentially from Patinkin's inconsistency of the "first kind." Even when Patinkin's type of inconsistency has been removed, the homogeneous functions—existence of money type of inconsistency may still exist. The latter kind of inconsistency is a sufficient, but not a necessary condition for Patinkin's inconsistency of the first kind.
Let us evaluate now the four systems which are described and discussed in Patinkin II.

1. The Cassalian System: This system is definitely inconsistent. To remove this, either the market functions must be transformed by entering the money stock as a new parameter into the functions. This breaks up the logical dichotomy in real and money relations. The resulting system is consistent and determinate in money and real prices. Or the money equation becomes meaningless, because money supply is zero, or prices are driven up to infinity -- and then we have a determinate and consistent real system on our hands.

2. A modified Cassalian System\textsuperscript{34}: This system is determinate and consistent in the real variables. The money equation is meaningless. Thus it is actually a real system.

3. The Lange System: This system is definitely inconsistent. To remove this, either stocks of money and bonds have to become additional parameters of the market functions, and thus these same functions become non-homogeneous in prices. This removes any possibility at dichotomisation in the sense defined in this paper. The resulting system is consistent and determinate in the money and real prices. Or the bond and money equation are meaningless; then we are left with a system determinate in real commodity prices and consistent.

4. A modified Lange System\textsuperscript{35}: This system is inconsistent. The remedy is as in 3.

As a general result we may state: Any attempt at constructing a monetary theory of the total economy by adding a monetary equation to a determinate and consistent system of

\textsuperscript{34} The characteristic features of this system are described in II,1, and on p. 16 & 17 of Patinkin II.

\textsuperscript{35} The characteristic features have been discussed in II,2 and in Patinkin II, p. 22.
real relations will result in a failure. Even if we find a mathematical solution, the economic logic involved remains faulty. To prove such a construction not to be a bogus invented to justify the present discussion we might quote from an outstanding book in the field of economic analysis: "It is obvious that the equilibrium position of any individual with given tastes and given initial quantities of the n commodities depends only upon the (n-1) ratios of exchange." "It is quite common in static theory to introduce an outside standard which serves as a unit of account and as a medium of exchange, but which is no part of the equilibrium system at all." "In traditional theory (the value of the numéraire in terms of money) is determined by a supplementary equation, the so-called equation of exchange."\(^{36}\)

There is no direct bridge from a real theory to a monetary theory by simply adding a new relation to the system. In classical economics we thus meet two types of theories: real ones and monetary ones. Both differ essentially. Both are determinate and consistent, but inconsistent with each other. Their common denominator is their being classical systems, that is, systems to which Say's law applies. This property assures that excess supplies will exist only in disequilibrium, and will tend to disappear.

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\(^{36}\) Quotations from Hosak, "General Equilibrium Theory in Internation Trade", p. 10 and 11, and p. 37.