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# A Note on Additivity in Linear Programming Models(*)

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At present we have two important types of models of production. The continuous production function model is formulated exclusively in terms of commodity flows and makes no attempt to distinguish different methods of production as such. More precisely, the method of production is assumed to have been "maximized out." Each point on the production surface is associated with the method of production which yields the maximal output for the given input, or the minimal input for the given output. No variable denoting the method of production appears explicitly.

The linear programming model is formulated so as to permit analysis of choice among alternative methods of production. Each set of input and output flows has associated with it a method of production. The solution to the maximum problem in this model is the set of methods employed. In other words, a solution consists of the set of methods in use at a maximal (efficient) point. Thus, that which is taken to be given in the continous production function model is part of the problem posed in the linear programming model. This being the case, the linear programming model must face up to certain problems assumed away in the other formulation. In particular, the continuous production function model assumes (either tacitly or explicitly) a given organization of production in producing units. In fact, the production function is defined for a particular producing unit or set of them. The linear programming model, having taken as part of its problem the specification of the methods of production employed in an optimal situation, does not take the producing unit as given. This comes out formally in that once the variable "the method" is introduced explicitly we need rules specifying how it may be manipulated. These rules of operating with the variable express certain theories about the structure of production. If the model containing them deserves to

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be called a theory, these rules must be interpreted as hypotheses having empirical content, i.e., as statements about future observations. In that case, it is of interest to know what theories the rules express, and how these theories affect, if at all, the limits of applicability of the model.

The linear programming model specifies two properties of the variable "method." The first is that methods are arranged in half-lines out of the origin in the space of commodity flows. Such half-lines (families of methods) are called activities. The variable $i = 1 \ldots n$ denotes the kind of activity and the variable $x_i \geq 0$, the particular point (the "method") in the $i^{th}$ family of methods. It is assumed that, if $y$ is the vector of commodity flows,

1.1) $y^i = g^i(x_i) \qquad x_i \geq 0$

such that

1.2) $\lambda y^i = g^i(\lambda x_i) \qquad \lambda \geq 0,$

I.e., if $y$ is a possible point, then $\lambda y \lambda \geq 0$ is also a possible point, and $\lambda y$ is associated with some $x$ in the set $\{x_i\}$. The empirical content of this assumption is that if a particular method is possible, all scale-models of it are also possible. I shall not discuss this assumption.

The second rule for operating with the variable denoting the method is the additivity postulate. It states that methods may be added, i.e., the sum

1.3) $\underset{i=1}{\mathcal{C}}(x_i) = x_k$

is defined for all $x_i$ such that,

1.4) $g[\underset{i=1}{\mathcal{C}}(x_i)] = g(x_k).$

This states that it makes no difference whether things are done alternatively or in combination, the results will be the same. The theory expressed here holds that one can neither gain nor lose from combining activities (or methods). Methods are strictly neutral; they neither complement or inhibit one another. It follows from this that the structure of producing units is indeterminate. Any distribution of functions among producing units will be optimal. Since this theory excludes no possible structure of plant, it cannot be used to analyse the distribution of functions among producing units or firms. More precisely, if we found a case in which combination of activities gave results different from, say, independent operation of the same activities, then the theory would have to be rejected. Consequently the problem of finding optimal distributions of functions among producing units would remain and the model including the additivity postulate would not be appropriate for treating that problem.

I shall give a very simple example where the theory underlying the addivitity postulate would not be appropriate.

Consider a planned economy (or a firm) in which two final commodities are to be produced, the rates of flow of which are denoted by $y_1$ and $y_2$. These commodity flows are produced separately by two activities 1 and 2 both of which require input flows $y_3$ and $y_4$ where $y_3$ and $y_4$ are in turn joint outputs of a third activity employing the single primary input flow $z$.

The managers of this economy (firm) must decide whether to build one plant to produce $y_1$ and another to produce $y_2$, i.e., to operate activities 1 and 2 independently, or to combine the two activities in a single plant to produce $y_1$ and $y_2$ jointly. The additivity postulate tells us that it makes no difference what they decide. However, suppose that the flows of intermediate products $y_3$ and $y_4$ deteriorate or dissipate very rapidly, e.g. heat, so that they must be used "on the spot" where and when they are produced, or they cannot be used at all. Therefore, separate plants to produce $y_1$ and $y_2$ must each produce $y_3$ and $y_4$ as well. Tables 1 and 2 show the technologies of these two plants respectively.

Table 1

$$
\begin{array}{c c}
 & \begin{array}{cccc} x_1 & x_3 & x_5 & x_6 \end{array} \\
\begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \\ z \end{array} &
\left[\begin{array}{cccc}
+1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\gamma_{13} & \gamma_{33} & -1 & 0 \\
-\gamma_{14} & \gamma_{34} & 0 & -1 \\
0 & -1 & 0 & 0
\end{array}\right]
\end{array}
$$

Table 2

$$\begin{array}{c c}
 & \begin{array}{cccc} x_2 & x_4 & x_7 & x_8 \end{array} \\
\begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \\ -s \end{array} &
\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
+1 & 0 & 0 & 0 \\
-\gamma_{23} & \gamma_{33} & -1 & 0 \\
-\gamma_{24} & \gamma_{34} & 0 & -1 \\
0 & -1 & 0 & 0
\end{array}\right]
\end{array}$$

Activities 5, 6, 7 and 8 reflect the fact that disposal by dissipation is costless. They are required since in general, the ratios in which $y_3$ and $y_4$ are required as inputs in each plant will be different from the ratios in which they are produced.

In order to decide whether to build these two plants or a single combined plant the managers of the economy must compare the results of the two situations. The additivity postulate would lead us to believe that we could represent the total technology with two independent plants by activities 1 and 2 and by the combined activity 3+4. But this would imply that $y_3$ and $y_4$ were freely transferable between the two plants contrary to the given technical facts. In order to represent this technology within the linear programming model we must distinguish $y_3$ in the first plant from $y_3$ in the second plant, and similarly with $y_4$. The matrix A in Table 3 shows this technology.

Table 3

$$\begin{array}{c c}
 & \begin{array}{cccccccc} x_1 & x_3 & x_5 & x_6 & x_2 & x_4 & x_7 & x_8 \end{array} \\
\begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ s \end{array} &
\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\gamma_{13} & \gamma_{33} & -1 & 0 & 0 & 0 & 0 & 0 \\
-\gamma_{14} & \gamma_{34} & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\gamma_{25} & \gamma_{45} & -1 & 0 \\
0 & 0 & 0 & 0 & -\gamma_{26} & \gamma_{46} & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0
\end{array}\right] \\
 & \underbrace{\hphantom{x_1 \quad x_3 \quad x_5 \quad x_6 \quad}}_{\text{Plant 1}} \quad \underbrace{\hphantom{x_2 \quad x_4 \quad x_7 \quad x_8 \quad}}_{\text{Plant 2}}
\end{array}$$

To find the efficient point set for this economy assuming $-z = -z_o$, we examine the net output equations.

2.1) $y_1 = x_1$

2.2) $y_2 = x_2$

2.3) $0 = y_3 = -\gamma_{13} x_1 + \gamma_{33} x_3 - x_5$

2.4) $0 = y_4 = -\gamma_{14} x_1 + \gamma_{34} x_3 - x_6$

2.5) $0 = y_5 = -\gamma_{25} x_2 + \gamma_{45} x_4 - x_7$

2.6) $0 = y_6 = -\gamma_{26} x_2 + \gamma_{46} x_4 - x_8$

2.7) $-z_o = -x_3 - x_4$

From which we obtain,

3.1) $$x_1 = \frac{\gamma_{33}(z_o - x_4)}{\gamma_{13}} - \frac{x_5}{\gamma_{13}}$$

3.2) $$x_1 = \frac{\gamma_{34}(z_o - x_4)}{\gamma_{14}} - \frac{x_6}{\gamma_{14}}$$

3.1a) $$x_2 = \frac{\gamma_{45} x_4}{\gamma_{25}} - \frac{x_7}{\gamma_{25}}$$

3.2a) $$x_2 = \frac{\gamma_{46} x_4}{\gamma_{26}} - \frac{x_8}{\gamma_{26}}$$

In efficient production at least one of the pair $x_5$ $x_6$ and one of the pair $x_7$ $x_8$ will be zero. If $x_5 = 0$ and $x_7 = 0$, then we have,

3.3) $$x_1 = \frac{\gamma_{33}(z_o - x_4)}{\gamma_{13}} = \frac{\gamma_{34}(z_o - x_4)}{\gamma_{14}} - \frac{x_6}{\gamma_{14}}$$

or,

3.4) $$x_6 = \frac{z_o - x_4}{\gamma_{13}}\left[\gamma_{13}\gamma_{34} - \gamma_{33}\gamma_{14}\right].$$

Therefore $x_6 = 0$ according as $\dfrac{\gamma_{13}}{\gamma_{14}} \gtrless \dfrac{\gamma_{33}}{\gamma_{34}}$.

Similarly,

3.5) $$x_2 = \frac{\gamma_{45}\, x_4}{\gamma_{25}} = \frac{\gamma_{46}\, x_4}{\gamma_{26}} - \frac{x_8}{\gamma_{26}}$$

or,

3.6) $$x_8 \gtreqless 0 \text{ according as } \frac{\gamma_{25}}{\gamma_{26}} \gtreqless \frac{\gamma_{45}}{\gamma_{46}} .$$

The managers wish to compare this with the results of building a single plant to produce $y_1$ and $y_2$ jointly. Since in a combined plant with activities 1 and 2 operated in coordination we may use, say, the surplus $y_3$ from the production of $y_1$ to help produce $y_2$, to distinguish $y_3$ from $y_5$ and $y_4$ from $y_6$ would be misleading.

Taking the matrix A of Table 3 we can introduce conversion or transfer activities 9, 10, 11 and 12 to obtain the matrix $\bar{\bar{A}}_1$ in Table 4.

Table 4

$$\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ -s \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} x_1 & x_3 & x_5 & x_6 & x_2 & x_4 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \end{array} \\
\left(\begin{array}{cccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\gamma_{13} & \gamma_{33} & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & +1 & 0 \\
-\gamma_{14} & \gamma_{34} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & +1 \\
0 & 0 & 0 & 0 & -\gamma_{25} & \gamma_{45} & -1 & 0 & +1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -\gamma_{26} & \gamma_{46} & 0 & -1 & 0 & +1 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}$$

We can reduce $\bar{\bar{A}}_1$ to $\bar{\bar{A}}$ below.

$$\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ -s \end{array}
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-\gamma_{13} & -\gamma_{23} & \gamma_{33} & -1 & 0 \\
-\gamma_{14} & -\gamma_{24} & \gamma_{34} & 0 & -1 \\
0 & 0 & -1 & 0 & 0
\end{array}\right)
\end{array}$$

The net output equations for this economy are,

4.1) $$y_1 = x_1$$

4.2) $$y_2 = x_2$$

4.3) $$0 = y_3 = -\gamma_{13} x_1 - \gamma_{23} x_2 + \gamma_{33} x_3 - x_4$$

4.4) $$0 = y_4 = -\gamma_{14} x_1 - \gamma_{24} x_2 - \gamma_{34} x_3 - x_5$$

4.5) $$-z_o = -x_3$$

or,

4.6) $$x_1 = \frac{\gamma_{33} z_o - \gamma_{23} x_2 - x_4}{\gamma_{13}}$$

4.7) $$x_2 = \frac{\gamma_{34} z_o - \gamma_{14} x_1 - x_5}{\gamma_{24}}$$

or,

4.8) $$x_4 = \gamma_{33} z_o - \gamma_{23} x_2 - \gamma_{13} x_1$$

4.9) $$x_5 = \gamma_{34} z_o - \gamma_{14} x_1 - \gamma_{24} x_2 .$$

With $x_4 = 0$ and $x_5 = 0$ equations 1.8 and 1.9 have a positive solution.

5.1) $$\begin{pmatrix} \gamma_{33} z_o \\ \gamma_{34} z_o \end{pmatrix} = \begin{pmatrix} \gamma_{13} & \gamma_{23} \\ \gamma_{14} & \gamma_{24} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

From which it follows that either,

5.2) $$\frac{\gamma_{13}}{\gamma_{14}} > \frac{\gamma_{33}}{\gamma_{34}} > \frac{\gamma_{23}}{\gamma_{24}}$$

or

5.4) $$\frac{\gamma_{13}}{\gamma_{14}} < \frac{\gamma_{33}}{\gamma_{34}} < \frac{\gamma_{23}}{\gamma_{24}}$$

At other efficient points i.e., efficient point such that $x_1 \neq x_1^o$ $x_2 \neq x_2^o$, either $x_4 > 0$ or $x_5 > 0$ but not both.

We can compare the efficient point sets under independent and combined production in terms of the following numerical example. Let $\gamma_{13} = 1$, $\gamma_{14} = 2$, $\gamma_{25} = 2$, $\gamma_{26} = 1$, $\gamma_{33} = \gamma_{34} = 1$ and let $z_o = 9$. These are the values of the coefficients of A. Let the coefficients of $\bar{A}$ be the same as those of A except that $\gamma_{23} = \gamma_{25} = 2$ and $\gamma_{24} = \gamma_{26} = 1$.

Figure 1 shows a simple graphical method for finding the efficient point sets shown in Figure 2.

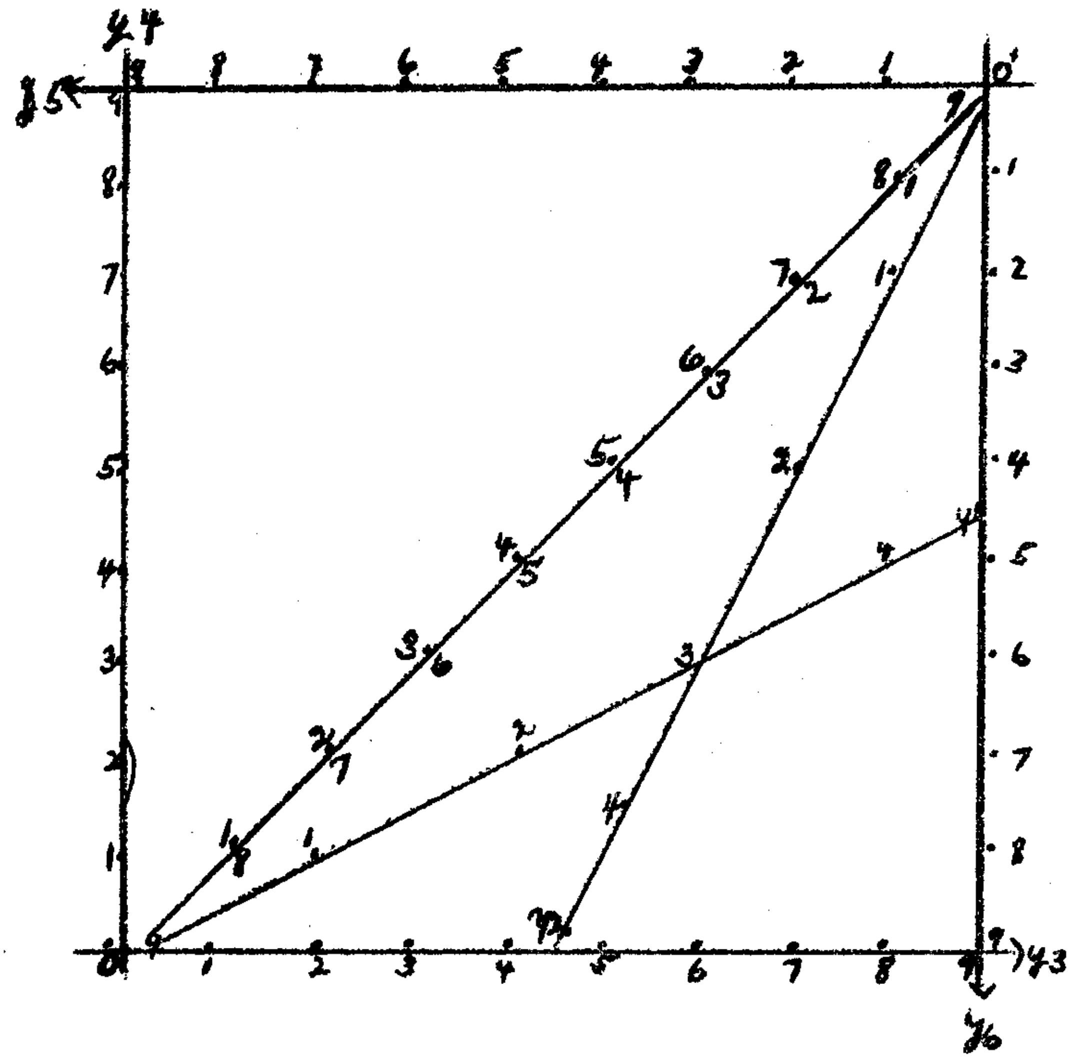

FIGURE 1

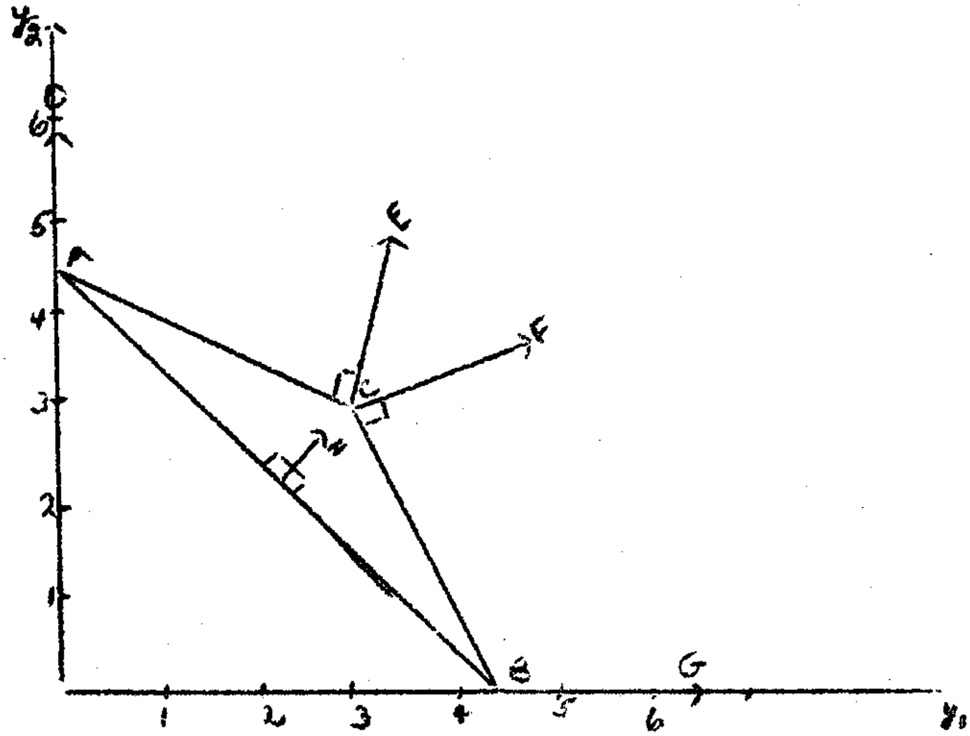


FIGURE 2

The axes in Figure 1 represent the amounts of intermediate commodities $y_3$ and $y_4$ used in each activity. The production of $y_1$ is measured along the line o $y_1$. The coordinates of a point on this line (referred to the coordinate system with origin o) are the quantities $y_3$ $y_4$ required to produce that amount of $y_1$. Similarly for $y_2$ referred to the coordinate system with origin $o^1$. The quantity of z is measured along the line $oo^1$. Starting from o the distance along $oo^1$ gives the quantity of z devoted to the production of $y_1$ (in the case of independent production), say, 3 units. These three units produce 3 units each of $y_3$ and $y_4$ which in turn produce at most $1^1/2$ units of $y_1$. The distance from $0^1$ to the point 0 + 3 along $00^1$ is the remaining quantity of z , 6 units. Referring now to the axes $y_5$ $y_6$ , 6 units of z produce 6 units each of $y_5$ $y_6$ , which in turn produce at most 3 units of $y_2$. Thus, one efficient point in the case of independent production is the point $y_1 = 1^1/2$, $y_2 = 3$ in Figure 2.

In the case of combined production the line $00^1$ may be ignored. E.g. if $1^1/2$ units of $y_1$ are produced we require 3 units of $y_3$ and $1^1/2$ units of $y_4$ leaving $7^1/2$ units of $y_4$ ($\simeq y_6$) and 6 units of $y_3$ ($= y_5$) for the production of $y_2$. With these quantities 4 units of $y_2$ can be produced, giving us the efficient point $y_1 = 1^1/2$ $y_2 = 4$ in Figure 2.

The line A B in Figure 2 is the efficient point set in the case of independent production; the broken line A, C, B is the efficient point set in the case of combined production.

In the present example complementarity of activities is due to the possibility of eliminating by combining activities, a barrier which prevents the movement of intermediate products from one activity to another. In this simple case we were able to handle the problem formally within the additive model by a rather artificial redefinition of commodities. However, unless we know in advance just where the barriers are, we shall either get the wrong answer by ignoring them, or, by defining each input and output flow of each activity to be a different commodity, make the model so general as to make it useless. In the latter case additivity will be preserved formally, but there will be nothing left to add. What seems to be needed is some criteria for deciding which barriers are important, and which enable us to mark off the class of cases in which they arise. Alternatively, if the problem is too complicated to be treated directly, we must distinguish the class of cases in which barriers can be ignored and limit the applicability of the additive model to those cases.

The case of non-additivity arising from the existence of barriers, we have seen, can be handled formally within the additive model by appropriate redefinition of commodities. But, in general, non-additivity or complementarity among activities arises in another fashion. Combining two or more activities may make possible further division of labor or specialization in the combined activity, and thereby make its coefficients different from the sums of the coefficients of the component activities. But if we assume that this is, in general, possible we must also question the assumption of constant returns to scale. For the possibility of gains from specialization depends on the existence of properties of commodity items which would render the assumption of constant returns to scale inappropriate.

Formally, if the appropriate assumption is

$$6.1) \quad g[C(X_i + X_j)] \gtrless g(X_i) + g(X_j)$$

then for $i = j$

$$6.2) \quad g[C(X_i, X_i)] \gtrless 2\, g(X_i)$$

and if $>$ ever applies, the assumption of constant returns to scale will be inconsistent with the assumption of complementarity among processes.

If it is important to treat such cases of non-additivity we need a rule of combination C which gives us for every set of methods $\{X_1 \ldots X_n\}$, the combined method $C(\{X_i\})$ where

$$6.3) \quad g[C(\{X_i\})] \gtrless g(X_1) + \ldots + g(X_n)$$

If what we really want is a model based on an assumption like (6.3), but for reasons of mathematical convenience are forced to keep the additive model, then I think the appropriate range of application of the additive model comes very close to that of the traditional continuous production function model. Neither model deals with the problem of finding optimal distributions of productive operations among plants. Both models take technology as given for a given distribution of functions among producing units. The fact that in the present trivial example for some relative prices [see Figure 2] the managers of our economy (firm) would be indifferent as between independent or combined organization suggests that it would be possible to construct cases in which independent organization would be preferred to combined organization for some price ratios and not for others. If this is the case, then the distribution of functions among plants will not be invariant under price changes (in the long-run). Thus, the matrix of activities in the additive model cannot be regarded as purely technological, and valid for all prices. It must be regarded as an approximation to the technology in some neighborhood of the existing or assumed distribution of plants.