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OUTLINE FOR SOME NOTES ON STOCHASTIC PROCESSES IN ECONOMICS

by

Merton P. Stoltz
Brown University

1. There will be no need here to discuss either the importance or the necessity for the construction of models whenever we desire to treat some real problem mathematically or otherwise. The function of the model is to represent in idealized form those features of reality which are deemed essential for the problem in question. The model, of course, becomes useless if the "realities" are not capable of identification. In economics, as well as in the most varied of other fields, a large and important class of problems may be reduced to the investigation of systems changing over time. The appropriate model in any instance may be either "well-determined" or "statistical." In the former the system is uniquely determined for any time, t, if the state of the system is known at one particular time, t_0. It is clear that such "causal" processes are applicable in a limited number of fields, e.g., classical mechanics, but is inappropriate in others wherein it is impossible or excessively difficult to apply. In economic problems we may well regard a well-determined system as fundamentally impossible. In these cases the appropriate model is statistical. In general, there have been two statistical approaches employed in economic problems - the empirical and the a priori. The former has had some success, but is rather to be condemned. We will here be concerned with the a priori method, i.e., with models for which the knowledge of certain initial conditions does not uniquely determine the state of the system for all t, but rather a certain probability of

the possible states.

2. By a stochastic or random process is meant a process in which enter one or more random variables possessing a probability distribution depending on one continuously varying parameter which may be taken, e.g., as time. The main advantage of this definition is that mathematically the probabilities of the distribution may be found as solutions of some differential equations, ordinary or partial, along with the initial conditions. In the general development of the theory of stochastic processes two different types of phenomena that may be characterized by random processes have been distinguished. The continuous process is characterized by small, incessantly occuring, random changes, e.g., Brownian movement. The discontinuous process is characterized by random changes occuring at discrete moments and with finite values. A mixed type might also be defined, but has not had, as yet, any practical application. I t will be supposed that the appropriate stochastic type in economic applications is discontinuous.

A stochastic process is termed stochastically definite if the influence of past history is entirely expressed by the initial conditions, i.e., the probability distribution of the system will be determined for all times t ($> t_0$) if the state is known at the time t_0. We shall, in what follows, always assume that the random process is stochastically definite. Whether such processes are sufficiently general for economic analysis remains to be investigated.

3. The probability scheme. The theory of stochastic processes is developed on the basis of the modern axiomatic theory of probability, fundamentally due to Kolmogoroff. The label of an event is defined as the point set S in Euclidean n-space, R_n, $n \in N$, and all sets are taken as Borel sets. The

probability of an event in the label set S is defined as an arbitrary real set function P(S), the probability function. P(S) satisfies certain well-known axioms, e.g., those due to Cramer.

Denote the stochastic variables by $\xi_1, \xi_2, \ldots$, the value taken by the random variables by $x_1, x_2, \ldots$, and the distribution parameter by t. If for some value t, the variable of a one-dimensional process takes a value lying inside a domain S of the given Euclidean space, the event $\xi(t) \in S$ has occured. Since the probability of this event has generally been conditioned by past history, the conditional probability of an event $\xi(t) \in S$, relative to the hypothesis that the variable takes the value x at the moment t_o, $\xi(t_o) = x$, is uniquely defined by a conditional pf.f. of the type

$$P(S,t;\, x,t_o) = \Pr(\xi(t) \in S \mid \xi(t_o) = x)$$

Where the variable has a given initial value, we may define the absolute probability of the event $\xi(t) \in S$, and the absolute pr.f.

$$P(S,t) = \Pr(\xi(t) \in S)$$

In this pr.f. is defined for $t = t_o$, and for a stochastically definite process, we have from the general composition rule that the absolute pr.f. of the random variable $\xi(t)$ is for every $t \geq t_o$ given by

$$P(S,t) = \int_R P(S,t;\, \xi,\, t_o)\, d\, P(X,t_o)$$

where the integral is a Lebesgue-Stieltjes integral taken over the entire space R, and the differential is referred to the set X. ξ denotes a point corresponding to X.

From the definition of the set function $P(S,t;\, x,t_o)$ which analytically completely describes a stochastic definite process it follows that certain properties must exist for all $t \geq t_o$. Every function $P(A,t;x,t_o)$ satisfying these conditions, together with another assumption on measurability, may be regarded as a conditional pr.f. defining a stochastically definite process. These conditions are called the fundamental conditions.

For the analytical treatment of the discontinuous process two additional functions will be defined, the density function $p(x,t)$ and the transitional probability function $\pi(S;x,t)$. If the variable $\xi(t)$ takes the value x at the moment t, the density function $p(x,t)$ is an asymptotic expression of the probability of a random change in the interval $t + \Delta t$. The transition pf.f. $\pi(S;x,t)$ is the conditional probability of the event $\xi(t+\Delta t) \in S$, relative to the hypothesis that a random change of the variable from x has taken place during Δt.

From these definitions, again, it follows that certain properties must exist for all $t \geq t_0$. These conditions, together with the adjoined condition of uniform boundedness, constitute the fundamental conditions for a discontinuous stochastical process. From these conditions we may deduce certain equations to be called the fundamental equations which are integro-differential in form. Feller's theorem covers the existence of a unique solution of the fundamental equations such that the solution constitutes a conditional fr.f. characterizing a stochastically definite process of the discontinuous type.

By an extension of these notions we may define stochastic process called Markoff chains, wherein the set functions characterizing such a process is reduced to a point function. Similarly, we may define the so-called elementary random process as a special application of the Markoff theory. Time homogeneous, space homogenous, and completely homogeneous processes may be defined. In all of these cases the fundamental conditions have been generalized to serve in applications to fields other than economics, for which they do not appear to be everywhere appropriate.

Of major importance in economic analysis, the theory of one-dimensional stochastic processes, which have been considered thus far, may be generalized to the multi-dimensional case.

The application of any of the stochastic processes which have been considered here to economic analysis still constitutes are area of uncertainty. Considerable research is still required, both in economics and mathematics, to develop more general conditions on the stochastically definite process, and to extend our technical knowledge concerning the theory of random equations and the theory of random distributions. However, the role of a stochastic process of the type considered may be easily visualized in economic problems. As a very general illustrative case we might approach an economic problem mathematically by the construction of equations in finite or infinitesimal differences, with coefficients which are random variables. These are thus random equations which embody the a priori hypotheses concerning the machinery of the economic processes under consideration. The solution of a system of stochastic equations cannot give a unique system of values of the $\xi_j(t)$, $(j=1, 2, .., n)$, for each moment t, $\xi_j(0)$ being fixed. Rather we have a probability distribution which in general depends on the values $\xi_j(0), ..,$ $\xi_j(t-1)$, which the variables $\xi_j(t)$ had at the previous moments. The appropriate fr.f. is derived according to the method outlined above. More concrete examples of stochastic processes in economics may be, of course, given.

It will be noted that the processes developed here are appropriate to the testing of hypotheses by some statistical procedure, e.g., that of Neyman-Pearson.