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THE RATIONALE OF THE DEMAND FOR MONEY AND "MONEY ILLUSION," SECTIONS 1.-4.

by

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THE RATIONALE OF MONEY DEMAND AND 'MONEY ILLUSION'

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1. The Problem Outlined

1. A. Unlike precious metals, paper money is neither a consumption nor a production food. The classical theory of market equilibrium was developed at a time when metallic money prevailed. It fails to relate the desired quantity of paper money and the exchange ratios between paper money and various commodities (the "absolute" prices of commodities) to assumptions of rational (utility-maximizing) behavior.*) As a makeshift, an "equation of exchange"

*) Patinkin [1].

was used by later authors.*) It appealed to the alleged existence of an

*) E.g., Divisia [1].

"institutional" constant (velocity of circulation) superimposed upon the assumption of rational behavior of individuals. More recently, this makeshift device was generalized, and slightly rationalized, into Keynes' "liquidity preference" equation.*) Like the rest of macro-economics, this equation is still in need

*) See Hicks [1].

of being related to assumptions of rational behavior. Should the "liquidity preference equation" and other Keynesian equations have purely empirical claims, these would be hard to establish: the observed time series of the relevant variables (quantity of money, interest rate, price level, consumption, national income and, possible, its distribution) are no doubt consistent with a large number of equation-systems other than the Keynesian one.*)

*) For an attempt to derive a macroeconomic system from the assumption that individuals maximize their utilities or profits see Klein [1], Technical Appendix pp. 192-199.

1. B. Patinkin [1] has proved -- in a manner to be somewhat modified in our Section 2 below -- the inconsistency of traditional microeconomics with paper money. After having contributed this important criticism, he has not attempted to give a better microeconomics improved upon Walras and Pareto. Instead, Patinkin [2] followed*) the more superficial approach of Cassel whose

*) Patinkin is aware of the loss of information due to this neglect of microeconomic assumptions: [2], footnotes 19, 20; [1], footnote 3.

aggregate demand and supply equations emerge from nowhere. Had these equations been derived from the maximization of each individual's utility subject to the limitation of his and the society's resources, these limitations would result in certain restrictions upon the macroeconomic system. Unless one derives these restrictions precisely, the determinacy of the system cannot be discussed.

1. C. Questions like the following arise in a paper money system: Why is a positive price (viz., 1) paid for a thing that is neither a consumption nor a production good; in other words, why the absolute prices of consumption and production goods are settled in the market at finite equilibrium levels? Why is the desired stock of money positive? Finally: if there are altogether N-1 consumption goods $(1,\ldots,N-1)$, is the demand for each of them a function of the N-2 price ratios $p_1/p_{n-1},\ldots, p_{n-2}/p_{n-1}$, or is it the function of the N-1 absolute prices $p_1,\ldots, p_{n-1}$? If the former is the case, i.e., if demand is homogenous of zero degree in absolute prices we say that there is no "money illusion." But then the number of independent equations in the system is larger by one than the number of variables to be determined. It has been pointed out, accordingly, that a consistent model of modern economy has to admit "money illusion" on the part of at least one group of decision makers -- be it workers or their unions (as explicitly assumed by Keynes),*) or be it

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The importance of this assumption was first pointed out by Leontief [1], I believe.

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consumers, entrepreneurs, banks, or the fiscal authorities.

1. D. Sections 3-5 below are devoted to such a microeconomic determination of the demand for paper money, and of the ratios of exchange of paper money against goods (so-called absolute prices). The essential steps beyond Walras-Pareto are three:

(1) In Section 3, the assumption is removed that the individual is concerned with only one marketing date. In the resulting more general, "multi-period" model stocks appear in an "essential" way (i.e.--they are not necessarily proportional to flows);

(2) In Section 4, a liquidity concept is introduced, in terms of market imperfections;

(3) In Section 5, uncertainty is introduced.

1. E. Of these three steps, the first, formulation of a simple multi-period model (Section 3) is sufficient to explain the occurrence of positive stocks of money, but is not sufficient to explain why the money stock of an individual takes values other than either zero or the money equivalent of all the resources. Step two, introduction of liquidity (Section 4), is sufficient for an explanation of this fact, even (contrary to the prevailing opinion) without introducing uncertainty. Accordingly, step three, introduction of probabilities (Section 5) merely adds an (important) touch of reality.

1. F. A further step towards reality is to introduce lending (and borrowing) or, more generally, claims (and liabilities) of all kinds. There are analogies, from the point of view of a single individual, between claims and physical assets. Therefore, production, not discussed in the preceding sections, must be introduced (in Section 6) before treating claims (in Section 7).

1. G. Finally, the aggregation problem is discussed in Section 8.

2. A Uni-Period Model of Pure Exchange Economy

2. A. We shall call a "good" anything that can be demanded in non-zero quantities. Let there be A individuals and N goods. The a-th individual ($a=1,\ldots,A$) enters the market with the initial quantity $\bar{x}_{na}$ of the n-th good ($n=1,\ldots,N$). He leaves it with a desired quantity x_{na}. The difference $x_{na}-\bar{x}_{na}$ is his demand for this good; this expression can also be called his negative supply," or the "excess of his demand over his supply," or his "net demand" for the n-th good. The sum $\sum_{a=1}^{A}(x_{na}-\bar{x}_{na})$ is best called "aggregate net demand."

2. B. We shall call a "consumption good" a good that enters the utility function of at least one individual. Dollar bills, factories, and bonds are goods but not consumption goods; (but factories and bonds will not be considered in a pure exchange economy).

2. C. A "flow" is the rate of use of a "stock," per unit of time. We shall formulate a uni-period model in terms of "flows." It will be presently seen (in 2.E.) that its reformulation in terms of "stocks" does not add anything essential. The same will not be true of a multi-period model (Section 3 and the following).

2. D. We shall first assume all goods to be consumers goods, and let the a-th individual maximize his utility function of the N flows,

$$(2:1)\qquad u^a(x_{1a},\ldots,x_{Na}),$$

subject to his "budget restriction"

$$(2:2)\qquad \sum_{n=1}^{N}(\bar{x}_{na}-x_{na})\,p_n = 0,\quad a=1,\ldots,A.$$

We obtain, denoting by u_{na} ("marginal utility") the partial derivative of u^a with respect to its n-th argument,

$$(2:3)\qquad u_{na}/u_{Na} = p_n/p_N;\ n=1,\ldots,N-1;\ a=1,\ldots,A.$$

These N-1 equations together with (2:2) determine, for any given set of the N-1 price ratios $p_1/p_N,\ldots,p_{N-1}/p_N$, the N desired flows $x_{1a},\ldots,x_{Na}$. Thus the individual's demand $(x_{na}-\bar{x}_{na})$ for the n-th good depends on the N-1 price ratios. Bargaining goes on (i.e. prices and exchanged quantities are subjected to trial-and-error adjustments, not further analyzed in this theory), till "the market is cleared," i.e., till the aggregate demand for each commodity balances its aggregate supply: its aggregate net demand vanishes:

$$(2:4)\qquad \sum_{a=1}^{A}(\bar{x}_{na}-x_{na}) = 0,\ n=1,\ldots,N.$$

The system (2:2--4) contains N-1 + NA independent equations, since equations (2:2) together with the first N-1 equations in (2:4) yield the N-th equation in (2:4). This number of independent equations equals the number of the following variables: N-1 price ratios and NA demands. If, now, the N-th consumption goods is used as a numeraire, an extra equation is added:

$$(2{:}5)\quad p_N = 1,$$

permitting to determine the N absolute prices instead of the N-1 price ratios.

2.E. The well known model just given has certain tacitly implied properties of great importance. Marketing takes place on a single date. The individual's satisfaction is not affected by flows other than those determined on that date; and no distinction is made by the individual between the different post-market dates at which consumption may take place. Suppose such an individual plans for a period of θ time units (a "horizon"--possibly his whole life). Suppose he enters the market with $\bar{x}_{na}$ initial flow units of a certain kind (future daily housing or heating services of a house or of a pile of coal, future daily labor services or annual crops) and leaves it with x_{na} desired flow units. We can then also say, under the assumption of the uni-period model, that he enters the market with $\bar{y}_{na}$ stock units (rooms, coal calories, expected man years of labor or bushels of grain) and leaves the market with y_{na} stock units, where

$$(2{:}6)\quad \bar{y}_{na}/\bar{x}_{na} = y_{na}/x_{na} = \theta \text{ time units.}$$

Because of this proportionality a uni-period model can be said to admit stocks "in a non-essential way only."

2.F. Suppose now the N-th good to be paper money. Equations (2:2), (2:4), (2:5) still hold. But the utility function (2:1) is replaced by

$$(2{:}7)\quad u^a\,(x_{1a},\ldots,x_{N-1,a});\ \text{hence}$$

$$(2{:}8)\quad u_{Na} = 0.$$

Return for a moment to a system with non-paper money. Assume that there is no "universal satiety": that is, $u_{na} \neq 0$ for at least one individual a and one consumption good $n \neq N$. Then by (2:3), (2:5), its absolute price,

$$(2{:}9)\quad p_n = u_{na}/u_{Na},\ n=1,\ldots,N,$$

approaches infinity as u_{Na} approaches zero. It follows that in a uni-period model with paper money, prices of consumption goods are bid up indefinitely. No equilibrium can be reached.

2.G. An additional solution (the one obtained by Patinkin [1]) is arrived at if one introduces new restrictions. Returning first again to the system in which the numéraire is a consumption good, observe that in a uni-period model no debts, or servicing of debts, can exist. Hence all stocks and flows are non-negative:

$$(2{:}10')\quad x_{na} \geq 0,\ n=1,\ldots,N.$$

To maximize (2:1) subject to (2:2) and (2:10') rewrite the latter restriction thus:

$$(2{:}10)\quad x_{na} - (r_{na})^2 = 0,\ n=1,\ldots,N,$$

where the r are known to be real. The a-th individual maximizes with respect to x_{na}, r_{na} $(n=1,\dots,N)$ the expression

$$(2{:}11) \quad u + \lambda \sum_n (\bar{x}_{na} - x_{na})\, p_n + \sum_n \mu_{na} \left(x_{na} - (r_{na})^2\right),$$

where the prices p are given, and the λ, μ_n are Lagrange multipliers. We obtain

$$\left.\begin{aligned} &(2{:}12) \quad u_{na} + \mu_{na} = \lambda p_n, \\ &(2{:}13) \quad \mu_{na} r_{na} = 0 \end{aligned}\right\} \quad n = 1,\dots,N.$$

By (2:12), for any two goods -- say n, N --

$$(2{:}14) \quad p_n/p_N = (u_{na} + \mu_{na})/(u_{Na} + \mu_{Na}).$$

This is a more general equation than the usual marginal utility theorem (2:3). It reduces to that theorem if the optimal flows x_{na}, x_{Na} of the two goods are known to be positive, since then by (2:10), (2:13) μ_{na}, μ_{Na} are both zero.

If N is paper money, (2:14) becomes

$$(2{:}15) \quad p_n = (u_{na} + \mu_{na})/\mu_{Na} \qquad n=1,\dots,N-1.$$

This permits determinate and finite prices for all goods provided $\mu_{Na} \neq 0$, and therefore provided the money flow $x_{Na} = 0$. Since this is true for all individuals, we have, by (2:4)

$$(2{:}16) \quad \sum_{a=1}^{A} \bar{x}_N = 0,$$

i.e., money occurs not physically but only as "money of account."

2. H. The uni-period model is the usual model of the general equilibrium theory. We can now answer, with respect to this model, the questions asked in 1.C. The coexistence of paer money and determinate and finite prices of consumption goods is not consistent with this model. This answer rules out the question whether in such a model the equilibrium demand for a consumption good ($x_{na} - \bar{x}_{na}$), n=1,...,N-1, is or is not homogenous of zero degree in absolute prices: the model itself excludes equilibrium. (The question is similar to this one: If a rectangle has five angles, is their sum 360° or 450°?).

3. A Simple Multi-Period Model

3. A. We shall now introduce the expectation of changing -- viz., falling -- prices as a simple but far from satisfactory explanation for the desire to hold paper money. It will be seen that such a model has important unrealistic features, not due to the paper money's lack of immediate utility but to other properties of the model. The necessary modifications will be introduced in the Section following the present one.

3. B. Suppose the individual plans for a finite sequence of time intervals devoted alternately to marketing and to consumption. The marketing intervals are neglibible in length (but see Section 8).). The length of each consumption interval is used as time unit. As in 2.E., denote flows by x and stocks by y. Assuming first that all goods are consumption goods, (with leisure among them) the plan of the a-th individual with respect to the n-th good is as follows:

	Instant	Stocks Brought	Traded at Price	Stocks Retained	Period	Consumed
(3:1)	0	$\bar{y}_{na}$	p_n^o	y_{na}^o	(0,1)	x_{na}^o
	1	$y_{na}^o - x_{na}^o$	p_{na}^1	y_{na}^1	(1,2)	x_{na}^1
	..	. . .		. .	. . .	. .
	t	$y_{na}^{t-1} - x_{na}^{t-1}$	p_{na}^t	y_{na}^t	(t,t+1)	x_{na}^t
	..	. . .		. .	. . .	. .
	T-1	$y_{na}^{T-2} - x_{na}^{T-2}$	p_{na}^{T-1}	y_{na}^{T-1}	(T-1,T)	x_{na}^{T-1}
	T	$y_{na}^{T-1} - x_{na}^{T-1}$	p_{na}^T	y_{na}^T		

$\bar{y}_{na}$ is the "initial stock" to be distinguished from the "starting stock" y_{na}^o.

The initial stock is given to the individual. He tries to retain and consume, respectively, optimal quantities of the following stock and flow variables:

(3:2) stocks : $y_{na}^o, y_{na}^1, \ldots, y_{na}^{T-1}, y_{na}^T$

(3:3) consumption flows : $x_{na}^o, x_{na}^1, \ldots, x_{na}^{T-1}$, $n=1,\ldots,N$.

3. C. The individual's satisfaction depends on the consumption flows (3:3) in each of the T considered periods; but it also depends on the final stocks y_{na}^T. To be sure, he is indifferent to the particular way in which he or his heirs will arrange their consumption in the future following the instant T. But he is not indifferent as to the resources--evaluated at the prices of instant T--with which he or his heir will start that future. The column headings of the Plan (3:1) have to be interpreted, with respect to the last line, as follows: "Traded at Price" means here "Evaluated at Price"; "Stocks Brought (Retained)" means "Stocks Brought (Retained) if Trading would take Place." Accordingly, we include the quantities y_{na}^T into the argument of the utility function, which we write thus:

$$(3:4) \quad u^a(x_{1a}^o, \ldots, x_{Na}^o, \ldots, x_{1a}^{T-1}, \ldots, x_{Na}^{T-1}, y_{1a}^T, \ldots, y_{Na}^{T-1}).$$

We shall write for every n: $\partial u^a/\partial x^t_{na} \equiv u^t_{na}$ $(t=0,\ldots, T-1)$; $\partial u^a/\partial y^T_{na} \equiv u^T_{na}$.

3. D. The plan (3:1) must satisfy the following conditions, each stating that stocks brought to the market are exchanged, at certain prices, for stocks with which the next consumption period is started:

$$(3{:}5)\quad \sum_{n=1}^{N}(\bar{y}_{na} - y^o_{na})\, p^o_n = 0;$$

$$(3{:}6)\quad \sum_{n=1}^{N}(y^{t-1}_{na} - x^{t-1}_{na} - y^t_{na})\, p^t_{na} = 0,\ t = 1,\ldots,T.$$

Note that the prices p^o_n settled at instant 0 -- analogous to (2:2)-- are the same for all individuals; but the expected prices p^t_{na}, $t > 0$, will, in general, vary from one man to another. The expected prices need not clear the market; but the prices of instant 0 must be such as to have the market cleared (analogous to (2:4)):

$$(3{:}6')\quad \sum_{a=1}^{A}(\bar{y}_{na} - y^o_{na}) = 0;\quad \sum_{a=1}^{A}(y^{t-1}_{na} - x^{t-1}_{na} - y^t_{na}) = 0,\ t=1,\ldots,T.$$

3. E. We rule out negative consumption flows, labor being replaced by leisure. For the present model we also rule out liabilities, i.e., negative stocks (but see Section 7 below). The stocks and flows in (3:1) obey therefore the following inequalities:

$$(3{:}7)\quad y^t_{na} - x^t_{na} \geq 0;\ x^t_{na} \geq 0;\ y^T_{na} \geq 0;\ n=1,\ldots,N;\ t=0,\ldots,T-1.$$

They can be rewritten thus:

$$\left.\begin{aligned}
&(3{:}7)\quad x^t_{na} - (r^t_{na})^2 = 0\\
&(3{:}8)\quad y^t_{na} - x^t_{na} - (s^t_{na})^2 = 0\\
&(3{:}9)\quad y^T_{na} - (r^T_{na})^2 = 0,
\end{aligned}\right\}\quad \begin{aligned}&t = 0,\ldots, T-1\\ &n = 1,\ldots, N\end{aligned},$$

where the r, s, are known to be real.

3.F. Let $p_N=1$ and write $n'=1,\ldots, N-1$.

Each of the optimal stocks and flows listed in (3:2), (3:3) is determined by the a-th individual as a function of the following givens: his N initial stocks $\bar{y}_{na}$, the N-1 market-clearing prices $p^o_{n'}$, and the T(N-1) expected prices $p^t_{n'a}$, $t=1,\ldots,T$. For each set of these givens, optimal stocks and flows are obtained by maximizing (3:4) subject to the conditions (3:5), (3:6), (3:7), (3:8), (3:9). Omit the index a (indicating the individual) for brevity and maximize with respect to x^t_n, y^t_n, y^T_n, r^t_n, r^T_n, s^t_n $(t=0,\ldots, T-1;$ $n=1,\ldots,N)$ the following expression:

$$(3{:}10')\quad u + \lambda^o\sum_n(\bar{y}_n - y^o_n)p^o_n + \sum_n\sum_{t=1}^{T}\lambda^t(y^{t-1}_n - x^{t-1}_n - y^t_n)p^t$$

$$+ \sum_n\sum_{t=o}^{T-1}\mu^t_n(x^t_n - (r^t_n)^2) + \sum_n\mu^T_n(y^T_n - (r^T_n)^2) + \sum_n\sum_{t=o}^{T-1}\nu^t_n(y^t_n - x^t_n - (s^t_n)^2),$$

where the λ, μ, ν, with appropriate sub and superscripts, are Lagrange multipliers, and summation $\sum_n$ is from 1 through N. After equating the partial derivatives to zero, and rearranging, we obtain the following equations:

$$\left.\begin{array}{ll}(3{:}10) & u_n^t + \mu_n^t = \lambda^t p_n^t = \lambda^{t+1} p_n^{t+1} + \nu_n^t \\ (3{:}11) & u_n^T + \mu_n^T = \lambda^T p_n^T \\ (3{:}12) & \mu_n^t r_n^t = \mu_n^T r_n^T = \nu_n^t s_n^t = 0\end{array}\right\} \quad t=0,\ldots,T-1;\ n=1,\ldots,N$$

To check and sum up, let us count the equations and unknowns (for a given individual, with prices given):

Equations		Unknowns (where t = 0,...,T-1	
(3:5)	1	x_n^t	TN
(3:6)	T	y_n^t, y_n^T	(T+1)N
(3:7)	TN	r_n^t, r_n^T	(T+1)N
(3:8)	TN	s_n^t	TN
(3:9)	N	λ^t, λ^T	T+1
(3:10)	2TN	μ_n^t, μ_n^T	(T+1)N
(3:11)	N	ν_n^t	TN
(3:12)	(2T+1)N		
	1 + T + 3N + 6 TN		1 + T + 3N + 6 TN.

As to the aggregate system, the same reasoning applies as in the last paragraph of 2.D., with equations (2:4) replaced by (3:6).

3\. G. For any two consumption goods, say n, N, we obtain by (3:10), (3:11)

$$(3{:}13)\quad p_n^t/p_N^t = (u_n^t + \mu_n^t)/(u_N^t + \mu_N^t),\ t = 0,\ldots,T;$$

$$(3{:}14)\quad p_n^{t+1}/p_N^{t+1} = (u_n^t + \mu_n^t - \nu_n^t)/(u_N^t + \mu_N^t - \nu_N^t),\ t = 0,\ldots, T-1.$$

(3:13) repeats, for each period, the generalized marginal utility proposition (2:14) of the uni-period model. Suppose now that the prices of the two goods are expected to change in different proportions between the two instants t, t+1. Then by (3:13), (3:14)

$$(3:15) \quad \nu_N^t(u_n^t + \mu_n^t) \neq \nu_n^t(u_n^t + \mu_n^t)$$

If both prices in (3:13) are finite and non-zero, so are the co-factors of ν_N^t, ν_n^t in (3:15). Therefore ν_N^t, ν_n^t cannot both be zero. Hence by (3:12), (3:8)

$$(3:15') \quad y_n^t = x_n^t \text{ or } y_N^t = x_N^t, \; t = 0, \ldots, T-1.$$

That is, (see Plan (3:1)), if the prices of two goods are expected to change in different proportions from one marketing instant to the next, the optimal stock of at least one of them must be just high enough to satisfy consumption needs till the next marketing instant. If $y_n^t = x_n^t$ we say that the stock of the n-th good is "unloaded."

3. H. Because of the restrictions (3:5), (3:6), (3:7) it is impossible for all stocks to be unloaded simultaneously (except in the trivial case when all stocks and flows vanish beginning at next marketing instant). Suppose the only good whose stock is not unloaded, is the N-th good: that is, by (3:7'), $y_N^t > x_N^t$. Suppose further that no two prices are expected to change in the same proportion. Then by the (3:15), $y_n^t = x_n^t$ $(n = 1, \ldots, N-1)$, i.e., all other stocks are unloaded. Then, for example at t=0, (3:5) becomes

$$(3:16) \quad y_N^o p_N^o = \sum_{n=1}^{N} \bar{y}_n p_n^o - \sum_{n=1}^{N-1} x_n^o p_n^o$$

the individual shifts all his resources (apart from those needed for immediate consumption) into stocks of a single good, N.

3. I. We have shown that if all prices are expected to change in different proportions, the stocks of all goods are unloaded in favor of one. If two prices are expected to change in the same proportion, the inequality (3:15) is replaced by an equation and therefore $\nu_n^t \neq 0$ implies $\nu_N^t \neq 0$ and conversely. Hence $y_n^t = x_n^t$ implies $y_N^t = x_N^t$, and conversely. Therefore

$$(3:16') \quad \text{Either } y_n^t = x_n^t,\; y_N^t = x_N^t, \text{ or } y_n^t \neq x_n^t,\; y_N^t \neq x_N^t,$$

i.e., either both stocks are unloaded or both are not unloaded. It follows that if all price ratios are expected to remain unchanged from one marketing instant to the next, then no stock is unloaded (since not all stocks can be unloaded simultaneously: see 3. H.).

3. J. Combining the results derived for equal and unequal relative rates of price changes, we see that if the set of all goods can be split into

two or more subsets, each consisting of goods whose prices change in the same proportion (a subset may have one or more elements) then the stocks in all the subsets but one will be unloaded in favor of the remaining subset. It is conjectured that this subset is the one whose prices are expected to rise in the largest proportion. (This conjecture is derived by listening to the "little voice of the rational man within us." To prove this formally, the maximization of the utility function subject to linear equations and inequalities (3:5), (3:6), (3:7') should be carried through with possibly different tools than the use of auxiliary variables r, s_{in} (3:7)-(3:9)).

3.K. Thus, in the multi-period model defined in 3.B., all resources (apart from those needed for the consumption of a single period) are invested, at any time, in one single commodity, and the shift from one commodity to another depends on slight changes in relative prices. This instability can be regarded as an idealization of the phenomena such as the alternating "flight into commodities" or "flight into money." These terms were coined, I believe, in Germany during the inflation of the 20-ies and the deflation of the 30-ies. These phenomena do not presuppose paper money: "flight into commodities" occurred also in times of "gold inflation."

3.L. If the N-th good is used as a numeraire and is paper money, then

$$(3{:}19)\quad p_N = 1,\quad x_N^t = u_N^t = 0,\quad y_N^T = u_N^T = 0^{*)}$$

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*) One may deny that $y_N^T = u_N^T = 0$, and argue that the man conceives of his final transactions (of instant T) as a "liquidation" of at least a large part of his wealth, his satisfaction is affected by the prospect of possessing, or leaving to his heirs, a sum of money, regardless of its equivalent in consumers' goods. This does not affect the following argument except in detail: the anticipation of possessing a sum of money upon retirement (or of bequesting it) can be regarded as a "consumption flow" of the last period (T-1, T).

Hence (3:13) becomes

$$(3{:}20)\quad p_n^t = (u_n^t + \mu_n^t)\,/\,\mu_N^t,\quad t = 0,\dots,T.$$

where by (3:7), (3:9), (3:12),

$$(3{:}21)\quad \mu_N^t \neq 0\ ,\quad t = 0,\dots,T.$$

This is analogous to (2:15) of the uni-period model: prices of consumption goods can be finite because consumption flows of money are zero. But while in the uni-period model this implied that money stocks (proportional to money flows: (2:6)) must also be zero, the same does not follow for the multi-period model. Here paper money stocks may be positive. For example, suppose $y_N^t > 0$. This implies, according to what was said in 3.G.-3.[illegible], that the price of no consumption good is expected to rise in the next marketing instant, and that the stocks of all consumption goods whose prices are expected to fall are unloaded in favor of stocks of money and of those consumption goods whose prices are expected to remain unchanged. If the price of at least one consumption good is expected to rise, the money stock is "unloaded," that is (since always $x_N^t = 0$), $y_N^t = 0$. If the price of no consumption good is ever expected to remain constant from one marketing instant to another, the planned money stock must jump, throughout the whole time of the plan, between two limits: the lower limit is zero, the upper limit is the money equivalent (at the then prevailing prices) of all stocks that are then available after providing for the consumption needs of the next interval. The planned money stock is zero whenever the price of at least one consumption good is expected to rise. The planned money stock is at its upper limit whenever all prices are expected to fall.

3. M. In (2:16), for the uni-period model, the aggregate of all optimal and of all initial money flows (and therefore stocks) was shown to be zero, as an implication of finite prices of consumption goods. This need not be the case in the multi-period model, unless all individuals desire to rid themselves of money stocks at the same instant 0; this would happen if all individuals expected at that instant that at least one consumption good was going to rise in price. Only then would the aggregate optimal stock of money vanish, and, by (3:) a positive aggregate initial stock would then be impossible. Thus the present model reconciles finite prices with finite aggregate money stocks. But this is based on a rather strong assumption that at least one individual should expect no consumption goods prices to rise.

3. N. In the present model, "money illusion" is rational, that is, each optimal flow and stock is a function of the set of actual and expected absolute prices p_n^t/p_N^t, $t = 0, \ldots, T$, (where $p_N^t = 1$). This is so not only when the numeraire N is a consumption good—as in 3.A.-3.K.; but also when the numeraire is paper money as in 3.L.-3.M., as seen from the reasoning in these sections.

3. O. To sum up Section 3: the assumptions of the multi-period model of this Section are sufficient to explain the existence of positive paper money stocks desired by individuals, and the rationale of "money illusion." But in this model the stocks of money, as well as of consumption goods, fluctuate in a discontinuous fashion under the impact of ever so slight expected price changes; in addition, a rather strong assumption has to be made about the diversity of the individuals' expectations (3.M).

4. Introducing Imperfect Liquidity

4. A. In the multi-period model of Section 3 no perfect knowledge of the future was assumed. However, perfect market was assumed in the following sense: The a-th individual ($a=1,\ldots,A$) considers the prices $p_n^o, p_{na}^1, \ldots, p_{na}^T$ ($n=1,\ldots,N$) as given to him, independent of his actions.

4. B. We now weaken this assumption. Let each price ratio p_{na}^t/p_{ma}^t depend on the size and sign of the transaction, in a manner characteristic of the a-th individual, the time t, and the two commodities n, m involved. The ratio p_{na}^t/p_{ma}^t with which the individual has to reckon is a non-decreasing function of the amount of the commodity n he acquires (from one or more other individuals), and a non-increasing function of the amount of the commodity m he gives up (to one or more other individuals). This is supposed to be true of the marketing instant 0, and the individual is supposed also to expect it to be true of the future marketing instants $1,\ldots, T$.

4. C. On the other hand we introduce the following additional assumption: one of the goods is an universal means of payment, -- i.e., this good is required and accepted in exchange against all other goods. This good is chosen as the numeraire, N.

4. D. Denote the amounts of goods (including the numeraire) given up (acquired), by

$$(4:1) \qquad z_{na}^o = \bar{y}_{na} - y_{na}^o,$$

$$z_{na}^t = y_{na}^{t-1} - x_{na}^{t-1} - y_n^t, \; n = 1,\ldots,N; \; t = 1,\ldots,T$$

(compare (3:5), (3:6)). Then our assumptions in 4.B. and 4.C. can be states thus:

$$(4:2) \qquad p_{na}^t = f_{na}^t\,(z_{na}^t),\; n = 1,\ldots,N,$$

$$(4:3) \qquad f_{Na}^t\,(z_{Na}^t) = 1,$$

where all f_{na}^t are non-decreasing functions, some-times called "selling schedules (curves)." Note that because of assumption 4.C. it was possible to render assumption 4.D. by N equations (4:2).

4. E. We shall assume the functions f_{na}^t continuous and differentiable and define

$$(4:4) \qquad df_{na}^t/dz_{na}^t \equiv g_{na}^t\,(z_{na}^t) \leq 0.$$

In an earlier paper,[*)] the function f^t_{na} was assumed discontinuous, the break

[*)] Marschak [1]

occurring at $z^t_{na} = 0$: a finite difference was assumed to exist between the buying price and the selling price, when the amounts purchased and sold that are considered approach zero. This assumption greatly complicates the mathematical analysis. I conjecture that the essential results of the present paper -- the conditions for existence of positive stocks of money -- are not affected by assuming continuity and differentiability. Observe that the behavior of price atthe point $z^t_{na} = 0$ (zero purchases) has nothing to do with "unloading of stocks" (in which case $y^t_{na} = x^t_{na}$, hence $z^t_{na} = y^{t+1}_{na}$) or with its particular case, the vanishing of money stocks ($y^t_{Na} = 0$), that is the subject of this paper.

4. F. By (4:3)

$$(4:4) \quad g^t_{Na}\,(z^t_{Na}) = 0$$

for every a, t. This can be described as "perfect liquidity" of money. The degree of liquidity of any commodity n, with respect to a given individual, can be identified with the "elasticity of demand to the individual," $\eta = p_{na}/(z_{na}g_{na})$; or the degree of "illiquidity" can be defined identical with the so-called degree of "market imperfection," $1-\frac{1}{\eta}$. Note that the perfect liquidity of money is due to the asymmetrical assumption 4.C. which resulted in (4:2): without this assumption, a strong increase in the ratio p_n/p_N due to an increase of purchases of n against the numeraire N might be interpreted as strong illiquidity of either n or N (or both).

4. G. The givens of the perfect market model of Section 3 were, for the a-th individual: his N initial stocks $\bar{\bar{y}}_{na}$, his utility function u^a, the N prices of zero-instant p^o_n (common to all individuals), and the N(T-1) expected prices $p^1_{na}, \ldots, p^T_{na}$. In the imperfect market model, the set of prices $p^o_n, p^1_{na}, \ldots, p^T_{na}$ is replaced by as many selling schedules $f^o_{na}, \ldots, f^T_{na}$, all varying from individual to individual.

For the market as a whole, the perfect model considered as the givens, the AN initial stocks, the A utility functions, and the ANT expected prices. The imperfect market model considers at the givens, the AN initial stocks, the A utility functions, and the AN(T+1) selling schedules. The latter do not form an independent set. Their forms and parameters are restricted by the condition (3:6') that the market be cleared. This is simply another way of saying that, in the market, one man's strength is another man's weakness. For A < 5, the theory of Games has discussed these restrictions.

4. H. We now maximize utility u^a with respect to the same variables as in Section 3-- x^t_{na} y^t_{na}, y^T_{na}, $t=0,\ldots,T-1$--and, to the new variables p^t_{na}, z^t_{na}; using the same restrictions (3:5), (3:6), (3:7') and the restrictions (4:1), (4:2). Corresponding to (3:10), (3:11), (3:12) we obtain now (dropping again the a-subscript for brevity)

$$(4{:}5)\quad u^t_n + \mu^t_n = \lambda^t (p^t_n + z^t_n g^t_n) = \lambda^{t+1}(p^{t+1}_n + z^{t+1}_n g^{t+1}_n) + \nu^t_n$$

$$(4{:}6)\quad u^T_n + \mu^T_n = \lambda^T (p^T_n + z^T_n g^T_n)$$

$$(4{:}7)\quad \mu^t_n r^t_n = \mu^T_n r^T_n = \nu^T_n s^T_n = 0,\ n = 1,\ldots,N;\ t = 0,\ldots,T-1.$$

Since $p^t_N = 1$, $g^t_N = 0$ for every t, equations (4:5), (4:6) are, for $n = N$,

$$(4{:}8)\quad u^t_N + \mu^t_N = \lambda^t,\ t = 0,\ldots,T;$$

$$(4{:}9)\quad \lambda^{t+1} - \lambda^t = \nu^t_N,\ t = 0,\ldots,T-1.$$

4.I. The perfect market model of Section 3 led to "corner solutions"; the individual was shown to unload all stocks (after provision for one period's consumption) so as to invest all resources in that commodity or commodities that promised the strongest price rise. This was true regardless of whether the numeraire was a consumption good or paper money, except that the unloading of paper money reduces its stock to zero. Thus the optimal money stock of each individual jumped between zero and the money equivalent of all his resources.

We shall now see that imperfect liquidity of goods other than money leads to other, continuous solutions.

4. J. It will suffice to prove that, in the present model, it is possible to have changing and finite prices for all consumption goods without the "unloading" of stocks occurring for any goods, including money. That is, the stocks of every good may exceed current consumption needs (and, hence, the stocks of paper money may be always positive):

$$(4{:}10)\quad y^t_n > x^t_n\ ;\ n = 1,\ldots,N;\ t = 0,\ldots,T-1.$$

This implies by (3:8), (4:7)

$$(4{:}11)\quad \nu^t_n = 0;\ n = 1,\ldots,N;\ t = 0,\ldots,T-1;$$

and therefore by (4:5), (4:8), (4:9)

$$(4{:}12)\quad p^t_n + z^t_n g^t_n = p^{t+1}_n + z^{t+1}_n g^{t+1}_n,\ n = 1,\ldots,N-1;\ t = 0,\ldots,T-1.$$

If all goods were always perfectly liquid, $g^t_n = 0 = g^{t+1}_n$ for every n and t, then (4:12) would imply that all prices are expected to remain constant. This would bring us back to the uni-period model. But if some $g^t_n \neq 0$, (4:12) admits of expectation of changing prices; at the same time, as (4:12) and (4:10) imply for all n, t, expectations of changing prices are consistent with stocks of single commodities that can exceed current consumption needs yet do not absorb all resources of the individual.

4. K. This result can now be applied to paper money. It is defined by

$$u_N^t = u_N^T = x_N^t = 0,\ t = 0,\ldots,T\text{-}1,$$

so that condition (4:10) is replaced by

$$y_n^t > x_n^t,$$

$$y_N^t > 0,\ n = 1,\ldots,N\text{-}1;\ t=0,\ldots,T\text{-}1.$$

Thus stocks of paper money, in a multi-period model with not all prices constant and with some consumption goods imperfectly liquid, can take values other than zero without making it necessary to reduce all other stocks to current consumption needs.

4. L. Using the expressions of old economic literature the results obtained so far can be summarized as follows: It was shown in Section 2 (confirming the finding of Patinkin [1] that the existence of paper money is not explained by its function as a numéraire, a unit of price measurement. In Section 3, the "store of value" function of money was investigated but was not proved sufficient to explain paper money stocks that take values other than 0 or the money equivalent of all resources. In Section 4, money was viewed (in addition to being the numéraire, and to "store value" in expectation of favorable prices) as the universal means of payment ("legal tender"), and all other goods were assumed to have imperfect markets. Under these conditions continuous positive stocks of paper money were explained. Note that it was not necessary to introduce uncertainty; and that production, borrowing and lending have not yet been discussed.