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The Economic Loss Associated With a Non-optimal Situation

by

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I

Let us first define our economic model by

1) the satisfaction functions of the n consumers

$$S^i\ (Ai,\ Bi\ \ldots\ Xi,\ Yi\ \ldots) \qquad i = 1, 2 \ldots n$$

Ai, Bi... are the quantities of goods, A, B,... consumed by the i^{th} individual.

Xi, Yi are the quantities of the different kinds of labor X,Y... supplied by this individual.

2) the production functions of the m industries

$$F_j\ (\mathring{A}^j,\ \mathring{B}^j, \ldots A^j,\ B^j \ldots) = 0$$

$\mathring{A}^j$, $\mathring{B}^j$... are the quantities of goods A, B... consumed by the j^{th} industry,

A^j, B^j... are the quantities of goods A, B,... produced by this industry.

3) the initial quantities $\bar{A}$, $\bar{B}$ which the economy has available.

Let us, secondly, choose as a definition of the optimum, the Paretian one: the situation is optimal if it is impossible to increase one satisfaction without making at least another one decrease.

It is obvious that an infinity of such situations exists and that any one of them can be obtained by choosing arbitrarily the values of $S^2\ S^3 \ldots\ S^n$ and then making S^1 as great as possible.

One recognizes easily a typical problem of maximum under certain constraints: it can be solved by the usual machinery of Lagrangian multipliers and once they have been partially eliminated there remain equations such as

$$i = 1, 2 \ldots n \qquad \frac{S'_{Ai}}{a} = \frac{S'_{Bi}}{b} = \ldots = \frac{-S'_{Xi}}{x} = \frac{-S'_{Yi}}{y} = \ldots$$

$$j = 1, 2 \ldots m \qquad \frac{F'^{j}_{\mathring{A}}}{a} = \frac{F'^{j}_{\mathring{B}}}{b} = \ldots \frac{F'^{j}_{\mathring{X}}}{x} = \frac{F'^{j}_{\mathring{Y}}}{y} = \ldots = \frac{-F'^{j}_{A}}{a} = \frac{-F'^{j}_{B}}{b} = \ldots$$

($a, b \ldots x, y$ are Lagrangian multipliers.)

The outcome is as if a system of prices $a, b, \ldots\ldots x, y, \ldots$ were used and if, at the same time, each individual tried to enjoy the maximum satisfaction possible with his income, each industry tried to produce its output at the smallest possible total cost and sell it at marginal cost, everybody considering prices as data.

A difference (positive or negative) can exist between the expenditures and the labor income of any individual; we call it

$$a\,Ai + b\,Bi + \ldots \qquad = x\,Xi + y\,Yi \ldots + \Delta i$$

The individual looks at this difference as independent of his economic activity.

II

When these conditions are fulfilled the situation is optimum, there is no economic loss; when the situation is not optimum the system incurs an economic loss to which we now try to give a quantitative form.

Let us compare 2 situations (E_o) and (E) and first the satisfactions of the i^{th} individual in these 2 situations:

$$S^i(\mathring{A}i, \mathring{B}i \ldots) \qquad S^i(Ai, Bi \ldots).$$

For this purpose let us define the quantity σ^i_A by

$$S^i(Ai - \sigma^i_A, Bi, \ldots) = S^i(\mathring{A}i, \mathring{B}i, \ldots),$$

thus σ^i_A is the quantity (positive or negative) of the good A that we must subtract from Ai in order to reduce (increase) the satisfaction S^i to its former level.

Though it is illegitimate to add or compare Satisfactions, it is legitimate to add the n σ_A^i since they all are quantities of the same good A dependent only on the indifference surfaces, not on the choice of the satisfaction function,

$$\sigma_A = \sum_{i=1}^{n} \sigma_A^i$$

is the result.

The necessary and sufficient condition for E_o to be optimum is that $\sigma_A \leq 0$ for all situations E that it is possible to achieve from the initial data.

In effect if $\sigma_A > 0$ it is possible from the situation E, without disturbing the productive sector at all, only by taking from or giving to certain individuals certain quantities of the good A, to have all satisfactions equal to the satisfactions in the situation E_o and, at the same time, to be left with a positive quantity of A which can be used to increase any chosen satisfactions.

The second part of the theorem is almost self evident.

This property of σ_A seems to be one of the main requirements of the definition of the economic loss. We will base our study on it.

If E_o is optimum, there exists, at least implicitly, a system of prices which allows us to choose a more satisfactory definition of the loss

$$a \cdot \sigma_A$$

which is now a value, instead of a quantity of the good A.

The choice of A rather than of B or C... is of course arbitrary and one can think of a more elaborate definition such as

$$\frac{1}{p} [a \cdot \sigma_A + b \sigma_B \ldots - x \sigma_X - y \sigma_Y \ldots]$$

p being the number of terms between the brackets.

The two first differentials of the loss have the utmost importance for us, since the use of the classical development $\sigma_A = \delta \sigma_A + \frac{1}{2} \delta^2 \sigma_A + \ldots$ gives an approximation of the 3rd order; they are

$$a.d\,\sigma_A = \sum_B b.d\,B_c$$

B_c is the total quantity of the good B consumed directly by consumers.

If E_o is not optimum, this provides a first order approximation $\sum_B b.\delta B_c$ of the social loss (or gain) associated with an economic change.

If E_o is optimum this expression equals zero and it is necessary to use the second differential which, in this case, equals

$$\frac{1}{2}\,a\,d^2\sigma_A = \frac{1}{2}\sum_B db\,d\,B_c + bd^2\,B_c - \frac{1}{2}\sum_i\left(\sum_B bd\,Bi\right)\sum_B\left(\frac{S'_{Bi}}{S'_{Ai}}\right)_{Ai} d\,B_i$$

This formula is much more simple if one makes the supplementary assumption that in the considered change, $\sum_B b\,d\,Bi = 0$ for all i; then there remains only,

$$\frac{1}{2}\,a\,d^2\sigma_A = \frac{1}{2}\sum_B d\,b\,d\,B_c + b\,d^2\,B_c\ .$$

III

This can be readily applied to the evaluation of the loss associated with a system of taxation.

Let us suppose that each industry has to pay a tax (positive or negative) t_A per unit of good A produced and $\dot{t}_A$ per unit consumed; t_B $\dot{t}_B$ ... are defined in the same way.

The equilibrium equation of the j^{th} industry is now

$$\frac{F'_{\dot{A}}j}{a+\dot{t}_A} = \frac{F'_{\dot{B}}j}{b+\dot{t}_B} = \frac{F'_{\dot{X}}j}{x+\dot{t}_X} = \dots = \frac{-F'_Aj}{a-t_A} = \frac{-F'_Bj}{b-t_B} = \dots$$

It follows from this that,

$$\sum_A d\,a\,d\,Ac + a\,d^2\,Ac = \sum_A d\,\dot{t}_A\,d\,\dot{A} + d\,t_A\,d\,A$$

where $\dot{A} = \sum_j \dot{A}^j$ and $A = \sum_j A^j$.

Using this result one finds the expression of the loss:

$$\frac{1}{2} a\, d^2 \sigma_A = \frac{1}{2} \sum_A d\, \mathring{t}_A\, d\, \mathring{A} + d\, t_A\, d\, A - \frac{1}{2} \sum_i \left(\sum_B b\, d\, B^i\right) \sum_B \left(\frac{S_B^{\prime}{}^i}{S_A{}^i}\right)'_{A^i} d\, B^i$$

and when the supplementary assumption " $\sum_B b\, d\, B^i = 0$ for all i" is added

$$\frac{1}{2} a\, d^2 \sigma_A = \frac{1}{2} \sum_A d\, \mathring{t}_A\, d\, \mathring{A} + d\, t_A\, d\, A$$

In the formula

$$\frac{1}{2} \sum_A \delta\, \mathring{t}_A\, \delta\, \mathring{A} + \delta\, t_A\, \delta\, A$$

$\delta\, \mathring{t}_A$, $\delta\, t_A$ are the taxes per unit of A, $\delta\, \mathring{A}$, $\delta\, A$ are the variations in $\mathring{A}$ and A resulting from the introduction of the ~~taxes~~, $\mathring{A}$ and A being the gross quantities consumed and produced by all the industries.

The supplementary assumption is essential, since it can be shown that when it is dropped the quantity $\frac{1}{2} \sum d\, \mathring{t}_A\, d\, \mathring{A} + d\, t_A\, d\, A$ can be positive, which seems to make it meaningless for the evaluation of a loss.

Moreover, although the terms $\delta\, \mathring{t}_A$, $\delta\, t_A$ ... are immediately known, the terms $\delta\, \mathring{A}$, $\delta\, A$ ... are far from being so.

The simple formula given above appeared ~~several~~ times in the economic literature often founded on a controversial basis, too simple assumptions, or obscure definitions. Professor Hotelling gave in his paper "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates" (Econometrica, July, 1938) a thorough criticism of the former proofs and one of his own which, however, still has some important shortcomings.

IV

After underlining the effective generality of the model we studied, we can indicate some uses of an evaluation of the economic loss.

For the layman who probably cannot understand the subtle nature of this hidden loss a number running in billions of dollars is the way to dramatize the issue; for the economist an order of magnitude at least can be useful.

A more specific issue is raised by industries (railroad transportation, for example) where the policy of sale at marginal cost would lead to a permanent deficit. If a balanced budget is a supplementary condition (various reasons can be imagined) an unescapable loss is incurred and the best policy seems to be to minimize this loss.

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The concept of loss leads easily to intricate theory and the more intricate the theory, the greater the need for a clear view of all the assumptions on which it rests.