Econometric Models: General Concepts and Definitions

When one thinks of science, one usually thinks also of experiments. In a typical experiment there is one variable whose behavior is studied under various conditions. The experimenter fixes at will the values of all the other variables he thinks are important, and observes the one in which he is interested. He then repeats the process, fixing different values of the other variables each time. The variables which he fixes are called independent; the one which he only observes is called dependent. The experimenter hopes to be able to find a single equation which describes closely the relationship he has observed.

In more complicated situations there may be more than just one relationship existing among the variables studied. This is the case when there is a determinate result and the experimenter fixes less than \( n - 1 \) of the \( n \) important variables (there may even be no experimenter). Economics abounds with such situations. The simplest is of course a competitive market, in which neither price nor quantity is fixed by an experimenter. The economist assumes that two relations between these variables, a supply curve and a demand curve, must be simultaneously satisfied.

In econometric work we accept this state of affairs. Accordingly we deal with systems of simultaneous equations, each of which is assumed to describe an economic relation. Such equations are called structural equations.\(^3\)

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3. Structural equations are divided by Koopmans into four classes:
(1) equations of economic behavior such as the consumption function,
(2) technical constraints such as the production function, (3) institutional constraints such as tax schedules or reserve requirements, and (4) definitional identities such as income equals consumption plus investment. Another possible type is made up of market adjustment equations, of which equilibrium conditions are a special case.
Each structural equation is assumed to describe a relation completely except for random shocks; hence each contains a nonobservable random disturbance (with mean assumed to be zero). Errors in measurement of variables are here assumed to be small relative to the disturbances (this kind of model is called a shock model, as distinct from an error model).

We might have $G$ equations in $G+K$ variables. Of the variables, $K$ are fixed in advance, not by an experimenter to be sure, but by society, or by nature, or even by the past operation of the system of equations; they are called predetermined variables. The $G$ variables which are not fixed in advance but are determined by the economic process we seek to describe (analogous to the dependent variable in the simple experimental case at the beginning) are called jointly dependent variables (sometimes they are also called endogenous).

From analysis of our observations of the jointly dependent and predetermined variables (often in several successive time periods) we try to find a system of structural equations, with specific numerical values for all parameters of the equations and also for all parameters of the joint probability distribution of the disturbances. Such a system of equations, including numerical values for all parameters, is called a structure. We hope to find a structure which not only explains past

4. This means we believe that either (1) there are systematic discoverable causes for all the observed variation of the variables, but we are satisfied for the time being if we can explain enough of the variation so that the residual appears random, or (2) there really are random elements in economic affairs. For present purposes we do not care which of these two is the case.

5. More precisely, variables which are statistically independent of the random disturbances in the equations are called exogenous; they may be arbitrarily fixed by some agency, or they may be random. Past values of jointly dependent (i.e., endogenous) variables are called lagged endogenous. Exogenous and lagged endogenous variables together are called predetermined. The remaining variables are called jointly dependent, or endogenous.
observations, but enables us to predict as well.\footnote{6}

Of course there is an infinity of structures which explain any given set of observations. Our problem is then to choose the "best" one. The best one is the one which gives the most accurate predictions of the future, and we cannot know which one this is until afterwards. Therefore if we are to choose now we must do so on the basis of immediately available criteria. Four such criteria suggest themselves:

1) generality
2) simplicity
3) correspondence with our theoretical ideas of what to expect (but if we have a poor theory, this criterion will mislead us).
4) accuracy of explanation of past observations (though we must be careful with this criterion, because it is necessary but not sufficient...remember that it is always possible to fit an \textit{n}th degree polynomial \textbf{exactly} to a set of \textit{n} + 1 pairs of points, and that this very seldom makes for good prediction).

Ideally, we might decide in advance, for each conceivable set of observations, which one of all the possible structures we would choose as most likely to be "best"; if we do this we have a set of a priori admissible structures, and a rule which, given our observations, tells us uniquely what structure to choose.\footnote{7}

For reasons to be explained in a moment, we will use instead the following procedure, which is common to most econometric studies. We indeed have a set of admissible structures and a rule which, given our

\footnote{6. I will not here go into the technical statistical methods which are used in arriving at a structure from a given set of observations. Some reference will be made to them later in the paper when the results of least squares estimation and limited information estimation of our model are compared.}

\footnote{7. This idealised procedure was first suggested to me about a year ago by Milton Friedman.}
observations, tells us uniquely what structure to choose. But we restrict our set of admissible structures so that it forms a model, i.e., so that if literal numbers (α, β, γ, ...) are substituted for the numerical values of the nonzero \( \gamma \) parameters in each admissible structure, all the admissible structures become indistinguishable. \(^9\) In other words, before we make any observations at all we construct a model consisting of \( G \) explicit simultaneous equations in \( G \) endogenous variables, with literal numbers for parameters. We do this in such a way that from statistical analysis of the observations we can get a unique set of numerical values (estimates) for its parameters; this is the technical meaning of the statement that all its parameters are identifiable.

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8. By this is meant parameters not required by assumption to be zero.

9. For example, the following two hypothetical structures belong to the same model:

\[
\begin{align*}
\text{I} & \quad \begin{cases} 
D = -2p + .10Y + 1.5 + u_1 \\
S = 3p - 2.5w - 0.3 + v_1 \\
u_1 \text{ and } v_1 \text{ are normally distributed with means zero, variances 1 and 4, respectively, and covariance 2.}
\end{cases} \\
\text{II} & \quad \begin{cases} 
D = -2.5p + .15Y + 1.2 + u_2 \\
S = 2.5p - 3w - 1.5 + v_2 \\
u_2 \text{ and } v_2 \text{ are normally distributed with means zero, variances 2.5 and 3, respectively, and covariance 1.}
\end{cases}
\end{align*}
\]

This model (and hence any structure belonging to it) can be represented by the following system of equations:

\[
\begin{align*}
D &= \alpha_1 p + \alpha_2 Y + \alpha_0 + u \\
S &= \beta_1 p + \beta_2 w + \beta_0 + v \\
u, v \text{ normally distributed} \\
E(u) = E(v) = 0 \\
E(u^2) = \sigma_{uu} & ; \quad E(v^2) = \sigma_{vv} & ; \quad E(uv) = \sigma_{uv}
\end{align*}
\]

The model will be changed if we restrict any of its parameters to zero, or add any new terms, or change the assumptions about the distribution of \( u \) and \( v \).
This procedure is preferable because of its comparative simplicity... the equations of the model can be set up once for all on the basis of previous knowledge, theoretical or empirical, and then the parameters can be estimated by straightforward statistical processes. Its disadvantage lies in the possibility that the structures one would choose as "best" for different sets of observations might not all be within the same model.\textsuperscript{10} I think that usually the econometrician knows nearly enough what observations to expect so that he can construct a model which does contain the "best" structure for each likely set of observations; if so, the disadvantage is not serious.

In order to simplify computational procedures, further restrictions are placed on the model: it is assumed to be linear in the unknown parameters (though not necessarily in the variables); it deals with aggregates, i.e., macro-variables instead of micro-; its disturbances are assumed to be normally distributed and serially uncorrelated. It is obvious that each of these restrictions makes the model a poorer approximation to the ideal. They will be dropped as soon as statistical theory and computational efficiency permit.

With these restrictions, a model consists of \( G \) simultaneous equations in \( G \) endogenous variables, thus:

\[ \text{10. This again is Milton Friedman's comment.} \]
\[
\sum_{i=1}^{I_G} \alpha_{gi} f^i \left( y_{i1}^i, \ldots, y_{iG}^i; z_{1k}^i, \ldots, z_{ik}^i \right) = u_g
\]

\( g = 1, \ldots, G. \)

\( I_G \) = number of functions \( f^i \) in the \( g \)th equation.

\( \alpha_{gi} \) = unknown parameters, \( g = 1, \ldots, G, i = 1, \ldots, I_G \).

\( y_{ge}^i \) = endogenous variables, \( g = 1, \ldots, G. \)

\( z_{k}^i \) = predetermined variables, \( k = 1, \ldots, K' \).

The \( z_{k}^i \) may include some lagged \( y_{ge}^i \).

\( f^i \) = functions containing no unknown parameters, for example

\( y_1^i y_2^i = s_1^i \). As a special case \( f^i \) might be equal to \( y_1^i \)

for \( i = 1, \ldots, G \) and equal to \( s_{1-G}^i \) for \( i = G + 1, \ldots, G + K' \).

Here \( I = G + K' \). This is the case of a model linear in the \( y_{ge}^i \) variables; we will not restrict our model to this extent.

\( u_g \) = random normal disturbance.

\( E(u_g) = 0 \)

\( E(u_g u_h) = \sigma_{gh} \)

\( E \left( (u_g)_{t} (u_g)_{t-1} \right) = 0 \)

We can rewrite (1) in a more convenient form if we separate all the \( f^i \) into two classes: (1) those which are dependent on some subset of the \( y_{ge}^i \) (whether or not they are also dependent on the \( z_{k}^i \)) and (2) those which are dependent only on the \( z_{k}^i \). We will call the first group \( y_{i1}^i, i=1, \ldots, I \), and the second group \( z_{k}^i, k = 1, \ldots, K \). Then our model becomes

\[
\sum_{i=1}^{I} \beta_{gi} y_{i} + \sum_{k=1}^{K} \gamma_{gk} z_{k} = u_g
\]
\( g = 1, \ldots, g \)

\( \beta_i = \alpha_i \) for all \( i \) such that \( f^i = y_i \)

\( \gamma_k = \alpha_i \) for all \( i \) such that \( f^i = z_k \)

\( y_i \) = endogenous variables (including functions with no unknown parameters).

\( z_k \) = predetermined variables (including functions with no unknown parameters).

This is the form in which we will use the model.

There remains one more general remark before we get down to the business of discussing and testing a specific model. This paper is to be presented at a conference on business cycles, and so I must indicate how the kind of econometric model I have described is suited to the study of business cycles. Very well. Our model contains lagged values of many of its variables, and therefore it is a set of simultaneous difference equations. Whereas the solution of a set of ordinary simultaneous equations is simply a set of numbers, each giving a single value for one of the variables, the solution of a set of difference equations is a set of functions of time, each giving a path in time for one of the variables. Such a time-path gives the future history which a variable of the model would have if future disturbances were zero and future exogenous variables were constant at their present values. It may behave in one of several ways as \( t \) increases indefinitely: \(^{11} \)

1. approach a finite limit smoothly.

2. approach a finite limit by oscillations of diminishing amplitude. \(^{12} \)

3. oscillate indefinitely with constant amplitude. \(^{12} \)

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11. We expect that the solutions of the models we construct for our economy will be of the first, second, or third kind, because we do not observe economic variables "exploding" to infinity in the real world.

12. The oscillations will have a constant period if future disturbances are zero and future exogenous variables are constant; otherwise they will not.
(4) approach infinity (positive or negative) smoothly.

(5) approach infinity (positive and negative) by oscillations of increasing amplitude.  

Therefore our econometric model is capable of generating cyclical fluctuations, and so it is a promising analytical tool for business cycle research.
Statistical Estimation Procedures

We have said we will construct our model (2) so that it contains exactly one a priori admissible structure corresponding to each possible set of observations of the variables $y_1$ and $z_k$, i.e., so that it is identifiable.\textsuperscript{13} This means that within our model there must be one structure, and no more, which is capable of giving rise to any given set of observations.\textsuperscript{14} It is the job of a statistical estimation procedure, given the observations, to find estimates of that structure. We will use two such procedures, least squares, and limited-information maximum-likelihood (which we will call limited information for short).\textsuperscript{15}

Least squares estimates are relatively simple to compute and have been traditionally used. For an equation containing only one dependent variable, they are the maximum-likelihood estimates and therefore have the optimal properties of consistency and efficiency.\textsuperscript{16}

\textsuperscript{13} A model is said to be identified, or identifiable, if all its equations are. A necessary condition for identification of an equation of (2) is that the number of predetermined variables in the other equations of the model but not in that equation is at least as great as the number of jointly dependent variables in that equation minus one. This condition (the order condition) also has a high probability of being sufficient - which is fortunate because the necessary and sufficient condition (rank condition) for identifiability is much more difficult to apply. See Koopmans (9).

\textsuperscript{14} The definition of identifiability which I have given here and on page 4 is not general; it applies only in case all equations are exact and there are no disturbances and no probability distributions. The general definition follows. A model is said to be identified (or identifiable) if, given complete knowledge of the probability distribution of all the observable variables (what we get from our observations is an estimate of this distribution, which for increasingly large samples approaches it), there is only one structure capable of generating this distribution. Thus the idea of identification is something apart from sampling variability; we use it to insure that we choose just one hypothesis (or structure) as a result of our observations. See Koopmans (7, 9).

\textsuperscript{15} The limited information method is due to Anderson and Rubin, (1).

\textsuperscript{16} These optimal properties of maximum-likelihood estimates, of course, exist only if the model used is the correct one, which means that it does contain the best structure in the sense defined on page 3. This note applies equally to the case where there are several jointly dependent variables.
But it has been proved by Haavelmo (4) for an equation containing more than one dependent variable that least squares estimates are not the maximum-likelihood estimates, and that they are inconsistent, and hence biased even for infinite samples.

The justifications advanced for the continued use of the least squares method in cases where it yields biased estimates are: (1) its bias may be small and convergence of the estimates as the sample size increases may be rapid, so that the error may be small for small samples; (2) it requires observations only for the variables in the equation which is to be estimated; (3) it is simple computationally. The validity of (1) is as yet a matter of conjecture.

The limited information method, since it is derived by maximizing the likelihood ratio, yields consistent and efficient estimates. It is a method adapted to the existence of several jointly dependent variables in the equation to be estimated, and hence also to the existence of other equations. The reason is as follows: in the equation to be estimated, it treats all the jointly dependent variables in the same way, considering them all dependent as a group upon the predetermined variables; and it uses observations of predetermined variables which are not in the equation to be estimated but which are in other equations of the system - at least as many of such predetermined variables as are required to satisfy the necessary condition for identifiability given in note 13. Thus while it does require some information and observations concerning equations of the system other than the one to be estimated, it does not require a knowledge of the complete system. For reasons of economy it is often used, however, even when
knowledge of the complete system is available.\textsuperscript{17/}

The limited information method appears preferable because it preserves the simultaneous-equations character of economic theory, rather than distorting it by forcing it into a framework designed for only one dependent variable. But until it is shown in what cases least squares bias in small samples is in fact so large as to make least squares a poorer method than limited information for small samples, it may be just as well to use both methods.

\textit{Klein's Model III}

Klein's model \textsuperscript{17:18/} is the most comprehensive piece of econometric analysis which has been published so far. It has sixteen equations, of which four are definitional identities containing no disturbances and no unknown parameters. Thus there are twelve equations to be estimated.

There are sixteen endogenous variables (in the sense of the $y_g$ in equation (1) on page 6), as follows:

- $C$ consumer expenditures, in 1934 dollars.
- $D_1$ gross construction expenditure for owner-occupied one-family non-farm housing, in 1934 dollars.
- $D_2$ gross construction expenditure for rented non-farm housing, in 1934 dollars.
- $H$ inventories at year end, in 1934 dollars.
- $\text{private I}$ net (nonhousing) investment in plant and equipment, in 1934 dollars.

\textsuperscript{17.} There is also the full-information maximum-likelihood method (see Koopmans (9)), which requires a knowledge of the complete system of equations and observations of all the variables. Because it makes use of more information, it yields estimates with smaller variances than the limited information method. But even if such information is available, the full information method is computationally much more expensive and for this reason has never (to my knowledge) been used for a system of more than three equations.

\textsuperscript{18.} Klein (5, 6).
average corporate bond yield, in percent.

K fixed capital at year end, in 1934 dollars.

\( W_1 \) active balances = demand deposits + currency outside banks, in current dollars.

\( W_2 \) idle balances = time deposits, in current dollars.

p general price level, 1934 = 1.0.

R_1 non-farm rentals, paid or imputed, in current dollars.

r rent index, 1934 = 1.0.

v \% of non-farm housing units occupied at year end.

W_1 private wages and salaries, in current dollars.

X private output (except housing services), in 1934 dollars.

Y disposable income, in 1934 dollars.

There are fourteen exogenous variables (in the sense of the exogenous e_k in equation (1) on page 6), as follows:

D_g gross construction expenditures for farm housing, in 1934 dollars.

D^n depreciation on all housing, in 1934 dollars.

\( \xi \) excise taxes, in current dollars.

\( \xi_R \) excess bank reserves, in current dollars.

\( \Delta F \) thousands of new non-farm families.

G government expenditure (except transfers and net government interest) + net exports + net investment of nonprofit institutions, in 1934 dollars.

N^s millions of non-farm housing units at year end.

q price index of capital goods, 1934 = 1.0.

q_1 construction cost index, 1934 = 1.0.

R_2 farm rentals, paid or imputed, in current dollars.

\( \rho_F \) base-year non-farm rent level, in thousands of base-year dollars per annum.
(exogenous variables, continued)

\[ T = \text{government revenue - not government interest - transfers} \]
\[ + \text{corporate saving}, \]
\[ - \text{not national product - disposable income, in 1934 dollars}. \]
\[ t \quad \text{time, in years}. \]
\[ W_2 \quad \text{government wages and salaries, in current dollars}. \]
The twelve equations and four identities are as follows (they are here grouped as they were by Klein for his limited information estimation, and renumbered by me):

1. Demand for investment
   \[ I = \beta_0 + \beta_1 \frac{\pi - \epsilon}{q} + \beta_2 \left( \frac{\pi - \epsilon}{q} \right)_{-1} + \beta_3 K_{-1} + u_2 \]

2. Demand for inventory
   \[ H = \gamma_o + \gamma_1 (X - \Delta H) + \gamma_2 P + \gamma_3 H_{-1} + u_3 \]

3. Output adjustment
   \[ \Delta X = \mu_0 + \mu_1 (u_3)_{-1} + \mu_2 \Delta p + u_{12} \]

4. Demand for labor
   \[ W_1 = \alpha_0 + \alpha_1 (p^x - \epsilon) + \alpha_2 (p^x - \epsilon)_{-1} + \alpha_3 t + u_1 \]

5. Demand for consumer goods
   \[ C = \delta_0 + \delta_1 Y + \delta_2 t + u_4 \]

6. Demand for owned hsg
   \[ D_1 = \epsilon_0 + \epsilon_1 \frac{F}{X_{-1}} + \epsilon_2 (Y_{-1} + Y_{-2}) + \epsilon_3 \Delta F + u_5 \]

7. Demand for dwelling space
   \[ V = \eta_0 + \eta_1 F + \eta_2 Y + \eta_3 t + \eta_4 \mu^S + u_7 \]

8. Rent adjustment
   \[ \Delta F = \theta_0 + \theta_1 v_{-1} + \theta_2 Y + \theta_3 \frac{1}{r_{-1}} + u_8 \]

9. Demand for rental housing
   \[ D_2 = \lambda_0 + \lambda_1 F_{-1} + \lambda_2 (q_1)_{-1} + \lambda_3 (q_1)_{-2} + \lambda_4 F + \lambda_5 \Delta F_{-1} + u_6 \]

10. Demand for active dollars
    \[ M_1 D = \zeta_0 + \zeta_1 p (Y + T) + \zeta_2 Y + \zeta_3 p (Y + T) t + u_9 \]

11. Demand for idle dollars
    \[ M_2 D = K_0 + K_1 i + K_2 i_{-1} + K_3 (M_2 D)_{-1} + K_4 t + u_{10} \]

12. Interest adjustment
    \[ \Delta i = \lambda_0 + \lambda_1 \varepsilon_R + \lambda_2 \varepsilon_{i-1} + \lambda_3 t + u_{11} \]

13. Definition of net national product
    \[ Y + T = C + I + \Delta H + D_1 + D_2 + D_3 - D^n + G \]

14. Definition of \( X \)
    \[ X = Y + T - \frac{1}{p} (W_2 + R_2 + R_3) \]

15. Definition of \( R_1 \)
    \[ R_1 = \frac{2}{200} \rho_0 r (w^S + \nu_{-1} w^S_{-1}) \]

16. Definition of \( \Delta K \)
    \[ \Delta K = I \]
Equations (1), (2), (3), and (4) are related to the market for goods and services, excluding labor and the construction of housing (these two markets will be treated separately immediately below). (5) Demand for consumers' goods is a linear function of income and trend.

(1) Demand for net investment in plant and equipment is a linear function of (a) present and lagged values of deflated (by capital goods prices) privately produced national-income-at-factor-cost excluding housing, and of (b) the stock of plant and equipment at the beginning of the year. This function is meant to show the dependence of demand for investment upon and (a) anticipated receipts from sales (not of excises) relative to costs, upon (b) existing capital.

(2) Demand for inventory stocks to hold is a linear function of sales, of expected price change (assumed to be given by a linear combination of current and lagged prices), and of the stock of inventories at the end of the year (an inertia factor).

Equation (3) expresses the change in private nonhousing output as a linear function of unintended inventory accumulation (assumed to be measured by \( w_3 \), the lagged disturbance in the demand-for-inventory-stocks equation (4)), and of the rate of change of general prices. It is essentially a supply equation.

Equation (4) gives the demand for labor, measured by the total wage bill, as a linear function of trend, and of current and lagged values of privately produced national-income-at-factor-cost excluding housing (which is supposed to reflect anticipated receipts from sales, net of excises). Observe that this equation could be omitted without impairing the completeness of the model, because the variable \( W_1 \) (wage-bill) does not appear in any other equation; in other words, if this equation were omitted, there would remain a system of 15 equations in 15 variables.
Equations (7) to (11) pertain to the housing market. (7) Demand for owner-occupied one-family non-farm housing construction, which is purchased by consumers, is a linear function of the real value of rents (where the deflator is construction costs), of accumulated cash balances (assumed to be proportional to the sum of incomes over the three most recent years), and of the increase in number of non-farm families.

(10) Demand for rented non-farm housing construction, which is purchased by entrepreneurs, is a linear function of lagged rents, of anticipated prices of housing (assumed to be given by a linear combination of construction costs lagged one and two years), of corporate bond yield, and of lagged increase in number of non-farm families.

Equation (9) describes the determination of the rent level, which occurs in the housing-construction equations (7) and (10), as a linear function of lagged rents, lagged occupancy rate, and income.

Equation (11) describes the change in corporate bond yield, which occurs in the housing construction equation (10) and in the idle-balances equation (17), as a linear function of excess reserves, of lagged interest rate, and of trend. Note that it has only one dependent variable.

(8) Occupancy rate (non-farm), which occurs in the rent adjustment equation (9), is a linear function of rents, of income, of trend, and of the supply of non-farm dwelling units.

Equations (12) to (15) are definitions containing no disturbances. Equation (12) is an identity defining net national product as a sum of demands for consumers' goods, net investment, increase in inventories, housing construction (net), and goods for government use. This sum might be regarded as an aggregate demand; the fact that it is called the definition of net national product indicates that there is implicit in the model an assumption that quantity supplied always equates itself to quantity demanded, except for unintended inventory (see equation (3)).
Equation (13) defines privately produced real output excluding housing services, which appears in equations (1) to (4).

Equation (14) defines non-farm rentals, which appear in the definition of private nonhousing output, equation (15).

Equation (15) defines stock of capital, which appears in the demand-for-investment equation.

Equations (16) and (17) could, like (4), be omitted without impairing the completeness of the model, since the variables \( u_1^D \) and \( u_2^D \) (active and idle balances, respectively) occur in no other equations. (13) Demand for active balances is a nonlinear function of disposable money income and trend. (17) Demand for idle balances is a linear function of current and lagged corporate bond yield, of lagged idle balances, and of trend.

Equations (1), (2), (4), (6), (7), (8), (10), (16), and (17) are demand equations, describing behavior of various economic groups in the population. Equations (3), (9), and (11) are market adjustment equations describing responses of certain market variables to disequilibria. Equations (12) to (15) are identities describing definitional relationships.

Klein's estimates of the parameters of his model, for both least squares and limited information methods, appear in a later section of this paper.19

The results which I am interested in presenting are those flowing from my revision of Klein's model. This revision is based upon a test of Klein's model carried out by Andrew W. Marshall. The next section discusses the test and its findings.

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19. The estimates appearing in Klein (6) have been revised due to the discovery of an error in the time series for \( X \). This revision is reflected in Klein (6) and in the present paper.
Marshall's Test of Klein's Model

The ultimate test of an econometric model, as of any theory, comes with checking up on the predictions it leads to. A model of the form of equations (2) on page 6 is not yet ready to make predictions, even after its parameters have been estimated. It must first be solved algebraically for the jointly dependent variables \( y_i \). This means that each of the \( y_i \) is expressed in terms of the predetermined variables \( z_k \) alone. It will then be said to be in the reduced form and it will look like this:

\[
y_i = \sum_{k=1}^{K} \Pi_{ik} z_k + v_i
\]

where \( i = 1, \ldots, I \).

\( \Pi_{ik} \) = parameters dependent upon the structural parameters \( \beta_{gi} \) and \( \gamma_{gi} \).

\( v_i \) = a random disturbance, equal to a function of the \( \beta_{gi} \) and \( u_{gi} \).

all other symbols have the meaning given on pages 6-7.

Using (18), predictions of \( y_i \) for time period \( t \) may be made simply by substituting the values of the predetermined \( z_k \) for time period \( t \) into the equation for \( y_i \).

20. Since the equations of the reduced form have only one dependent variable each, they are automatically identified, and the least squares method will give unbiased (and consistent and efficient) estimates of their parameters \( \Pi_{ik} \). Thus it appears that if we desire to predict the \( y_i \) we need only estimate the equations of the reduced form, not bothering with the structural equations (2) on page 6 at all. This is true as long as no change in the parameters of the structural equations occurs between the period for which the reduced-form equations (18) are estimated and the period for which predictions are to be made. If such a change of structure does occur, then the reduced-form parameters \( \Pi_{ik} \) will change also. In order to find what they change to, we must know (a) the values of the structural parameters \( \beta_{gi} \) and \( \gamma_{gi} \) before the structural change and (b) the effect upon them of the structural change. Thus for prediction under changes in structure, the reduced form (18) is not sufficient; we must also estimate the structural parameters.
A reasonable test of the model would then consist of constructing, for each equation of (18), a tolerance interval for the calculated value of the disturbance $v_i$, i.e., the difference between the predicted and observed values of $y_i$. If the calculated value of each $v_i$ fell within its tolerance interval, we would feel happy about the model. If very many fell outside, we would not feel happy— we would conclude that the year for which we tried to predict was not drawn from the same population as the years from which we estimated the structure, so that our estimates could not have given us good predictions except by accident.

The trouble with such a test of the reduced form, if prediction is poor, is that it does not tell us which structural equations we should change and which we should keep as they are. This is because each equation of the reduced form depends upon several or even all of the structural equations. It would be desirable to have a test applicable to each structural equation separately.

Marshall (10) has used just such a test. He has constructed tolerance limits for the calculated disturbances of each structural equation (we follow him in calling these calculated disturbances $u^*$. He chooses $\delta = 0.99$ and $p = 0.95$. This means there is a probability of 0.99 that the interval contains at least 0.95 of the population of the calculated disturbances $u^*$.

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21. A tolerance interval is an interval which encloses, with a certain probability $\delta$, at least a certain proportion $p$ of the individuals in a given probability distribution. This, and not a confidence interval, is what we want here: we are interested in predicting a future drawing from our population of years, not in the true mean. Tolerance limits for the normal distribution have been developed by Wald and Wolfowitz (13), and tables have been prepared for constructing them (12). The size of a tolerance interval in a case like this depends on an estimate, using Hotelling's formula, of the variance of the calculated disturbance, i.e., it depends partly on the estimates of the parameters of the equation.

22. "Autonomy" of an equation is the name given to a concept which is useful here. It is not numerically defined, but it corresponds to the degree to which the equation is invariant under possible changes in structure. Structural equations are the most autonomous, since each depends on the structural parameters of only one equation, namely itself. Reduced form equations are the least autonomous. The advantage of autonomous equations is obvious, for prediction under changes of structure (see note 20, above). See Koopmans (9).
The tolerance interval is of the form $\bar{x} \pm ks$, where $\bar{x}$ and $s$ are the mean and standard deviation computed from a sample of $N$, and $k$ is a number depending upon $\bar{x}$, $P$, and $u$. In this case, $\bar{x}$ is $\bar{u}$, which is zero by the construction of the estimates of the structural parameters. For each structural equation, Marshall uses in place of $s$ an estimated approximation to the standard deviation of the calculated disturbance $u^*$, analogous to the Hotelling formula. This approximation, which we will call $s^*$, is given by Rubin (11). For the $g$th structural equation and the year $t$, it looks like this:

$$
(19) \quad s^*_{g,t} = E(u^2) \\
= s^2 + \frac{\sigma^2}{T} + \text{tr} \Lambda \cap + \frac{\sigma^2}{T} z_t^* z_t^* M^{-1} z_t^* z_t^* \\
+ z_t^* W^* W^* W^* W^* z_t^* \\
$$

where

- $s^2 = E(u^2)$
- $T$ = number of years in sample.
- $\Lambda$ = covariance matrix of the estimates of parameters of those endogenous variables $y_i$ appearing in the $g$th structural equation.
- $\cap$ = covariance matrix of disturbances $v_i$ of those reduced-form equations containing endogenous variables $y_i$ which appear in the $g$th structural equation.
- $z_t^*$ = vector of values in year $t$ of all those predetermined variables $z_k$ appearing in the $g$th structural equation, measured from the sample mean.
- $M_{z^* z^*}$ = moment matrix of $z^*$ with $z^*$.
- $z^*$ = vector of residuals of the regression of the $z^*$ (i.e., those predetermined variables $z_k$ appearing in the system as a whole but not in the $g$th structural equation, measured from the sample mean) over the $z^*$; i.e., $z_t^* = z_t^* - M_{z^* z^*} M^{-1} z_t^* z_t^*$.  

\[ \Pi^{**} = \text{matrix of reduced-form parameters of the } z^{**} \text{ in those reduced-form equations containing endogenous variables } y_i \]

which appear in the \( g \)th structural equation.

For each structural equation the values of \( T, z_t^*, \) and \( z_t^{**} \) are known, and estimates are available for \( c^2, \Lambda, \Omega, M_{z^2}, M_{z^2 z^2}, \) and \( \Pi^{**} \) [see Anderson and Rubin (1)]. Thus an estimate of \( c^2 \) is available for each structural equation. We will call this estimate \( s^* \).

The test for year \( t \) then takes the form of constructing a tolerance interval, \( \pm ks^* \), for each structural equation, and rejecting the equation if its calculated disturbance \( u^* \) falls outside the interval. I will call it the \( ks^* \) test.

The advantage of the \( ks^* \) test over a similar test of the reduced form is that it tests each structural equation separately and enables us to tell good ones from bad ones. (In this form it is not strictly a test of predictability, though; it is rather a test of goodness of fit.)

In applying the \( ks^* \) test, Marshall computed \( ks^* \) in five steps: \( ks_1^* \), \( ks_2^* \), \( ks_3^* \), \( ks_4^* \), and \( ks_5^* = ks^* \), corresponding to the first term of the estimate of (19), the first two terms, ..., and all five terms. For each equation he compared each of these successively with \( u^* \), and stopped as soon as he got a region \( \pm ks_1^* \) which enclosed \( u^* \). In this way he saved some computational effort, because he did not have to compute all the terms of \( s^* \) for every equation.

The \( ks^* \) test tells us whether the fit of an equation, for a new year outside the sample, is as good as we could expect on the basis of the estimated past distribution of the calculated residuals. We would also like to know that the equations of our model give a better fit than do certain very naive models (as Marshall calls them). If this is not true, we had better use the naive models instead of our complicated econometrics, or at least revise our
econometric analysis at certain points.

Marshall uses two naive models. One says that any variable \( y_t \) which appears on the left hand side of an equation of Klein's model is equal in year \( t \) to the previous year's value plus a random normal disturbance \( w_t \); the other says that any such variable in year \( t \) is equal to the value in year \( t-1 \), plus the difference between the values in years \( t-1 \) and \( t-2 \), plus a random normal disturbance \( w_t \). He looks to see whether each Klein calculated residual \( u^* \) is within the tolerance intervals \( \overline{w} \pm k_w \) constructed on the basis of the corresponding equation of the naive models, using the same \( \overline{v} \) and \( P \) as before. For both naive models we have

\[
\begin{aligned}
\overline{w} &= \frac{1}{T} \sum_{t=1}^{T} w_t \\
\left( \begin{array}{l}
\overline{w} = \frac{1}{T} \sum_{t=1}^{T} w_t \\
\end{array} \right)
\end{aligned}
\]

\[ s_w^2 = \frac{1}{T-1} \sum_{t=1}^{T} (w_t - \overline{w})^2 \]

and for the first and second naive models we have, respectively,

\[ w_t = y_t - y_{t-1} \]

\[ w_t = y_t - (y_{t-1} + \Delta y_{t-1}) \]

A Klein equation for year \( t \) is rejected by a naive model test if its calculated residual \( u^* \) falls outside the tolerance interval \( \overline{w} \pm k_w \) for the naive model.

Marshall's results for these tests are given in Table 1 below.

24. These have also been suggested by Hilton Friedman.

25. Marshall realises that the naive model tests, unlike the \( k_s^* \) test, are not independent of the choice of variable to put on the left side of the equation and to normalize the equation on. It was merely for convenience that he chose as he did.
Table 1. Marshall's Results of Testing Klein's
Model IIIa, b, c.

<table>
<thead>
<tr>
<th>Eq.</th>
<th>Var. at left</th>
<th>Yr.</th>
<th>u*</th>
<th>$k_s^*$</th>
<th>$k_s^*$</th>
<th>$k_s^*$</th>
<th>Naive Model I Tolerance Limits</th>
<th>Results</th>
</tr>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>k* test</td>
</tr>
<tr>
<td>1</td>
<td>I</td>
<td>46</td>
<td>-5.6</td>
<td>2.0</td>
<td>3.4</td>
<td>8.0</td>
<td>(-5.2, 3.3)</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>47</td>
<td>-2.3</td>
<td>2.9</td>
<td>3.6</td>
<td></td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>H</td>
<td>46</td>
<td>-7</td>
<td>2.6</td>
<td></td>
<td></td>
<td>(-3.8, 4.6)</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>47</td>
<td>-1.9</td>
<td></td>
<td></td>
<td></td>
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<td>A</td>
</tr>
<tr>
<td>3</td>
<td>$\Delta x$</td>
<td>46</td>
<td>-81.0</td>
<td></td>
<td></td>
<td></td>
<td>(-17.8, 20.8)</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>47</td>
<td>-37.9</td>
<td></td>
<td></td>
<td></td>
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<td>A</td>
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<td>4</td>
<td>$V_1$</td>
<td>46</td>
<td>7.2</td>
<td>4.4</td>
<td>9.2</td>
<td>15.9</td>
<td>(-15.2, 16.6)</td>
<td>A</td>
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<td></td>
<td>47</td>
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<td>8.7</td>
<td>8.9</td>
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<td>R</td>
<td>A</td>
</tr>
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<td>C</td>
<td>46</td>
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<td></td>
<td></td>
<td></td>
<td>(-7.2, 10.2)</td>
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<td>14.0</td>
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<td>-3</td>
<td>.9</td>
<td>.9</td>
<td></td>
<td>(-7, 8)</td>
<td>A</td>
</tr>
<tr>
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<td>47</td>
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<td>1.3</td>
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<td></td>
<td>R</td>
<td>A</td>
</tr>
<tr>
<td>8</td>
<td>$v$</td>
<td>46</td>
<td>1.5</td>
<td>3.6</td>
<td>4.5</td>
<td></td>
<td>(-5.9, 2.0)</td>
<td>A</td>
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<td></td>
<td></td>
<td>47</td>
<td>4.6</td>
<td>4.4</td>
<td>8.1</td>
<td></td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>9</td>
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<td>46</td>
<td>-1</td>
<td>.1</td>
<td></td>
<td></td>
<td>(-2, 2)</td>
<td>A</td>
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<td></td>
<td></td>
<td></td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>10</td>
<td>$D_2$</td>
<td>46</td>
<td>-6</td>
<td>1.2</td>
<td></td>
<td></td>
<td>(-1.0, 1.0)</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>47</td>
<td>.5</td>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>11</td>
<td>$\Delta i$</td>
<td>46</td>
<td>-8</td>
<td>2.1</td>
<td></td>
<td></td>
<td>(-2.0, 2.0)</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>47</td>
<td>-6</td>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>16</td>
<td>$M_1$</td>
<td>46</td>
<td>1.9</td>
<td>5.3</td>
<td>6.4</td>
<td></td>
<td>(-6.3, 8.3)</td>
<td>A</td>
</tr>
<tr>
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<td>47</td>
<td>-3.0</td>
<td>6.5</td>
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</tr>
<tr>
<td>17</td>
<td>$M_2$</td>
<td>46</td>
<td>12.2</td>
<td></td>
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<td></td>
<td>(-5.4, 4.5)</td>
<td>A</td>
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<td>47</td>
<td>12.0</td>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>A</td>
</tr>
</tbody>
</table>

a See Marshall (10).

b In the last two columns, A and R mean "accept" and "reject" respectively.
I have not shown the tolerance limits for naive model II
because they are close to those for naive model I and in every
case lead to the same decision concerning acceptance or rejection.

Less than .05 in absolute value, and negative.

The $L^2$ test was not applied to equations already rejected for
both years by the naive model tests.

The table shows that equations (3), (6), and (17) are completely rejected;
equation (16) is on serious probation for having two rejects out of four;
equations (1), (4), (7), and (8) are on milder probation for having one reject
out of four; and equations (2), (9), (10), and (11) have a clear record so far.

We have seen that three out of twelve equations were rejected for both
years by the naive model tests, and three more were rejected for one year. This,
loosely speaking, is a record of fifteen successes out of twenty-four. But if
we compare the actual sizes of residuals $v$ of the naive models (not shown here)
with $u^*$ for each of the twenty-four cases, we find the following:

- $u^*$ exceeds both naive-model residuals: 15 cases
- $u^*$ between the naive-model residuals: 6 cases
- $u^*$ under both naive-model residuals: 3 cases

This paints a dimmer picture of the usefulness of Klein's model for prediction
purposes. But I believe we should judge an equation by comparing it with the
results that naive models can be expected to yield in a large number of cases,
rather than with the results of just one or two cases. Therefore, I believe
the naive-model tests based on tolerance limits are more like what we want to use.
Revisions of Klein's Equations

In this section I will present several equations designed to replace those of Klein's which fared badly in Marshall's tests.

The tests, as shown in Table I on page 23, rejected three equations: (3) output adjustment, (6) demand for consumption, and (17) demand for idle cash balances.

The ks test alone cast some doubt upon two additional equations: (4) demand for labor and (16) demand for active cash balances.

The above five equations are the ones to be revised or changed here. It is permissible to change the number of variables and equations in the model if this is theoretically justified, but the number of equations must not exceed the number of endogenous variables, and the two must be the same if the system is to be complete.

Consider first the demand for money equations, (16) and (17). The function of these equations is to determine two variables, \( M_1^D \) and \( M_2^D \) (active and idle money balances, respectively), which are purely symptomatic in the model. Since they enter into no other equations, \( M_1^D \) and \( M_2^D \) cannot mathematically affect the model but can only be affected by it. We have no interest in the quantity of money per se if it has no effect. Therefore we will drop equations (16) and (17), together with the variables \( M_1^D \) and \( M_2^D \). This cuts the number of equations by two, but still leaves a complete model.

---

26. I believe that a good economic theory will not say that the quantity of money is merely a symptom having no effect upon economic affairs. Accordingly, Klein's theory is amended below, at least with respect to the consumption function. It is only the lack of time and thought which prevented further changes involving the quantity of money in other parts of the model, and dictated the dropping of (16) and (17) rather than their revision.
Consider the consumption function next. Klein's equation (6) underestimated consumption in 1943-47 by some 13 to 14 billions of 1934 dollars, or about 15 to 16% (see Marshall (10)). At the same time the real value of the stock of money at the beginning of the year was 110 to 105 billions, approximately twice the largest value it attained during 1921-39. For the interwar years no significant effect of real cash balances has been found (Klein (5a)), but this was to be expected because real balances were almost constant during that period except for a smooth trend, and so their effect if any could not be discovered. The postwar data suggest that real balances may have been important in the consumption function all along. The skewness of the distribution of real balances might also be important; we might expect to find that an increase in the holdings of richer people would stimulate consumption less than an equal aggregate increase in the holdings of poorer people. Time series data are not available here, however.

The proper definition of cash balances for this purpose is total consumer holdings of currency, demand deposits, time deposits, and probably also series E U. S. savings bonds as long as they are guaranteed to be immediately redeemable in cash at no loss. Holdings by individuals and unincorporated businesses might be a good approximation, but suitable figures do not exist as far as I know, especially if series E bonds are included. Therefore the above definition seems best.

27. Defined as the sum of currency outside banks plus demand deposits adjusted plus time deposits, but not including government deposits, deflated by the 1934-base price index of output as a whole.
Lagged disposable income has often been mentioned as a candidate for membership in the consumption function. It is recommended by the fact that people do not adjust themselves right away to changes in income.\textsuperscript{28}

Accordingly we shall experiment with fitting the following consumption functions:

\begin{align*}
(6.0) \quad C &= \int Y_{-1} + \int \frac{M}{p} + \int t + u_6 \\
(6.1) \quad C &= \int + \int Y_{-1} + \int t + u_6' \\
(6.2) \quad C &= \int + \int \frac{M}{p}_{-1} + \int t + u_6'' \\
(6.3) \quad C &= \int + \int t + u_6'''
\end{align*}

where $M =$ currency outside banks + demand deposits adjusted + time deposits, at the end of the year.

other symbols are as defined on pages 11-13.

Observe that \((M/p)_{-1}\) is a predetermined variable since it is lagged (the same is of course true of \(Y_{-1}\)). Thus we have added no new endogenous variables to the system by these modifications of the consumption function.

There remain two equations, (3) output adjustment (which is really a supply function, as mentioned before) and (4) demand for labor. They are closely related theoretically, because under the assumption of profit maximizing, the firm's demand for factor equations are deducible from the profit function and the production function, and then the supply function is deducible in turn.

\textsuperscript{28} Lagged consumption has also been suggested on the same ground. I have not tried it because it can be expected to have very nearly the same influence as lagged income.
from these demand for factors equations and the production function. Equivalently, if the demand for factor equations and the supply equation are given, then the production function is determined. Thus if we are concerned only with the logical completeness of the model, it does not matter whether it is the production function or the supply function which we include, provided the demand for factor equations are present. We choose here to use a production function, because Klein's output adjustment equation is so far off (overestimating output by 61 and 38 billions of 1934 dollars in 1946 and 1947, respectively) and because the production function is more autonomous (see note 22, page 19).

29. Given competitive conditions, a production function
(1) \( y_i = f_1(y_1, \ldots, y_n) \)
and a profit function
(2) \( \Pi = px - \sum_{i=1}^{n} p_i q_i y_i \)
where \( x \) is output and \( p \) is its price, \( y_i \) is a factor of production and \( q_i \) its price, \( i=1, \ldots, n \), and \( \Pi \) is profit. Then the firm maximizes (2) with respect to the \( y_i \), subject to the restraint (1), to get

(3) \( \frac{\partial \Pi}{\partial y_i} - p = 0, \quad i=1, \ldots, n \).

If the set of simultaneous equations (1) is solved for the \( y_i, i=1, \ldots, n \), the result is the demand-for-factor equations

(4) \( y_i = f_1 \left( \frac{q_1}{p}, \ldots, \frac{q_n}{p} \right), \quad i=1, \ldots, n \).

The supply equation is obtained by substituting \( y_i \) from (4) into (1), \( i=1, \ldots, n \). Results are similar in the noncompetitive case, but elasticities of product demand and factor supply enter in then.

30. It is interesting to note that under assumption of profit-maximizing, with a profit function like (2) in note 29, the production function for the firm can be derived from given demand-for-factor equations, uniquely except for a boundary condition such as \( f(0, \ldots, 0) = 0 \), even with no knowledge of the supply function, thus: because the profit function contains each \( y, \) only once, and then as a product with its \( q_i \), it is possible to pass uniquely from (4) to

(5) \( \frac{q_i}{p} = \frac{\partial f_i}{\partial y_i}, \quad i=1, \ldots, n \)
which can be integrated to obtain \( f \) uniquely except for a constant term (subject to certain integrability conditions which in our case are satisfied), \( y, \) E. D. This proof is due to Koopmans.

This system is not likely to be made overdetermined by the inclusion of a production function (or alternatively a supply function), however, since an additional variable \( x \) is brought along at the same time.
Variables must be chosen to represent capital input and labor input in the production function. Capital input can be measured by depreciation charges, and this would be ideal if depreciation were really able accurately to reflect the services of capital. But depreciation is a very arbitrary thing, subject to various legal and accounting pressures, so that in practice it is not a satisfactory measure of capital use. Another possible measure is the stock of producers' capital at the beginning of the year, defined as the sum over time of not investment. This is not free from the effects of the arbitrariness of depreciation charges but it is less sensitive to them because stock of capital is so large in relation to depreciation charges for any one year. It also measures capital existing, not capital in use, which is unfortunate, but we will try it anyway, perhaps together with some device for indicating the extent to which available capacity is being used.

Labor input, which might also appear in the demand for labor equation, should ideally be measured in man-hours. But data difficulties deter us here; the BLS series for average weekly working hours previous to 1932 is for manufacturing and railroads only, and does not cover all industries even now. The concept of full-time-equivalent persons engaged in production, used by S. Kuznets and the Department of Commerce, is the next best thing. However, it does not regard overtime work as an increase of labor input: it measures roughly the

31. It is private labor input which concerns us here, by the way, and not total, because it is only in the private sector that production is assumed to be guided by the desire for profit.
number of persons engaged full time or more (where full time for any person means simply the current customary work week in his job, whether it is 39 hours or 55), plus an appropriate fraction of the number of persons engaged part-time (to convert them to full-time equivalents). A time trend term will then approximately take care of the secular decrease in weekly working hours which has occurred.

We might choose any one of several forms for the production function. The Cobb-Douglas function, linear in logarithms, is one possibility. A simple linear function might also be used as an approximation. Investment during the current year might be included on the theory that new capital is more productive than old capital even after depreciation has been deducted, because of improvements in the design of equipment.

In attempting to make the production function reflect the fact that output can be increased if existing capital is used more intensively, we might break our sample into two samples -- one containing in boom years in which capital was being used at approximately full capacity, and the other containing slack years in which it was not -- and then fit two production functions, one to each. The sign of net investment could be used as a crude indicator for classifying the years: in boom years one would expect demand for capital services to exceed exist-

32 Cyclical fluctuations in weekly working hours will be an important source of error here if their effect is not largely explainable by cyclical changes in full-time equivalent persons engaged plus a time trend, i.e., if data on weekly working hours (which we do not have), full-time-equivalent persons engaged, and time trend are not approximately linearly related.
ing supply and so to stimulate an increase in the stock of capital so that net investment would be positive, and in slack years the opposite. This scheme is disagreeable because it sets up a dichotomy where there should be a continuum, and because it reduces the already too small sample size. An alternative is to make each parameter of the production function a linear function of net investment, thus: 33, 34

\[(3.0) \quad X = (\mu_0 + \mu_1 I) + (\mu_2 + \mu_3 I)N + (\mu_4 + \mu_5 I)K_{-1} + \mu_6 t + u_3 \]

Where \( N \) = private labor input, in millions of full-time-equiva-

\[\text{least man-years (endogenous). Other variables are defined on pages 11 - 13.} \]

We would expect \( \mu_2 \) to be positive; a large positive net investment I can be presumed to indicate that capital is being used at a high per-

\[\text{centage of capacity, and existing capital } K_{-1} \text{ can be expected to con-}\]

\[\text{tribute more to output then than otherwise, so that its coefficient}\]

\[(\mu_4 + \mu_5 I) \text{ then should be high. We might expect } \mu_3 \text{ to be negative}\]

\[\text{because the marginal product of labor is probably less in boom times}\]

\[\text{than otherwise. Of course we expect } \mu_2, \mu_4, \text{ and } \mu_6 \text{ to be positive}\]

\[\text{(though } \mu_6 \text{ would be negative if the abovementioned secular drop in}\]

\[\text{working hours were enough to overbalance the increase in per-man-hour}\]

33. I am indebted for this suggestion to discussions with Jacob Marschak.

34. Such a production function suffers in autonomy (page 19, note 22), because it is now dependent upon the assumption that net investment occurs in response to near-capacity use of existing capital. If some-

\[\text{thing happens so that this is no longer true, then the production function}\]

\[\text{changes. But nothing of this sort is likely to happen unless the profit-}\]

\[\text{maximizing assumption becomes invalid, in which case several other equa-}\]

\[\text{tions will go by the boards too.} \]
productivity). We have no presumptions about $\mu_0$ and $\mu_1$, except that $\mu_1$ should probably be positive not very large.

Besides (3.0), we try the following less complex but more autonomous production functions:

\begin{align*}
(3.1) \quad X &= \mu_0^{'} + \mu_1^{'} I + \mu_2^{'} N + \mu_3^{'} K - 1 + \mu_4^{'} t + u_3^{'} \\
(3.2) \quad X &= \mu_0^{''} + \mu_1^{''} N + \mu_2^{''} K - 1 + \mu_3^{''} t + u_3^{''} \\
(3.3) \quad \log X &= \mu_0^{''''} + \mu_1^{''''} \log N + \mu_2^{''''} \log K - 1 + \mu_3^{''''} t + u_3^{''''} \\
(3.4) \quad X &= \mu_0^{IV} + \mu_1^{IV} N + \mu_2^{IV} t + u_3^{IV}
\end{align*}

Here we would expect all parameters (except the $\mu_0$'s) to be positive.

Observe that by replacing Klein's output adjustment equation with any of the production functions (3.0) to (3.4), we will have added a new endogenous variable, $N$. Before we finish our revisions we must therefore find a corresponding additional equation, if we are to end with a complete system.

Now that wage-salary bill and labor input are both in the model, it is natural also to include wage rate:

\begin{align*}
(23) \quad w &= \frac{\text{w}}{N} \\
\end{align*}

where $w =$ private wage-salary rate, in thousands of current dollars per full-time-equivalent man-year (endogenous).
\( w \) and \( N \) are defined on pages 12 and 31 respectively. We have here one new equation and another new endogenous variable, \( w \), so we still need to find an additional equation.

If the wage rate is to mean merely total labor earnings per unit of labor input, then the definition is satisfactory in the simple form \((25)\). However if it is to mean hourly wage, the thing over which workers and employers bargain, allowance must then be made for overtime payments, premiums for night-shift work, etc.; furthermore labor input must be measured in man-hours. The advantage of using hourly wage is that it enables us to introduce an equation describing the bargaining process and its dependence upon price movements, level of employment, and any other relevant variables. This wage adjustment equation could also serve as the additional one required by the introduction of the two new endogenous variables, \( w \) and \( N \), with only the one equation \((23)\). But existing data do not permit us to incorporate overtime payments and premiums for shift-work into the wage rate, nor (as we have seen) to define labor input in man-hours.

Accordingly we will retain \((23)\) as it stands. We will assume that our \( w \) is closely representative of hourly wage, \( \text{35} \) and we will use a

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35. A study in the Monthly Labor Review for November, 1942, pp. 1053-56, shows the estimated average number of overtime hours per worker per week in manufacturing in 1942, as a function of average total hours per worker per week. Using this study and the B.L.S. series for average weekly hours in manufacturing, and assuming that the 1942 study is good in all years and that all time over forty hours is paid at time and a half, one concludes that if overtime pay had been the only cause of the difference between our \( w \) and the straight-time hourly wage, this difference would have been less than 2% in all interwar years and less than 3% in 1946 and 1947. Thus we are not risking more than about 5% from this cause. Shift premiums probably do not contribute a larger error than this. And we are made more comfortable if we remember that the manufacturing industries probably had more extensive shift premiums and more complete observance of the time-and-a-half-overtime rule than did the economy as a whole.
wage adjustment equation like one of these:

\[
(5.0) \quad w = \kappa_0 + \kappa_1 \Delta p + \kappa_2 (N_L - N) + \kappa_3 w_{-1} + \kappa_4 (N_L - N)_{-1} + \kappa_5 t + u_5
\]

\[
(5.1) \quad w = \kappa'_0 + \kappa'_1 \Delta p + \kappa'_2 (N_L - N) + \kappa'_3 w_{-1} + \kappa'_4 t + u'_5
\]

Where \( N_L \) = labor force, including work relief employees but excluding other government employees,\(^36\) in full-time-equivalent man-years (exogenous).\(^37\) Other variables are previously defined.

These wage adjustment equations tell us that the wage level depends upon past wage level (reflecting the downward rigidity of wages), upon price change (reflecting wage increases following increases in the cost of living and in the prices received by employers), upon unemployment (reflecting the state of the labor market), and upon trend (reflecting the growing strength of unions?).

By the addition of a wage adjustment equation, we have completed our system again.

The demand for labor equation (4) is still to be considered. This equation was put on probation, not completely rejected. Therefore we

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\(^{36}\) \( N_L - N \) is meant to measure employment including relief workers, and \( N \) excludes government workers. Therefore if \( N_L - N \) is to measure correctly, \( N_L \) must exclude government non-relief workers.

\(^{37}\) Labor force is the only measure we have for labor supply, and it is expressible in man-years, but not in man-hours except by some trick assumption. Hence unemployment and employment, which add up to labor force, must be in man-years instead of man-hours too. Hence there appears another advantage in defining labor input \( N \) in man-years as we have done.
will try it again, but we will also try two alternatives which express the demand for labor in real terms as a function of real wage rate, of real output, and possibly of trend. Real wage appears as a result of the profit-maximizing assumption. Output appears on the theory that if producers receive more orders they will demand more labor even if the real wage does not fall. Trend appears in order to reflect the long-period rise in per-man-hour productivity. Our alternative equations are:

\[(4.0) \quad w_1 = \alpha_o + \alpha_1 (px - \varepsilon) + \alpha_2 (px - \varepsilon)_{-1} + \alpha_3 t + u_4\]

\[(4.1) \quad n = \alpha_o + \alpha_1 \frac{w}{p} + \alpha_2 x + \alpha_3 t + u_4\]

\[(4.2) \quad n = \alpha_o + \alpha_1 \frac{w}{p} + \alpha_2 x + u_4\]

Klein's equation (4.0) is not as different from the others as it looks at first: if we divide (4.0) through by \(w\) we get

\[\frac{w_1}{w} = \frac{n}{w} = \frac{\alpha_o}{w} + \frac{\alpha_1 x}{w/p} - \frac{\alpha_1 \varepsilon}{w} + \ldots\]

which also depends on real wage \(w/p\) and on real output \(x\), though there is only one parameter, \(\alpha_1\), to take care of both of them, and there are other terms involving \(w\) and \(\varepsilon\).

Whether we finally choose (4.0) or (4.1) or (4.2), we have still a complete system of sixteen equations (including five definitional identities) in sixteen endogenous variables. We have discarded two of Klein's variables, \(w_1^P\) and \(w_2^P\), and added two new ones, \(n\) and \(w_3\).

Computational results are presented in the next section.

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38. It should be acknowledged that Klein (6) at least hints at most of the changes made in this section, and even includes some exploratory computations on some of them.
Computational Results

Klein has estimated the equations of his model (page 14) by the least squares and limited information methods; the estimates are given in Klein (6). Certain of these equations have been rejected by Marshall's test, on the basis of Klein's limited information estimates and the data for 1946 and 1947; the results are given on pages 23 and 24 above. The rejected equations have been revised, replaced, or eliminated; these changes are described on pages 25-35.

The estimates presented in this paper are for the unrejected Klein equations, and for the new or revised equations of the previous section. They are based in each case on a sample consisting of the years used by Klein for his limited information estimates, with the addition of the two years 1946 and 1947. These years were added in order to bring the estimates up to date and give the model a fairer chance to do a good job of describing 1948; the war years 1942-45 were omitted because ordinary economic relationships were set aside in favor of direct government controls during that period.

All the unrejected and new equations are estimated by least squares, and the estimated standard errors of the disturbances and of the estimates are computed. Then one form is chosen from among the theoretically acceptable alternative forms of each equation (e.g. one production function from among equations (3.0) to (3.4), etc.), and estimated by the limited information method. In addition for each equation which is estimated by the limited

39. This means that my sample is 1922-41 plus 1946-47 for all equations except (10.0) and (11.0), for which it is 1921-41 plus 1946-47.

40. The choice is made partly on theoretical grounds (but not wholly, or else it could be made before any empirical work is done), and partly on the basis of the least squares estimates. There is a presumption that if an equation fits well by least squares (i.e. if its residuals and the estimated standard
information method, estimates are prepared for the standard error of the disturbance, for the covariances of the estimates of the parameters, for the successive values of $k_s$ required for the $k_s$ test, and for $S^2/\sigma^2$ which is used in testing for serial correlation of disturbances. The calculated disturbances for 1948 have been computed for Klein's limited information estimates, for my least squares estimates, and for my limited information estimates. The $k_s$ test has been applied to the latter.

The results of all of these computations appear in Table 2 below.

Table 2 is really the heart of this paper.

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40. (continued) errors of the estimates of its parameters are small), then there is likely to be a relation among its variables which can be consistently estimated by the limited information method. I realize that this procedure is not satisfactory to the uncompromising advocate of consistency in estimation. Ideally all the alternative forms of each equation should be estimated by the limited information method, but this is an expensive process, so the least squares estimates are used as a kind of screening device. How misleading they can be is shown in the cases of equations (10) and (4.0), which will be discussed below.

41. For each equation, $S^2$ is the mean square successive difference of the disturbances $u$, estimated by

$$S^2 = \frac{1}{T - F - 1} \sum_{t=2}^{T} (u_t - u_{t-1})^2$$

and $\sigma^2$ is the variance of the disturbances $u$, estimated by

$$\sigma^2 = \frac{1}{T - F - 1} \sum_{t=1}^{T} u_t^2$$

where $T$ is the sample size and $F$ is the number of parameters to be estimated in the equation. The question of the proper number of degrees of freedom is not solved, so we use $T - F$ arbitrarily, as if we were dealing with least squares.

The distribution of $S^2/\sigma^2$ has been tabulated in Hart and von Neumann (43).
Table 2. Results of Computations on revised and Unrevised Equations (a)

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Notes to Table 2.

(a) The equations are numbered here just as they are in the text, except that equations (1), (2), (4), (7), (8), (9), (10) and (11) in the text appear here as (1.0), (2.0), ..., (11.0).

The variables occurring in each equation are listed in the same row with the equation number, endogenous variables first and predetermined variables next.

The units in which each variable is measured are given in Appendix B, except that $c_R$ is converted to billions of current dollars, so that all quantities whose dimensions are current or 1934 dollars are measured in billions.

KL1 and KL1 refer to Klein's estimates by the least squares and limited information methods, respectively, based on a sample period ending in 1941, and found in Klein (6). The sample was 1921-41 for all KL1 equations except (1.0) and (9.0); and for KL1 equations (10.0) and (11.0); the sample was 1922-41 for all other KL1 equations.

CLS and CLI refer to Christ's estimates by the least squares and limited information methods, respectively, based on the KLI sample plus the years 1946 and 1947. (Thus the sample was 1921-41 and 1946-47 for CLS and CLI equations (10.0) and (11.0), and 1922-41 and 1946-47 for all other CLS and CLI equations).

Relatively few significant figures are given for estimates of parameters attached to variables having small numerical values, in the interest of not wasting effort in accurate computation of small quantities which will be added to larger and less accurate ones.

(b) The numbers in parentheses in columns (1) to (7) are estimates of the standard errors of the estimates of the parameters. The numbers not in parentheses are the estimates of the parameters. They are arranged in such a way that any equation may be read off directly in the form in which it is given in the text. For example the CLI estimate of the consumption equation (6.2) is seen to be $C = .543Y + .315 \left(\frac{N}{\bar{N}}\right) - .27t + 8.56$.

(c) Column (10) gives the observed 1948 value of the variable appearing on the left side of each equation, i.e. the variable in column (1) of the table. Column (11) gives the variable of the linear combination on the right side of each equation. Column (12) gives the difference (10) - (11), which is the calculated disturbance. If this is positive, the equation has underestimated the variable on its left side. (10) - (11) here may not equal (12) exactly because of rounding.

(d) $A$ and $B$ mean accept and reject, respectively.

(e) The constant term in the CLS estimate of equation (3.0) is -84.76.

(f) CLS* and CLS** under equation (3.2) refer to some special exploratory computations based on different series for K-1. They are discussed in the text.

(g) The LS and LI estimates of equation (11.0) are identical since it has only one dependent variable.