Some Elementary Properties of Convex Polyhedral Cones

by

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August 17, 1949

In a paper by T. C. Koopmans (A Mathematical Model of Production, LPC: 410) a number of questions are raised about certain elementary properties of convex polyhedral cones, a discussion of which seems to be of economic interest. Abstracted from context these questions may seem disconnected. This paper, therefore, is not intended to contain a logical development of a theory of convex polyhedral cones, but only to discuss certain specific points as they appear necessary.

The cones treated here will lie in some N-dimensional Euclidean space E^N. A half-line through the origin O may be specified by giving any point other than O which lies upon it. This point, and hence the half-line, may be represented as a column vector with N components. If every component of the vector be multiplied by the same positive real number, the half line which this vector represents is unchanged. When, therefore, a column vector is used to represent a half-line, that vector may be considered as not different from any of its positive multiples. Likewise, the vector may be considered as determined uniquely if determined up to a positive multiple.

When a finite number of half-lines through the origin are given in E^N, their convex hull is a polyhedral cone. This cone may be represented by a matrix by representing each half-line by a column vector and setting the vectors side by side. Conversely, any N-rowed matrix corresponds to a cone in E^N, the cone being the convex hull of the half-lines represented by its columns. If A is the matrix, the cone is the set of all points y such that $y = Ax$ with $x \geqq 0$ (i.e., all components of x non-negative) and will be denoted by (A). When we have occasion to consider the set of all points y such that $y = Ax$ with $x > 0$ (i.e., all components of x strictly positive) it will be denoted by)A(.

If some column of A is a positive linear combination of the others, (A) and the cone determined by A with that column deleted are identical. Let the columns of A be ordered arbitrarily. Delete the first column if it is a positive linear combination of any of the remaining ones, and proceed so, each time deleting a column if it is a positive linear combination of those that remain. In this way we arrive at a new matrix, called a frame of A and denoted A_F such that $(A) = (A_F)$. No column of A_F is a positive linear combination of any of the others. In general A_F depends on the manner in which the columns of A are ordered. Suppose, however, that (A) lies (except for the origin) entirely on one side of a hyperplane through the origin. Both A and (A) are then called pointed, and in this case one may assert the following:

<u>Theorem 1</u>: the frame of a pointed matrix is unique.

Proof: That (A) lies entirely on one side of a hyperplane through the origin (i.e., is pointed in the ordinary sense) may be expressed so: there is a vector p such that $pA < 0$ (i.e., all components strictly negative). The vector p is a normal to the hyperplane. From this it follows that if x is a vector whose components are all non-negative and of which at least one is positive (denoted $x \geq 0$), then $Ax \neq 0$. For if $Ax = 0$ then $pAx = 0$, which is impossible since we must have $pAx < 0$.

Suppose now that $A_F = (a_i^j)$ and $\bar{A}_F = (\bar{a}_k^l)$ are both frames of A. (The subscript is taken to index the rows, the superscript the columns.) Any frame of A has the property that every column of A is representable as a positive linear combination of columns of the frame. In particular, the columns of A_F may be expressed in terms of those of $\bar{A}_F$ and vice versa. There must then exist two matrices, $B = (b_m^n)$ and $\bar{B} = (\bar{b}_r^s)$ such that $b_m^n \geqq 0$ for all m and n,

$\bar{b}_r^s \geqq 0$ for all r and s, and

$$a_i^j = \sum_m \bar{a}_i^m \bar{b}_m^j \qquad \text{all } i \text{ and } j$$

$$\bar{a}_k^l = \sum_n a_k^n b_n^l \qquad \text{all } k \text{ and } l$$

But then $\bar{a}_k^l = \sum_m \sum_n \bar{a}_k^m \bar{b}_m^n b_n^l$, or

(1) $$\sum_m \sum_n \bar{a}_k^m (\delta_m^l - \bar{b}_m^n b_n^l) = 0 \qquad \text{all } k \text{ and } l$$

where δ_m^l is the Kronecker delta. Now it is asserted that we must actually have

$$\sum_n (\delta_m^l - \bar{b}_m^n b_n^l) = 0 \qquad \text{all } m \text{ and } l$$

For suppose we give the index l some fixed value l_o. Then

$\sum_n (\delta_m^{l_o} - \bar{b}_m^n b_n^{l_o})$ is a column vector with components indexed by m.

If this vector be denoted by x, then equation (1) reads: $\bar{A}_F x = 0$. Since all the b's and $\bar{b}$'s are positive or zero, all components of x with one possible exception are negative or zero. The exception is the l_o'th component, where the Kronecker delta makes its contribution. If this component is positive, then the equation $\bar{A}_F\, x = 0$ gives the l_o'th column of $\bar{A}_F$ as a positive linear combination of some of the other columns of $\bar{A}_F$. This is impossible from the way $\bar{A}_F$ was constructed. Therefore, this component is zero or negative, whence all components are zero or negative. If they are not all zero, the equation $\bar{A}_F \cdot x = 0$ is impossible, for it was assumed that A was pointed, and hence $\bar{A}_F$, which determines the same cone, is pointed also. So all components of x are zero, and as the index l_o was arbitrary, the assertion is proved:

$$\sum_n (\delta_m^l - \bar{b}_m^n b_n^l) = 0 \quad \text{or} \quad \sum_n \bar{b}_m^n b_n^l = \delta_m^l$$

By interchanging the roles of B and $\bar{B}$, we get likewise

$\sum_n b_m^n \bar{b}_n^l = \delta_m^l$ from which it follows that B and $\bar{B}$ are in fact square, $B\bar{B} = \bar{B}B = I$, the identity matrix, and B and $\bar{B}$ are both non-singular.

Consider now the matrix $\bar{B} = (\bar{b}_r^s)$. It must be that for any fixed value r_o of r there is at least one value s_o of s such that $\bar{b}_{r_o}^{s_o} > 0$, for otherwise B is singular. It must also be the case that $b_{rm}^{r_o} \bar{b}_{r_o}^{s_o} = 0$ if $rm \neq s_o$. For

$\sum_r b_m^r \bar{b}_r^{s_o} = \delta_{rm}^{s_o}$. All the terms in the summation are non-negative. If there were actually a positive term present, the sum would have to be positive, but $\delta_{rm}^{s_o} = 0$ if $rm \neq s_o$. Since $\bar{b}_{rm_o}^{s_o} > 0$, it follows that $b_m^{r_o} = 0$ if $rm \neq s_o$.

Still, for some m we must have $b_{rm}^{r_o} > 0$; or B would be singular, whence $b_{s_o}^{r_o} > 0$. This, however, proves the theorem, for it shows that any column of $\bar{A}_F$ is a positive multiple of exactly one column of A_F, and clearly the same holds with A_F and $\bar{A}_F$ interchanged.

Therefore, if a matrix A is pointed, one may refer to "the frame of A", which will be denoted A_F.

If a convex cone C is not pointed, it must contain a whole linear subspace of E^N. In this case it will be called lineal. If A is an N-rowed matrix and (A) is lineal, then A will be called lineal as well. Since C is convex there is a largest linear subspace of E^N contained in C which contains all other linear subspaces of E^N also contained in C. This space will be called the lineality space of C. Its dimension will be called the lineality of C and will be denoted by $\chi(C)$. We may define the lineality space and the lineality of A by setting them equal to those of (A). With this concept it is possible to give a criterion telling when a column vector belongs to the frame of a pointed matrix.

<u>Theorem 2</u>: Given a pointed matrix A, a necessary and sufficient condition that a column vector a belong to the frame A_F of A is that the matrix $[-a\ A]$

have lineality 1. (The matrix $[-a\ A]$ is the matrix whose first column is $-a$, remaining columns the columns of A.)

Proof: The necessity will be proved first. Suppose then that a is a column vector of A_F. Then $[-a\ A]$ is certainly lineal. For if x is a vector whose first two components are equal and positive and all other components zero, then $x \geq 0$ and $[-a\ A]x = 0$. If $[-a\ A]$ were pointed this would be impossible as shown in the proof of theorem 1.

Let the components of a vector x be denoted by x_i. If the only vectors $x \geq 0$ such that $[-a\ A]x = 0$ are of the form $x_1 = x_2 > 0$, $x_i = 0$ for $i > 2$, then the lineality of $[-a\ A]$ is one, for its lineality space is then spanned by the one vector a. This must, in fact, be the case, for if not, there would exist a vector $x \geq 0$ such that $[-a\ A]\ x = 0$ and $x_i > 0$ for some $i > 2$. But then if $x_2 - x_1 \geq 0$ we have a contradiction to the assumption that A is pointed for we would have a positive linear combination of some of its columns equal to zero. On the other hand, if $x_2 - x_1 < 0$ we have a given as a positive linear combination of some of the other columns of A. Then, however, a could not be in the frame of A, contrary to assumption.

To prove the sufficiency, we assume that the lineality of $[-a\ A]$ is one. Since $(A) = (A_F)$ the lineality space of $[-a\ A]$ is the same as that of $[-a\ A_F]$, and $[-a\ A_F]$ also has lineality one. Then there is an $x \geq 0$ such that $[-a\ A_F]\ x = 0$. Suppose $x_r, x_s > 0$ for some $r, s > 1$ and $r \neq s$. Then the r'th and s'th columns of $[-a\ A_F]$ which are actually columns of A_F since $r, s > 1$, must both be in the lineality space of $[-a\ A_F]$. For if we denote the columns of $[-a\ A_F]$ by a^i, from the equation $[-a\ A_F]\ x = 0$ we get

$$-a^r = \frac{1}{x^r} \sum_{i \neq r} a^i x_i \quad ; \quad -a^s = \frac{1}{x^s} \sum_{i \neq s} a^i x_i .$$

Therefore the cone $([-a\ A_F])$ contains not only a^r and a^s but their negatives as well. Now the vectors a^r and a^s are linearly independent. For if

$a^r = \lambda a^s$ and $\lambda > 0$, then a^r and a^s could not both be in the frame A_F, and if $\lambda < 0$ then A_F is not pointed for then there exists an $x \geq 0$ such that $A_F x = 0$. But then the lineality space of $[-a \; A_F]$, containing two independent vectors, would have dimension two, contrary to assumption.

So the supposition is impossible and it must be the case that if $x \geq 0$ and $[-a \; A_F] x = 0$, then there is a unique $r > 1$ such that $x_r > 0$. That is to say a is the $r-1^{st}$ column of A_F, or A is in the frame of A. This ends the proof of the theorem.

Up to this point the dimensionality of the convex cone in question has been of no consequence. That is, it has not mattered what the dimension of the least linear subspace of E^N containing the cone has been. This becomes important now, however, since we wish to discuss properties of the boundary of the cone. A convex cone C will have interior points if and only if its dimensionality is N. Given a matrix A, the dimensionality of (A) is just the rank of A. The dimensionality of C will be denoted d(C). For simplicity, d((A)) will be denoted d(A), and occasionally reference may be made to "the dimension of a matrix A", meaning its rank.

By an abuse of language an N-rowed column vector not identically zero may be identified with the half-line in E^N which it represents. It becomes possible then to speak of a vector A as being in the interior of (A).

If it happens that the cone C is the whole space, it will be called solid. If (A) is solid, A will be called solid as well. It should be noted that we are tacitly assuming always that C is made up entirely of half-lines through the origin.

The interior of a convex cone C may be characterized simply. Let the N-rowed column vector whose i'th place is 1 and all other places zero be denoted by δ^i. Then a vector c is in the interior of C if and only if there exists an $\epsilon > 0$ such that the 2N vectors $c \pm \epsilon \delta^j$ ($j = 1 \ldots N$) are all in C.

For since C is convex, if all these vectors are in C, their positive linear combinations, which fill out a whole neighborhood of c, are also in C. If c is not in the interior of C, given any $\epsilon > 0$ there will exist a j_o such that either $c + \epsilon \delta^{j_o}$ or $c - \epsilon \delta^{j_o}$ is not in C. In fact, from the convexity of C, if c is not in the interior of C, there is a j_o and a fixed sign (+ or -) such that $c + \epsilon \delta^{j_o}$ or $c - \epsilon \delta^{j_o}$ is never in C, regardless of ϵ so long as it be positive and not zero. The interior of C may be denoted $\overset{o}{C}$. The boundary of C consists of all vectors not in the interior of C. We may now state:

<u>Theorem 3</u>: If the dimension of a matrix A is N ($\geqq 2$) and any frame A_F of A be given, then a necessary and sufficient condition that every column vector of A_F be in the boundary of (A) is that $\chi(A) < N - 1$.

Proof: The necessity is demonstrated first. If $\chi(A) = N$, then the cone is solid and any vector of A_F is in the interior of (A). If $\chi(A) = N - 1$, the lineality space of A is a hyperplane and the cone a half-space bounded by the hyperplane. Since $d(A) = N$, there must be a vector of A_F not in the hyperplane. This vector is in the interior of (A).

To prove the sufficiency, let us note that if $d(A) = p$ and $\chi(A) = q$, then $p \geqq q$ and there must be at least p-q vectors in any A_F whose negatives are not contained in A. In particular, under the hypotheses of the theorem, there are at least two vectors in A_F whose negatives are not in (A). Let $A_F = (a_i^j)$, the column vectors be denoted by a^j, and the i'th component of the j_o'th column be denoted $a_i^{j_o}$.

Now let a be an arbitrary column vector of A_F. By renumbering the columns of A_F we may assume $a = a^1$ and that a^2 is a column vector of A_F whose negative is not in (A). Suppose now the theorem false and a^1 actually in the interior of (A). Then there is an $\epsilon > 0$ such that $a^1 \pm \epsilon \delta^j$ is in (A) for all j. Now a^2 may be represented so:

$$a^2 = \sum_{i=1}^{N} a_i^2 \delta^i$$

There is an $\epsilon' > 0$ such that $\epsilon' |a_i^2| < \epsilon$ for all i. But then all the vectors $a^1 - \epsilon' a_i^2 \delta^i$ are in (A), so

$$\sum_{i=1}^{N} (a^1 - \epsilon' a_i^2 \delta^i)$$ is in (A). That is, $Na^1 - \epsilon' a^2$ is in A and may be written as a positive linear combination of some of the columns of A_F :

$$Na^1 - \epsilon' a^2 = \sum_{j=1}^{M} \lambda_j a^j$$, where $\lambda_j \geqq 0$ for all j, and M is the number of columns of A_F. If $\lambda_1 < N$, this gives a as a positive linear combination of some of the other columns of A_F, which is impossible. If $\lambda_1 = N$ the equations would state that $-a^2$ is in (A), contrary to assumption. If $\lambda_1 > N$, the equation shows that $-a^1$ is in (A). We may then add $-Na^1$ to both sides of the equation and conclude as before that $-a^2$ is in (A), contrary to assumption. The supposition that the theorem is false having led to a contradiction, the theorem is proved.

Several remarks may be made about the cases not covered by the theorem. If $d(A) = \chi(A) = N$, the cone is the whole space and is rather uninteresting. If $d(A) = N$ and $\chi(A) = N-1$, then there can be at most one frame vector in the interior of (A). For the above proof breaks down because there need be no more than one frame vector whose negative is not in (A). (There is in fact exactly one.) The vector $a = a^1$ may then be this vector, leaving no a^2 with which to work. For all other frame vectors the proof succeeds. Since it was observed in the first part of the proof that in this case there must be at least one frame vector in the interior of (A), there must be exactly one (in any choice of frame). If $N = 1$, the statement of the theorem is meaningless and the situation is obvious. If $d(A) < N$, then (A) has no interior; the boundary of (A) and (A) coincide.

We have so far been dealing only with convex polyhedral cones determined by matrices, that is, we have tacitly adopted the definition that a convex polyhedral cone is the convex hull of a finite number of half lines through the origin. Equivalently they may be regarded as the intersection of a finite number of closed half spaces each determined by a hyperplane through the origin. (cf. for example H. Weyl, Comm. Math., Helv., 1935). The intersection of two convex polyhedral cones is then also such a cone, may be represented by a matrix and has a (not necessarily unique) frame.

Given two matrices A and B, theorem 3 gives some information about the frame of $(A) \cap (B)$. If $d((A) \cap (B)) < N$ or $d((A) \cap (B)) = N$ and $\chi((A) \cap (B)) < N-1$, then every frame vector of $(A) \cap (B)$ must lie in the boundary of (A) or the boundary of (B). For if it lay in the interior of (A) and the interior of (B) it would lie in the interior of $(A) \cap (B)$. But in these cases we know every frame vector of $(A) \cap (B)$ lies on the boundary of $(A) \cap (B)$.

It is possible to give a criterion telling when $d((A) \cap (B)) = N$.

<u>Theorem 4:</u> Suppose the matrices A and B be given, and that $d(A) = d(B) = N$. Then a necessary and sufficient condition that $d((A) \cap (B)) = N$ is that $[-A\ B]$ shall be solid. In this case, there is a vector lying in both the interior of (A) and the interior of (B).

The necessity will be proved first. Suppose $d((A) \cap (B)) = N$. Then it has an interior vector c, and for sufficiently small ϵ the 2N vectors $c \pm \epsilon\,\delta^j$ $(j=1 \ldots N)$ are all in $(A) \cap (B)$. In particular they are in (B). But (-A) contains -c, so all the vectors $\pm \epsilon\,\delta^j$ are in $([-A\ B])$. Any vector is a positive linear combination of the $\pm \epsilon\,\delta^j$, so $[-A\ B]$ is solid. Let it be noted that if we know only that (B) has dimension N and that there is a vector of (A) in the interior of (B), then again, $[-A\ B]$ is solid.

To prove the sufficiency, assume $[-A\ B]$ solid. Since it was assumed that $d(A) = d(B) = N$, (A) and (B) have interiors, and these will be denoted $(\mathring{A})$ and

$(\overset{\circ}{B})$. Since (A) and (B) are convex, so are $(\overset{\circ}{A})$ and $(\overset{\circ}{B})$, and (A) is the closure of $(\overset{\circ}{A})$, (B) that of $(\overset{\circ}{B})$. Let $(-(\overset{\circ}{A})\ (\overset{\circ}{B}))$ denote the (open) convex hull of $-(\overset{\circ}{A})$ and $(\overset{\circ}{B})$. Since $\lceil -A\ B \rfloor$ is solid, every point in E^N is a limit of points of $(-(\overset{\circ}{A})\ (\overset{\circ}{B}))$; i.e., $(-(\overset{\circ}{A})\ (\overset{\circ}{B}))$ is dense in E^N. Since it is also convex, it is in fact all of E^N. We may then take a vector $-b^1$ in $-(\overset{\circ}{B})$ and $(-(\overset{\circ}{A})\ (\overset{\circ}{B}))$, i.e., $-b^1 = -a + b^2$ where $-a$ is in $(\overset{\circ}{A})$ and b^2 is in $(\overset{\circ}{B})$, or $a = b^1 + b^2$. But a is in $(\overset{\circ}{A})$, $b^1 + b^2$ in $(\overset{\circ}{B})$. So $(\overset{\circ}{A}) \cap (\overset{\circ}{B}) \neq 0$, whence $(A) \cap (B)$ has interior points and (A) and (B) have a common internal vector. As was remarked before, for a cone to have the dimensionality of the space in which it is embedded and for it to have interior points are equivalent, so the theorem is proved.

Some further remarks may be made about the case where $\lceil -A\ B \rfloor$ is solid. By dropping all considerations of interiors, the proof above shows that there is a vector common to (A) and (B). If only (B) is assumed to be of dimension N, then there is a vector common to $(\overset{\circ}{B})$ and (A).

From these remarks and the last remark in the proof of the necessity we may draw the following corollary:

<u>Theorem 5</u>: Suppose matrices A and B be given such that $d(B) = N$ and $(A) \subset (B)$. Then a necessary and sufficient condition that (A) lie in the boundary of (B) is that $\chi(\lceil -A\ B \rfloor) < N$, i.e., $\lceil -A\ B \rfloor$ be not solid.

The proof has, in effect, already been given.

Given a convex polyhedral cone, this theorem tells, in particular, when a number of frame vectors span a facet of the cone, that is, when the cone they span does not penetrate into the interior (if there is any).

This preliminary discussion of the properties of convex polyhedral cones ends here and will be resumed in a later paper.