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A Mathematical Model of Production

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1. Introduction

1.1 Technology and Choice. The concept of a production function occupies a central place in the literature on production theory. In some discussions this concept is associated with a particular technological process. The function is then supposed to represent the output of one commodity (say) as a function of the quantities of various factors of production, combined according to a given technological principle or formula. Further elaboration of this concept has led to the distinction between situations where the set of technically possible factor combinations is unrestricted (allowing for continuous substitution between factors) and situations where some factors can only be combined, within the technological principle involved, in fixed ratios to each other (limitational factors). The second situations can only be reconciled in with the notion of a production function defined in the whole factor space by allowing the production manager to discard parts of the factor quantities specified as being available.

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* This model was first developed in a more special form relating to the transportation industry. In conversation, Dr. George B. Dantzig introduced me to the wider applicability of models involving constant production coefficients to the discussion of allocation problems. At this stage, I also learned of and benefited from the literature on similar models reviewed in 1.12 below. An earlier version of this paper was presented at the Madison meeting of the Econometric Society, September 1948. The present version has gained from the reading of G.B. Dantzig and M.K. Wood, "The Programming of Interdependent Activities".

In order to save time, earlier sections of this article are already being hectographed before later ones are completed. This will inevitably result in some inconsistencies and in shifting ground. The provisional character of this manuscript should be stressed.

References are made to the LPC bibliography.

The corresponding production functions have kinks at the points where the ratios of available factor quantities coincide with the technical ratios specific to the process in question.

A second concept of the production function has not always been distinguished sufficiently from the first. In this second concept, the production function incorporates the effects of production decisions of an entrepreneur or manager who exercises maximizing choice between alternative processes. Depending on the specified factor quantities which constitute the arguments of the production function, the manager chooses one process or another so as to maximize production (which for simplicity we still assume to be one-dimensional). The production function is thus represented by the locus of points of technologically maximum achieveable production for each quantitative factor combination. This view will be adopted in the present study.

1.2 Complete, Partial and Restricted Production Functions. Other distinctions have to do with the choice of variables (factors of production) which appear as arguments of the production function, and the assumptions made with regard to other factors affecting production but not shown as arguments.

Let us start from the ideal notions of a complete production function involving all variables actually or possibly affecting production of a commodity. This would include a flow of fresh air for the workers to breathe in, the exchange of heat between the workshop and its surroundings as determining workshop temperature, and many other things not ordinarily included among factors of production.

A partial production function, say, is obtained by omitting factors of production which are free goods, such as air in most cases, water in may situations. This does not mean that the amounts of the free factors applied in production are regarded as constants, but rather that they are optimally adjusted to the amounts of other factors.

A <u>qualitatively restricted</u> production function, say, is obtained by omitting, or rather regarding as given by nature, situation characteristics such as climate, daylight conditions, etc. These can be regarded as characteristics of the particular type of land applied in production. Similarly, the qualities of labor and other factors may be specified as part of the definition of a production function, which thereby ceases to depend on the variables that are essentially quality characteristics of these factors.

A <u>quantitatively restricted</u> production function, say, is obtained by prescribing, as given constants, the amounts applied of some of the factors of production. While logically any subset of factors could be so selected, the analytical application of this device has mostly been to those factors which in the arrangement of a productive process need to be chosen quantitatively at an early stage, because their variation at a later time would involve higher cost. These are the variables involved in the choice of plant and equipment.

So far we have used a terminology implying production of one commodity only. Almost all productive processes involve joint production. It is not difficult to reformulate the foregoing distinctions in terms of joint production. The concept of the production function now requires that, besides the amounts of all factors, the amounts of all but one of the products are also specified. The value of the production function is then defined as the maximum achievable amount of the remaining product.

<u>1.3 Aggregate Production Functions.</u> Finally, production functions have been discussed in an aggregative sense, relating to an industry or to the economy as a whole, and involving inputs and outputs which themselves are defined as aggregates of commodities (goods and services).

<u>1.4 Elements of the Production Problem.</u> In this article, a model of production will be developed in which the following circumstances or considerations are treated

as distinct elements of the production problem:

a) the purely technical possibilities of production,

b) the quantitative limitations on basic resources (primary factors of production) available to the economy,

c) the general objective to be served by production,

d) the optimizing choice whereby the technical possibilities are exploited in a coordinated manner toward that objective.

The production function to be derived from this model will be inclusive rather than aggregative. That is, while no aggregation of commodities will be introduced, the production function will be thought of as expressing the productive potential of an entire economy, or of any technically well-defined part thereof. We shall use the term "economy" to refer to either the whole or a defined part of what is usually called the economy. The production function represents the most favorable achievable relationship between the inputs of individual primary factors of production and the final outputs of individual commodities through that "economy." The existence of such a general transformation function of an economy was postulated by Lange [ ]. We shall from here on refer to it as "transformation function" rather then as "production function." When applied to a sufficiently wide concept of the economy, it involves a broader choice of combinations of productive activities than is available to the individual firm.

1.5 Static and Dymanic Models. For reasons of exposition, and in deference to a venerable tradition in economic literature, we shall first study a static model in which the elements of technology, scarity, objective, and choice are formalized in terms of variables thought of as remaining constant curing an indefinite period. Thereafter, we shall generalize the model to include planning for variation over time of outputs.

1.6 Commodities and Activities. The formalization of the technical possibilities involved only two basic concepts, the commodity and the activity. Each commodity is assumed to be homogenous qualitatively and continuously divisible quantitatively. Commodities include primary factors of production, such as labor of various kinds, the availability of land of various grades, access to mineral resources; intermediate products such as coal, pig iron, steel; and final products the production of which is the objective of the economy under study. We shall denote by

(1.1) $y_n, \quad n = 1, \ldots, N,$

the total net output of the n-th commodity by the productive system under study. A negative value of y_n signifies a net input of the n-th commodity.

In the static model an activity consists in the combination of certain qualitatively defined commodities in fixed quantitative ratios as "inputs," to produce as "outputs" certain other commodities in fixed quantitative ratios to the inputs. The k-th activity is defined by a set of coefficients

(1.2) $\gamma_{nk}, \quad n = 1, \ldots, N,$

indicating the rate of flow per unit of time of each of the N commodities, involved in the unit amount of that activity. Negative coefficients γ_{nk} indicate that the commodity involved is used up by the activity, positive coefficients that the commodity is produced. A value $\gamma_{nk} = 0$ indicates that the n-th commodity is not involved in the k-th activity.

Two basic assumptions are associated with the notion of an activity. In the first place, we assume that each activity is capable of continuous proportional expansion or reduction. If any non-negative scalar quantity x_k is selected to be the amount of the k-th activity, the corresponding commodity flows are assumed to be given by

(1.3) $x_k \cdot \gamma_{nk}, \quad n = 1, \ldots, N.$

this assumption implies the neglect of indivisibilities in production.

In the second place, we assume that any number of activities can be carried out simultaneously without modification in the technical ratios by which they are defined, provided only that the total resulting net output y_n of any commodity, whenever negative, is within the limitations on primary resources to be discussed in the next section. This assumption, taken together with the previous one, implies the neglect of economies or diseconomies of scale (except diseconomies resulting from scarcity of primary factors).

The two assumptions can be fused in the statement that we postulate the existence of a set of <u>basic activities</u>, represented by vectors

(1.4)
$$\gamma_k = \begin{bmatrix} \gamma_{1k} \\ \vdots \\ \gamma_{2k} \\ \vdots \\ \gamma_{Nk} \end{bmatrix},$$

such that any possible <u>state of production</u> can be represented by a linear combination of basic activities with non-negative coefficients x_k. The resulting net outputs y_n can be written as

(1.5)
$$y_n = \sum_{k=1}^{K} \gamma_{nk} x_k \quad ; n = 1, \ldots, N; \; x_k \geqq 0; \; k = 1, \ldots, K.$$

<u>1.7 The Limitations on Primary Factors.</u> We shall assume that certain commodities, called primary factors, can be made to flow into the economy from nature (or from the "outside world"), at a rate, possibly limited by a constant depending on the commodity:

(1.6)
$$\eta_n \leqq y_n$$

The constant η_n is algebraically smaller then the rate of flow y_n because a net inflow into the economy is represented by a negative number y_n, which cannot exceed the corresponding bound η_n <u>in absolute value.</u> Certain commodities, such as water and air, may be available in greater abundance then required for any conceivable objective of the economy. If this can safely be asserted before analysis, the commodity in question is certain to be free good, which does not give rise to any restrictions on allocative decisions. Its perfunctory role in the model can be expressed by writing $\eta_n = -\infty$, or the commodity can be omitted from the model, i.e., from all activities in which it is physically involved. Whenever its character as a free good is subject to doubt before analysis, the effective bound η_n on its availability should be incorporated in the model.

While a commodity could not be a primary factor without being an input to at least one activity, there is no reason why a primary factor could not also appear as output of some other activities, the sum of its outputs being less than the sum of its inputs in the several activities.

<u>1.8 The Objective of Allocative Decisions.</u> In order to cover a wide variety of cases, we shall assume as little as possible with respect to the aims pursued by the economy. We shall postulate only that there is a specified set of commodities, to be called <u>desired commodities</u>, which are required by the economy in the following sense: an addition to the total net production of one or more of the desired commodities which does not entail a reduction in the net production of any other desired commodity is regarded as an improvement. As long as such improvements are possible, the allocation of resources in production is not efficient.

It is clear that this postulate provides only a partial ordering of points in the space of which the coordinates are flows of desired commodities. No preference is expressed between alternatives A and B if A involves more of one desired commodity, B more of another. We can therefore not expect our model to produce a unique solution to the allocation problem. The postulate will prove sufficient for our more modest aim: to derive the transformation function representing all possibilities of efficient production.

It should be readily admitted that our assumption regarding the valuation of desired commodities ignores the possibility of saturation. To make allowance for saturation would required much more detailed specification of consumers' preferences than it is our present purpose to make.* The transformation function derived without regard to saturation will be relevant in all those portions of the space of desired commodity flows in which saturation is actually not reached for any desired commodity.

It may be thought that we can specify $\eta_n = 0$ in (1.4) for every desired commodity that is not at the same time a primary factor. However, it may occasionally be useful to allow for the fact that certain effects or conditions of production are negatively valued, such as smoke pollution. We can maintain the formal applicability of the objective as stated above by introducing these effects as negative outputs (i.e., inputs) of "desired" commodities, of which the algebraic increase (i.e., the absolute reduction) is deemed desirable.

<u>1.9 The Treatment of Labor.</u> In some models considered by Leontief [ ] and von Neumann [1935], labor has been treated as the output of an activity, of which the consumption of various commodities constitutes the set of inputs. The model thus becomes a closed one. We intend to study an open model which does not specify the structure of preferences between desired commodities, and between consumption and leisure.

One possibility is to treat labor, like smoke pollution, as a negatively desired commodity which is also a primary factor, and which can be involved in any activity as input but not as an output. We shall adopt this conception of labor whenever we wish to avoid such explicit bounds η_n as occur in (1.6).

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* An assumption whereby saturation in one commodity arises at a level of flow independent of the flows of other commodities could still be accomodated in our model at small cost in mathematical complication, but would add very little to the degree of realism attained.

Alternatively, we can treat manpower as a gift of nature, flowing into the economy at a rate bounded by the given negative number η_n, if n refers to manpower. Besides being used in all productive activities, manpower is then introducted as the sole input in the activity "recreation," of which the sole output* is the positively desired commodity "leisure."

In a more refined model, one may treat unskilled manpower as a gift of nature, and manpower of various skills as the outputs of a variety of educational and training activities. The consumption aspects of these activities can be recognized by stipulating that, besides skilled manpower, these activities also produce "educational services" which are desired for their own sake.

1.10 Intermediate Products. If the sets of primary factors and of consumers' goods do not together cover all commodities, the remaining commodities will be called intermediate products. For these must be true, what may but need not be true for primary factors or for desired commodities: they are simultaneously an input to at least one activity and an output of at least one other activity.

It may be thought that this statement ignores the existence of waste products. In fact, waste products can be disregarded if they are not used even in part for further production or consumption, and provided their disposal does not require the use of other commodities. Otherwise, waste products can be regarded as intermediate products by introducing a disposal activity, in which they appear as inputs, and with which no useful** outputs are connected. This treatment is desirable in particular if it is not known in advance whether, or in what part of the space of desired commodity flows, the commodity involved will actually come out of the

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* We return later to the question how to make allowance for the effect of recreation on the productivity of manpower when applied in production. For the present, that effect is disregarded.

** The meaning of the term "useful" will become clear below when prices are introduced.

analysis as a waste product.* In this respect the position of possible waste products in the analysis is similar to that of possible free goods.

For intermediate products we can specify in (1.6)

(1.7) $y_n = 0$

if n refers to an intermediate product, because negative net flows are not compatible with the static assumption of constant flows during an indefinite period. However, the incorporation of disposal activities in the model makes possible a sharpening of the restriction (1.6). Instead of $0 \leq y_n$, which follows from (1.7), we must write

(1.8) $y_n = 0$

if n refers to an intermediate product. This can be understood to mean that, whenever an "intermediate" y_n would otherwise turn out best to be positive, we undertake to reduce it to zero by the appropriate amount of a disposal activity. This does not really restrict allocative decisions beyond what is already implied in (1.7), except (and rightly so) in the case where the disposal activities necessary to ensure (1.8) consume other useful** commodities.

<u>1.11 Managerial Choice.</u> We shall define managerial choice as the selection of non-negative amounts (or levels) $x_1, \ldots, x_k$ for all possible activities. This choice is here studied in the abstract. We do not inquire whether it is exercised by one entrepreneur or manager, by a number of independently acting entrepreneurs, by a public planning body, or by a combination of these. A study of the formal nature of optimizing choice is believed to be a useful preliminary to the study of the effectiveness of alternative social and institutional arrangements in realizing or approaching optimal choice.

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*In the early days of the oil industry, kerosene was the main desired product, and gasoline was disposed of by burning.

**See footnote on p.

Managerial choice is exercised subject to the restrictions (1.6) and (1.8)and is assumed to be guided by the objective formulated in 1.8. We have already indicated that in general more than one state of production can result from the pursuit of that objective. The main purpose of our discussion is to study the set of all productive states that realize the broad objective formulated.

1.12 Similar Models in Economic Literature. Discussions of models similar to that here considered have followed two main streams of thought. The first group of articles started from theoretical models without limitations on the numbers of activities and commodities. The second group of publications was from the beginning concerned with aggregative models conceived so as to permit quantitative measurement of the coefficients γ_{nk}.

The first group of articles originated in discussions in Karl Menger's mathematical seminar in Vienna, 1933-1936. Schlesinger [1933] suggested that economic theory be required to explain not only prices and quantities produced of scarce goods, but also which goods are scarce and which are free(i.e., have a zero price). Wald [1933, 1934] proved the existence of a unique answer to this question in a static model, where each produceable commodity is produced by only one activity requiring fixed ratios of primary factors of production, of which finite total amounts are given by nature. In the first study demand is described* by monotonically decreasing demand functions for eash desired good depending on price of that good only. In the second study, the demand price for each desired good is a function of the quantities of all desired goods. In the model of Schlesinger and Wald, desired goods appear as outputs only, primary factors as inputs only, so that circularity in the use of commodities in production cannot

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* The monotonicity assumption is then weakened to the statement that a certain finite variation Δy in the vector of flows of desired goods will call forth a variation Δp in the price vector such that $\Delta y' \cdot \Delta p < 0$.

occur. Prices are introduced as observed phenomena, and a condition of zero profits is imposed presumably as a consequence of competition. The main concern is with the existence and uniqueness of a solution, not with the efficiency of the allocation of resources.

Von Neumann [1938, 1945] generalizes the model of production by introducing alternative methods of producing given commodities singly or jointly, each method involving fixed ratios between inputs and outputs. Thus, he derives not only which goods are free but also which productive activities go unused. A commodity may appear simultaneously as input of one activity and as output of another. This circularity idea is extended even to the demand side by the somewhat forced concept of an activity producing labor by the absorption of consumption goods in fixed propositions. The model thus becomes a closed one, with no inflow of primary factors from outside or outflow of final products out of the system considered. Any non-consumed "surplus" is used for capital formation to obtain a continuous proportional expansion of production under unchanging technology. Von Neumann's model is therefore dynamic in the limited sense that change over time is described by one scalar coefficient of expansion.

Von Neumann likewise treats prices (including an interest rate) as formed in competitive markets so as to satisfy as zero profit condition on all activities engaged in. This model has a remarkable symmetry in prices and quantities. It is noted at the end of the article that any solution achieves efficiency of allocation in the sense of a maximum rate of expansion of production compatible with the given (productive and consumption) technology. However, von Neumann's main concern is with the existence of a solution, which under his assumptions is no longer necessarily unique.

Dantzig's model [1949] belongs in this same school of thought, although it was, I believe, independently conceived, and is also designed with a view to measurement of the technical coefficients. The model is more truly dynamic than von Neumann's in that it permits change over time in the relative amounts of various activities in order most efficiently to achieve a stated objective. It combines consideration of stocks and flows, as against the use of flows* only in the Wald model, and of stocks (at the beginning and at the end of each production period) only in von Neumann's model. Dantzig's model is an allocation model that does not depend on the concept of a market. It does not introduce prices except implicitly in the formulation of the objective, which is to maximize a given linear function of the amounts of certain activities or (equivalently) commodities. It explicitly introduces limitations on the amounts of primary (or initially available) factors of production. It allows the same commodity to appear as input in one activity, as output in another.

The model developed in this article builds forth on, and extends, the work just described. As was said already, it is an open model in which the nature of demand is specified only through the efficiency objective in the space of desired commodities. Limitations on primary factors are introduced explicitly, but circularity in the use of commodities in production is allowed for. It is an allocation model independent of the concept of a market. However, a price concept applicable to all commodities is derived from the requirement of efficient allocations. The ratios between these prices, whenever determinate, are marginal rates of substitution in efficient production. In terms of those prices, the two conditions of non-positive profits on all activities, and of zero profits on all activities engaged in, are found to be necessary and sufficient for efficient use of resources.

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*The article is not explicit on this point, but seems to be more naturally interpreted in that way.

The second stream of thought has been initiated, developed, and stimulated largely by Leontief [ ]. In his models, the "commodity" stands for the aggregate output of an industry, the "activity" for the act of production by that industry. Homogeneity of productive operations within an industry is aimed for by as detailed an industrial classification as is permitted by available data and computation methods and equipment. The method of measurement of technical coefficients has mostly been the observation of the money value of all goods and services delivered by each industry to each other industry in a Census year. This method precludes the separate measurement of alternative processes to produce the same commodities, or the recognition of joint production. Nevertheless, the models so obtained have been important tools in the evaluation of economic policies in situations in which substitution between activities is not quantitatively important, notably the discussion of the employment effects by industries of stimulating expenditures (of given composition by industries) in a situation of unemployment and unused capacity in all relevant industries. A related particular applicationhas been the anticipation of levels and compositions of expenditure at which limits to capacity or to the labor force are reached in specific industries, or generally.

2. Mathematical Tools

Note: In this review of tools, proofs are incomplete and sometimes absent. It is believed that all statements are valid, but where no rigor and proofs are given, they must be considered as conjectures. I am indebted to M. Gerstenchaber for discussions of these mathematical tools.

2.1 Notations. We shall introduce certain mathematical notations and concepts that will be useful in what follows. We introduce a number of partial ordering relations between vectors a, b with components a_i, b_i, respectively, in N-dimensional space, as follows:

Symbol	Meaning	If b = 0, a is called
$a > b$	$a_i > b_i$ for all i	positive
$a \geq b$	$a_i \geqq b_i$ for all i; $a_i > b_i$ for some i	semi-positive
$a \geqq b$	$a_i \geqq b_i$ for all i	non-negative
$a \not\geq b$	either $a_i = b_i$ for all i, or, if $a_i > b_i$ for some i, then $a_i < b_i$ for some other i.	non-semipositive

A vector a without transposition sign denotes a column vector (one-column matrix), a vector a' a row vector (one-row matrix). We denote by $s(\Gamma)$, $r(\Gamma)$, $c(\Gamma)$ the rank, the number of rows, and of columns, respectively, of a matrix Γ. We use as interchangeable terms the point y in N-dimensional space, and the vector y connecting the orgin with that point. We use the notation $y \in Y^*$ to indicate that a point y belongs to a set Y^*.

2.2 Convex cones. A point set Y^* is called convex if, for any two points $y^{(1)}$, $y^{(2)}$ belonging to Y^*, the points

$$(2.1) \qquad y = \lambda y^{(1)} + (1-\lambda) y^{(2)}, \; 0 \leqq \lambda \leqq 1, \; \lambda \text{ scalar},$$

located on the interval $[y^{(1)}, y^{(2)}]$ of the straight line connecting $y^{(1)}$ and $y^{(2)}$ also belong to Y^*. The non-negative set P_y defined by

$$(2.2) \qquad y \geqq 0$$

is convex.

A convex set is called closed if it contains its boundary. All convex sets we shall consider are closed, and we shall often omit the adjective closed as superfluous in this context.

A hyperplane S defined by

(2.3) $$\theta' y = \theta_0, \qquad \theta_0 \text{ scalar},$$

is called a supporting plane of a closed convex set Y* if it has at least one point $y^{(0)}$ in common with Y*, while Y* contains no points to one side of S. With the proper choice of sign for the vector $\begin{bmatrix} \theta \\ \theta_0 \end{bmatrix}$ this is expressed by

$$\theta' x^{(0)} - \theta_0 = 0 \quad \text{for some } y^{(0)} \in Y^*,$$

(2.4) $$\theta' x - \theta_0 \leqq 0 \quad \text{for all } y \in Y^*.$$

The half line

(2.5) $$y = y^{(0)} + \lambda\theta, \quad 0 \leqq \lambda, \ \lambda \text{ a scalar parameter},$$

out of $y^{(0)}$ orthogonal to S will be called an (outward) normal to Y* on $y^{(0)}$. It has only the point $y^{(0)}$ in common with Y*.

The <u>convex hull</u> of a point set Q is the set containing all points (2.1) such that $y^{(1)}$ and $y^{(2)}$ belong to Q. The convex hull of Q contains Q. The convex hull of a convex set Y* is Y* itself. We shall often consider the convex hull of the <u>union</u>

(2.6) $$U = Q_1 + Q_2 + \dots,$$

of point sets $Q_1, Q_2, \dots$, i.e., of the set U containing all points belonging to at least one of the sets $Q_1, Q_2, \dots$. Briefly, we shall call H the convex hull of the sets $Q_1, Q_2, \dots$.

A point set C is called a <u>cone</u> if, for any point $y^{(1)} \neq 0$ belonging to C, the points

(2.7) $$y = \lambda y^{(1)}, \quad 0 \leqq \lambda$$

located on the half-line out of the origin through $y^{(1)}$ also belong to C. The orgin 0 is called the <u>vertex</u> of the cone. The non-negative set P is a convex cone.

A supporting plane S of a convex cone C necessarily contains the vertex. If S has only the vertex in common with C it is called a vertical supporting plane. A supporting plane S having at least a half-line out of the vertex in common with C is called a lateral supporting plane. A normal to a convex cone C is called a vertical normal if it is normal to a vertical supporting plane, a lateral normal if it is normal to a lateral supporting plane. A vertical normal to C is necessarily a normal on the vertex. A lateral normal may be a normal on the vertex or on another boundary point of C.

The cone C* consisting of all normals on the vertex to a convex cone C will be called the conjugate cone to C. It is convex. We shall make use of a theorem, proved in another paper by D. Gale:

Theorem 2.2: If a convex cone C has no point, other than the origin, in common with the non-negative set P_y, then its conjugate cone C* contains an internal point

(2.8) $\quad y > 0$

of P_y.

2.3 The linear transform of the non-negative set. We will extensively use homogeneous linear transformations

(2.9) $\quad y = \Gamma x$

from the space of a vector x on to p the space of a vector y with possibly a different, often a smaller, number of dimensions. Such a transformation maps a cone into a cone, and preserves convexity. We shall particularly study the convex cone Y in the y-space into which the non-negative set X = P_x in the x-space is transformed by (2.9).

Denote by $e^{(1)} \ldots e^{(K)}$ the basic vectors

(2.10) $\quad e^{(k)} = (0_1 \; 0_2 \; \ldots \; 0_{k-1} \; 1_k \; 0_{k+1} \; \ldots \; 0_K)$

in the space of x. Then X is defined by

(2.11) $$x = \sum_{k=1}^{K} x_k e^{(k)}, \quad x_k \geq 0; \quad k = 1, \ldots, K.$$

The transforms by (2.9) of the vectors $e^{(k)}$ are the column vectors $\gamma^{(k)}$ of Γ.

Therefore the set Y can be defined by

(2.12) $$y = \Gamma x = \sum_k x_k \gamma^{(k)}, \quad x_k \geq 0; \quad k = 1, \ldots, K.$$

Thus Y contains all linear combinations, with non-negative weights, of the column vector $\gamma^{(k)}$ of Γ.

The cone Y is called <u>pointed</u> if all vectors $\gamma^{(k)}$ are internal to some half-space through the vertex, i.e., if there exists a vector θ such that

(2.13) $$\theta' \Gamma < 0.$$

The cone Y is called <u>solid</u> if it consists of the entire space of the vectors y. There are a number of intermediate cases, each corresponding to a set of measure zero in the space of Γ, which we shall briefly describe. In each of these cases, the $<$ sign in (2.12) needs to be replaced by a $\leq$ sign.

By the <u>dimensionality</u> of Y we mean the rank $\rho(\Gamma)$ of Γ. We shall confine ourselfes to cases where

(2.14) $$\rho(\Gamma) = r(\Gamma),$$

i.e., where the vectors $\gamma^{(k)}$ are not contained in some linear subspace of the y-space. By

(2.14a) $$\rho\left[\gamma^{(k)}\right] = 1, \ k = 1, \ldots, K$$

we exclude the trivial possibility that any column of Γ should consist of zeros only (exclusion of empty activities).

The cone Y will be called <u>lineal</u> if it contains at least one straight line (not just a half-line) through the vertex. This is equivalent to the existence of a solution x to

(2.15) $$y = \Gamma x = 0, \quad x \geq 0.$$

For if Y contains a line through the vertex, it contains two vectors

(2.16) $y^{(1)} = \Gamma x^{(1)} \neq 0, \quad y^{(2)} = \Gamma x^{(2)} \neq 0, \quad x^{(1)} \geq 0, \quad x^{(2)} \geq 0$

such that

(2.17) $0 = y^{(1)} + y^{(2)} = \Gamma \left[x^{(1)} + x^{(2)}\right] = \Gamma x,$

say, where

(2.18) $x = x^{(1)} + x^{(2)} \geq 0.$

Conversely, if (2.15) holds for some x with components given by

(2.19) $x' = \left[x_1 \ldots x_k \ldots x_K\right],$

then there exists a pair of integers (n_o, k_o) such that

(2.20) $\gamma_{n_o k_o} x_{k_o} \neq 0,$

because (2.14a) precludes the vanishing of an entire column of Γ. By defining

(2.21) $x'^{(1)} = \left[0_1 \ldots 0_{k_o-1}\ x_{k_o}\ 0_{k_o+1} \ldots 0_K\right] \geq 0$

and

(2.22) $x^{(2)} = \left[x_1 \ldots x_{k_o-1}\ 0\ x_{k_o+1} \ldots x_K\right] \geqq 0$

we satisfy (2.16), (2.17), (2.18) because (2.20) ensures that $y^{(1)} \neq 0$, and hence (2.17) implies that $y^{(2)} \neq 0$, and hence $x^{(2)} \geq 0$.

From (2.17) it follows that

(2.23) $y^{(1)} = \lambda y^{(2)}, \lambda < 0$ and scalar.

The positive linear combinations of $y^{(1)}$ and $y^{(2)}$ therefore form a full straight line.

By the <u>lineality</u> χ of Y we mean the dimensionality of the linear space L containing all straight lines in Y through the vertex 0. This space L is called the <u>lineality space</u> of Y. We have

(2.24) $0 \leqq \chi \leqq N.$

If $\chi = N$, L consists of the entire space of y, and Y is solid. If $\chi = N-1$, and (2.13) holds, Y is a half-space and is called <u>half solid</u>. If $\chi = 0$, Y is pointed. For $1 \leqq \chi$, Y is lineal with various possible degrees of lineality. As long as $\chi \leqq N-1$, there exists a vector G such that

(2.25) $\quad \theta' \Gamma \leq 0$

[the possibility $\theta'\Gamma = 0$, $\theta' \neq 0$, being excluded by (2.14)].

We are not concerned in this article with the difficult problem of finding computational procedures to determine the lineality of Y, but we shall utilize the lineality concept for purposes of classification of cases.

For brevity we shall use the expression "lineality of the matrix Γ" as equivalent to "lineality of the convex cone $Y(\Gamma)$ spanned by the column vectors $\gamma^{(k)}$ of Γ."

2.4 The Frame of a Matrix. The Intersection of Convex Cones. The cone $Y = Y(\Gamma)$ spanned by the columns of a matrix Γ is not affected by the deletion of a particular column vector $\gamma^{(k)}$ if and only if that column can itself be expressed as a linear combination of the other columns with non-negative coefficients. Any such column vector $\gamma^{(k)}$ is said to be an inessential vector. Every vector internal to $Y(\Gamma)$ is inessential. A vector in the boundary of $Y(\Gamma)$ can be inessential by being internal to a facet which is part of that boundary. A vector which is on the boundary of every facet of which it is a part can be inessential only if it is the positive scalar multiple of another column vector of Γ.

Delete from Γ, in arbitrary order, all columns found inessential (with reference to the part of Γ not yet deleted). Any submatrix Γ^* of Γ containing no further inessential columns is called a frame of Γ. Since the only column of a one-column matrix of rank 1 is always essential, every matrix Γ posesses at least one frame. We state two theorems without proof:

Theorem 2.4.1: The frame Γ^* of a pointed matrix Γ is unique except possibly for permutations of the columns of multiplication of columns by positive scalars.

We shall use the term frame of Γ and frame of the cone $Y(\Gamma)$ as interchangeable.

Theorem 2.4.2: A necessary and sufficient criterion for a column vector $\gamma^{(k)}$ of a pointed matrix Γ to belong to the frame of Γ (or to be a scalar multiple of a vector belonging to the frame) is that the matrix $\left[-\gamma^{(k)}\ \Gamma\right]$ have lineality 1.

The concept of frame is important in studying the intersection of two convex cones with common vertex. We state without proof:

Theorem 2.4.2: Let Γ_1 and Γ_2 be matrices of the same finite number N of rows, and of finite numbers K_1 and K_2 of columns respectively. Let Γ_1 and Γ_2 not be both solid. Then the intersection of the cones $Y_1 = Y_1(\Gamma_1)$ and $Y_2 = Y_2(\Gamma_2)$ is a convex cone of which the frame $\Gamma^*_{1,2}$ consists of a finite number of vectors, each contained in the boundary of Y_1 or of Y_2. The linealities of Y_1, Y_2 and the intersection Y_1Y_2 satisfy

$$(2.26)\qquad \lambda(Y_1Y_2) \leqq \lambda(Y_i), \qquad i = 1, 2.$$

The concept of boundary, occurring in Theorem 2.4.3, must be taken as referring to N-dimensional space. Thus, if $\rho(\Gamma_1) < N$, every point in Y_1 is on the boundary of Y_1.

Theorem 2.4.4: Let Γ_1 and Γ_2 have rank N. Then, for the dimensionality $d(Y_1Y_2)$ of the cone Y_1Y_2 referred to in Theorem 2.4.3 to equal the dimensionality N of the y-space, it is necessary and sufficient that the matrix obtained by adjoining $-\Gamma_1$ and Γ_2 be solid:

$$(2.27)\qquad \rho(\Gamma_1) = \rho(\Gamma_2) = N = \lambda\left[-\Gamma_1\ \ \Gamma_2\right].$$

In this case, Y_1 and Y_2 have a vector in common which is internal to both cones.

We shall also consider the case where Γ_2 is confined to a subspace of the y-space.

Theorem 2.4.5: *If*

(2.28) $\rho(\Gamma_1) = N, \qquad \rho(\Gamma_2) < N,$

then, for the dimensionality $d(Y_1 Y_2)$ *of the cone* $Y_1 Y_2$ *to equal the dimensionality* $d(Y_2) = \rho(\Gamma_2)$ *of* Y_2 *it is necessary and sufficient that the lineality space of*

(2.29) $\left[-\Gamma_1 \quad \Gamma_2\right]$

include all column vectors of Γ_2.

3. A Static Model With Capital Saturation

3.1 Constant Amounts of Activities. We shall assume that the rates x_k, $k = 1, \ldots, K$ remain constant over time for an indefinite period. This implies that the net flows y_n, $n = 1, \ldots, N$ of all commodities, as given by (1.5) or (2.8) are constants over time.

3.2 The Set Y of Achievable Points. A point y with coordinates $y_1, \ldots, y_N$ in the commodity space is called achievable if there exists a set of positive amounts of some (at least one) or all activities of which the joint effect is the net outputs $y_1, \ldots, y_N$ of the respective commodities. Mathematically, y is called achievable if there exists a point x in the activity space satisfying

$$(3.1) \qquad y = \Gamma x, \qquad x \geq 0.$$

The set of all achievable points y has already been denoted by Y. To indicate its dependence on Γ we may write

$$(3.2) \qquad Y = Y(\Gamma).$$

According to the analysis of section 2.2, Y is the convex hull of all half-lines

$$(3.3) \qquad y = \lambda \gamma^{(k)}, \qquad \lambda \geq 0 \text{ and scalar},$$

defined by the columns $\gamma^{(k)}$ of Γ, and therefore is a convex cone.

3.3 Fundamental Properties of Production. Before investigating the application of the notion of productive efficiency introduced in section 1.8 to the set of achievable points, it will be useful to formulate mathematical conditions on the set Y of achievable points, to express some properties of production which are presumed to be fundamental. The order in which these properties are introduced is suggested by reasons of mathematical exposition only. While the properties are formulated as conditions on Y, we shall derive their implications in terms of the matrix Γ of technical coefficients of all productive activities. We reiterate here the rank conditions (2.14) and (2.14a) saying that the set Y is not confined to a linear subspace of less than N dimensions and that the matrix Γ does not contain empty activities.

3.4 Irreversibility of Production. If labor is regarded as a primary input limited in total amount, rather than as the output of a "consumption" activity, then the empirical fact that labor is an input for all productive activities entails that the labor row of Γ contains only negative coefficients. If more than one kind of labor is distinguished, of which one is a primary input, the others intermediate products, the elements of Γ in the primary labor row are negative or zero. However, in any column (activity) with a zero element in that row, primary labor enters in some sense indirectly as the input to training activities (other columns) of which the output is an input to the activity in question. Since this indirectness may involve the telescoping of several training activities, it is desirable to specify mathematically what property of Γ is involved in what may be called the primary-input character of labor in all activities. While in the following postulate labor is not explicitly mentioned at all, the remarks just made about labor seem to provide one sufficient justification for its adoption.

Postulate A: It is impossible to find a set of positive amounts of some or all activities, of which the joint effect is a zero net output for all commodities.

Mathematically, the postulate says that there exists no vector x satisfying (2.15). The name irreversibility seems appropriate because, if (2.15) did have a solution x, then there would, according to (2.16), (2.17), exist two semi-positive vectors $x^{(1)}$, $x^{(2)}$ of activities, of which each would exactly reverse the effect of the other. From the analysis of section 2.2 we have

Theorem 3.4: The irreversibility postulate A is equivalent to the condition that the cone $Y(\Gamma)$ of achievable points is pointed.

Figure 3.4 illustrates five simple cases, involving three activities and two commodities only. In cases I and V production is irreversible. In cases II, III, and IV production is reversible. The diagrams are drawn in the two-dimensional commodity space. The activities are represented by the column vectors $\gamma^{(k)}$ $k = 1, 2, 3$, of Γ. The circular arcs serve to indicate the achievable point set. In cases II and IV this is a half-space, in case III the entire commodity space, in cases I and V the sides and the interior of an angle of less than 180°.

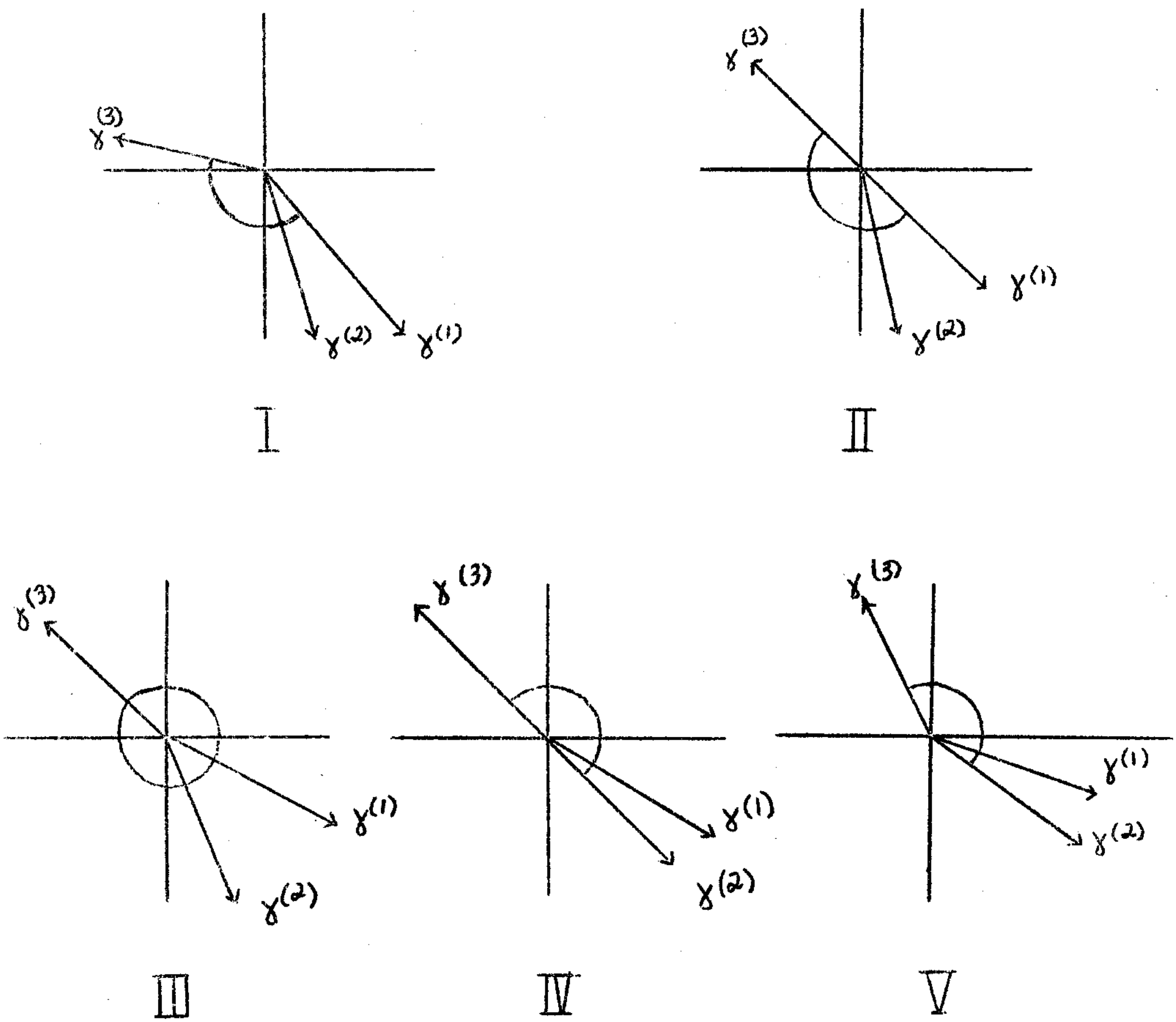

Figure 3.4

3.5 Impossibility of the Land of Cockaigne. The next postulate, while mathematically independent of the first, is related to it in its economic interpretation. To common sense it appears "even more true."

Postulate B: It is impossible to find a set of positive amounts of some or all activities, of which the joint product consists of a positive net output for at least one commodity, without causing a negative net output for at least one other commodity. This postulate admits cases I and II but rules out cases III, IV and V of figure 3.4 because the achievable point set contains points of the positive quadrant in these cases.

Mathematically, postulate B says that there exists no vector x satisfying

$$(3.4) \qquad y = \Gamma x \geq 0, \qquad x \geqq 0.$$

This is equivalent to requiring that

$$(3.5) \qquad \left.\begin{matrix} y \geq 0 \\ x \geqq 0 \end{matrix}\right\} \text{ excludes } \begin{bmatrix} -I & \Gamma \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = 0,$$

if I is the unit matrix of order N. For (3.5) to be true we must have either

$$(3.5a) \qquad \begin{bmatrix} y \\ x \end{bmatrix} \geq 0 \quad \text{excludes} \quad \begin{bmatrix} -I & \Gamma \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = 0, \text{ or}$$

$$(3.5b) \qquad \begin{cases} \left.\begin{matrix} y = 0 \\ x \geq 0 \end{matrix}\right\} \text{ permits } \begin{bmatrix} -I & \Gamma \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = 0, \text{ (which then implies } \Gamma x = 0\text{), for some } x\text{, but} \\ \left.\begin{matrix} y \geq 0 \\ x \geq 0 \end{matrix}\right\} \text{ excludes } \begin{bmatrix} -I & \Gamma \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = 0. \end{cases}$$

(The third logical possibility,

$$(3.5c) \qquad \begin{cases} y \geq 0 \\ x = 0 \end{cases}$$

does not permit $\begin{bmatrix} -I & \Gamma' \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = 0$ for any Γ and can therefore be disregarded).

(3.6) $\theta = \left[-I \quad \Gamma\right] = \left[-\theta^1 \quad \theta^1\Gamma\right] < 0,$

or, equivalently, a vector θ satisfying

(3.7) $\theta^1\Gamma < 0, \quad \theta > 0.$

Hence, the cone $Y = Y(\Gamma)$ is pointed and possesses a positive normal on the vertex in this case.

Case (3.5b) cannot occur if postulate A holds, but may be investigated for a moment because of its own interest. In this case $Y(\Gamma)$ is lineal, but not solid, because it does not intersect the non-negative set R_y except in the origin. Hence, it follows from theorems 2.2 that $Y(\Gamma)$ again possesses a positive normal on the vertex, i.e., a vector satisfying

(3.8) $\theta^1\Gamma \leq 0, \quad \theta > 0.$

The only difference with case (3.5a) is that the $<$ sign in (3.7) is replaced by a $\leq$ sign because of the lineality of Γ. Hence the normal θ is lateral instead of vertical.

These results can be summarized as follows:

Theorem 3.5.1: If postulate B (the impossibility of Cockaigne) holds, the cone $Y = Y(\Gamma)$ is non-solid and possesses a positive normal on the vertex.

Theorem 3.5.2: If postulates A and B hold simultaenously, the cone Y is pointed and possesses a positive normal on the vertex, which is a vertical normal.

3.6 The Possibility of Production Without Intermediate Products. For the formulation of postulates A and B, it was not necessary to specify which commodities are desired products, which are primary factors. Rather, these postulates expressed the necessity for the availability of primary factors, without identifying them with particular rows of Γ. In order to express in a postulate that production of some desired commodities is possible, it is necessary to specify in advance of what commodities net inputs are made available by nature. For simplicity we shall first assume that every commodity is either primary or desired, but not both. This rules out intermediate products. It also requires, for instance, that labor be treated by the device, described in section 1.9, of regarding manpower as a primary factor which is input to two kinds of activities, productive activities in the narrower sense, and a recreation activity producing desired leisure.

We then presuppose a partitioning of the rows of y and Γ

(3.9) $$y = \begin{bmatrix} y_D \\ y_P \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \Gamma_D \\ \Gamma_P \end{bmatrix}, \qquad y_D = \Gamma_D x, \quad y_P = \Gamma_P x,$$

given in advance. Of the net output vector of primary factors we require

(3.10) $$y_P \geqq \eta_P, \qquad \eta_P < 0.$$

On the vector y_D we may wish to impose the strong requirement that positive production of all desired commodities is possible,

(3.11) $$y_D > 0.$$

Or we may leave that question to be answered by analysis and insist only on the weaker requirement that it is possible to produce at least one commodity.poWe thus have the following postulates:

(3.12) $$y_D \geq 0.$$

(I am indebted to Herbert A. Simon for a discussion of this condition on Γ. A special case of this condition has been formulated by Simon and Hawkins [ ].

Postulate C_1 (The Strong Postulate): It is possible to find a set of amounts of some or all activities, of which the joint primary factor requirements are within the bounds set by nature, and of which the joint product consists of positive net outputs for all desired commodities.

Postulate C_2 (The Weak Postulate): It is possible to find a set of amounts of some or all activities, of which the joint primary factor requirements are within the bounds set by nature, and of which the joint product consists of non-negative net outputs for all desired commodities, including a positive net output for at least one such commodity.

To illustrate, let y_1 in Figure 3.4 correspond to a desired commodity, y_2 to a primary factor of which a flow of one unit is available. Then all five cases in Figure 3.4 satisfy both the weak and the strong postulate, because the achievable point set includes in each case points with positive values of y_1 above the line $y_2 = -1$.

A more instructive illustration is obtained if we reinterpret Figure 3.4 differently as follows. Let there be two desired commodities with net outputs y_1 and y_2, and one primary factor with net output y_3, limited by

(3.13) $$y_3 \geq \eta_3 = -1.$$

Let this primary factor be required for each of the three activities, and normalize each column of the coefficient matrix Γ by

(3.14) $$\gamma_{3k} = -1, \qquad k = 1, 2, 3.$$

The matrix can then be denoted

(3.15) $$\Gamma = \begin{bmatrix} \gamma^{(1)} & \gamma^{(2)} & \gamma^{(3)} \\ -1 & -1 & -1 \end{bmatrix} 1$$

where the $\gamma^{(k)}$ are vectors with two elements. Let Figure 3.4 now represent, in five possible cases, the configurations of the vector $\gamma^{(k)}$ in the space of the vector $y_D = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ of desired commodities. Then cases I and II are ruled out by either postulate C_1 or C_2, because no achievable point is found in or on the boundary of the positive quadrant. Cases III, IV and V are admitted by either postulate. Figure 3.5 throws another case (VI) admitted by both postulates, and a borderline case (VII) admitted by the weak postulate C_2 but excluded by the strong postulate C_1. (See next page for diagram.)

It should be added that if, in the present interpretation of Figures 3.4 and 3.5, we define the achievable point set in the desired commodity space as limited by (3.10), i.e. in this case by (3.13), but not by (3.11), that space is no longer represented by an entire angle, but instead by a convex polygon spanned on the origin and the end-points of the three vectors $\gamma^{(1)}$, $\gamma^{(2)}$, $\gamma^{(3)}$, as suggested in Figure 3.5 by dotted lines. The available primary factor input is fully used in any point on the dotted lines, and partially used in any point in which the origin enters with a positive weight. However, a negative net output of a non-primary commodity is possible only temporarily, if there is some stock to draw

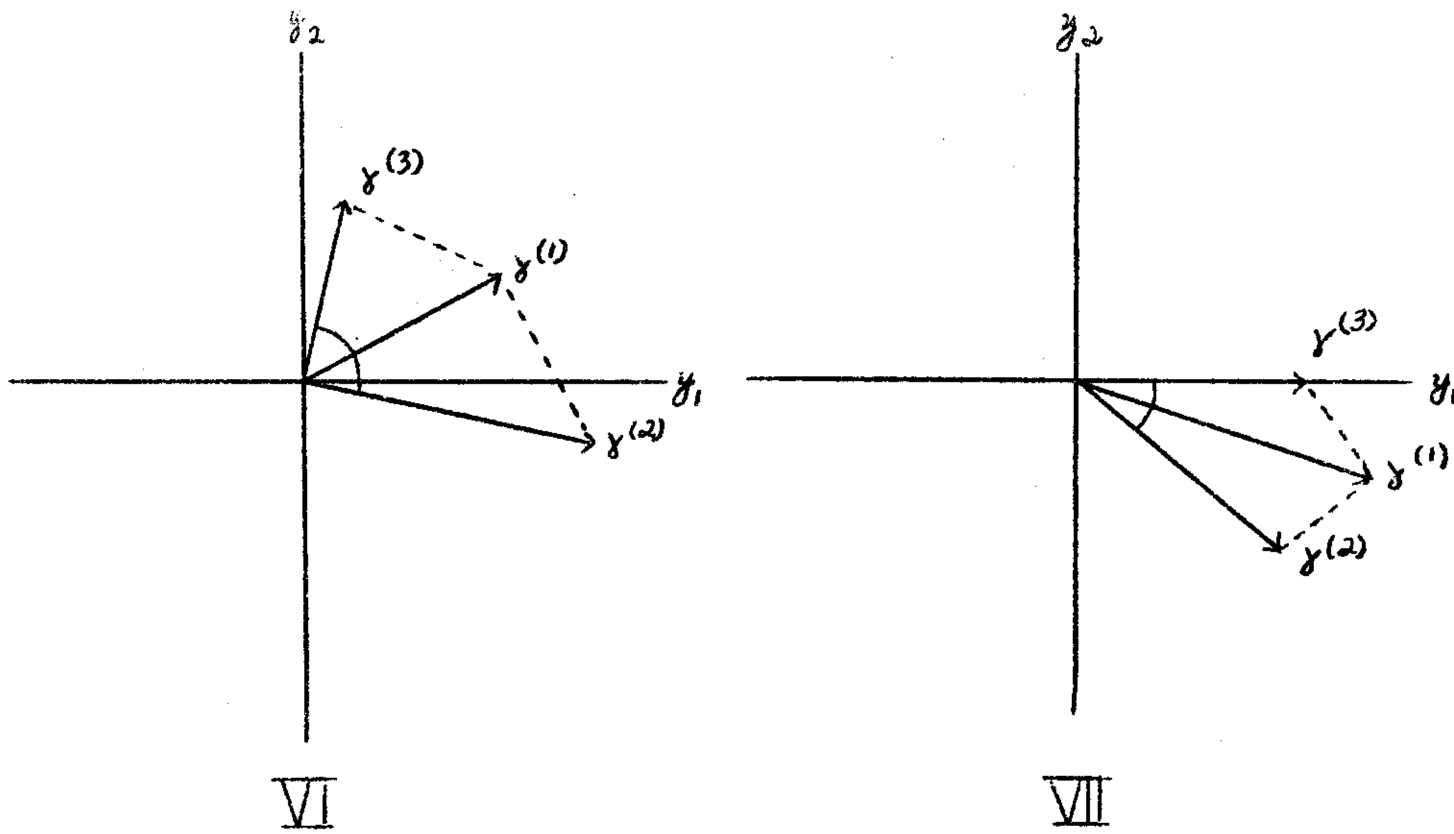

Figure 3.5

upon, and is impossible in a static model. If, more properly, we include the requirement (3.11) (i.e., in this case, $y_1 \geqq 0$ $y_2 \geqq 0$) in the definition of the achievable point set, that set contains only the origin in those cases in Figure 3.4 ruled out by the strong postulate, and consists only of a line segment in case VII of Figure 3.5.

Let us now translate the postulates in mathematical terms. The strong postulate C_1 says that there exists a solution y_D, y_P, x of (3.10) and

(3.16) $$y = \begin{bmatrix} y_D \\ y_P \end{bmatrix} = \begin{bmatrix} \Gamma_D \\ \Gamma_P \end{bmatrix} x = \Gamma x,\ y_D > 0,\ x \geqq 0,$$

we note that (3.10) does not restrict Γ under this requirement. If there exist vector x, y satisfying (3.16) but not (3.10),

(3.17) $$y_P = \Gamma_P\, x\ \eta_P,$$

then $\bar{x} = \lambda^{-1}x$, $\bar{y} = \lambda^{-1}y$, where λ is the largest of the (finite!) numbered y_n/η_n, with n referring to any primary factor, satisfy (3.10) as well as (3.16). Thus, postulate C_1 merely requires the existence of a solution x of

(3.18) $$y_D = \Gamma_D x > 0 \qquad x \geqq 0.$$

This is equivalent to the requirement that the cone $Y_D = Y_D(\Gamma_D)$ defined by

(3.19) $$y_D = \Gamma_D x, \qquad x \geqq 0,$$

have an internal vector in common with the cone P_{y_D} defined by

(3.20) $$y_D \geqq 0.$$

Postulate C_2 is equivalent to Y_D and P_{y_D} having a vector, not necessarily internal, in common. From Theorem 2.4.4 we derive

Theorem 3.5.1: The strong postulate C_1 of the possibility of production without intermediate products is equivalent to the requirement that the matrix

(3.21) $$\begin{bmatrix} -I_D & \Gamma_D \end{bmatrix}$$

be solid. We shall not attempt to find a similar criterion for the weak postulate C_2.

3.7 The Possibility of Production With Intermediate Products. The situation is somewhat more complicated in the presence of commodities which are neither desired nor given by nature. In this case, the vector y is partitioned according to

(3.22) $$y = \begin{bmatrix} y_D \\ y_I \\ y_P \end{bmatrix},$$

and the restriction

(3.23) $$y_I = 0$$

on the net output vector of intermediate products must be added to (3.10) and (3.11). This leads to the following possibility postulates:

Postulate D_1 (The Strong Postulate): It is possible to satisfy postulate C_1 in a manner involving zero net outputs of all intermediate products.

Postulate D_2 (The Weak Postulate): It is possible to satisfy postulate C_2 in a manner involving zero net outputs of all intermediate products.

Mathematically, postulate D_1 requires the existence of a solution x of

(3.24) $y_D = \Gamma_D x > 0, \qquad x \gtreqqless 0,$

satisfying

(3.25) $y_I = \Gamma_I x = 0.$

Postulate D_2 replaces (3.24) by

(3.26) $y_D = \Gamma_D x \geq 0, \qquad x \gtreqqless 0.$

For lack of time, we shall not attempt to pursue the mathematical contents of these postulates with the help of Theorem 2.4.5.

One interesting conclusion, however, can be drawn easily. Since both (3.2) and (3.2) are incompatible with $x \leqq 0$, either postulate requires the existence of a solution x of (3.24) and

(3.27) $x \geq 0.$

From this we conclude:

Theorem 3.8. A necessary condition for the satisfaction of either postulate D_1 or postulate D_2 is that each row of Γ_I contains at least one pair of elements of opposite sign. This expresses, in fact, the essence of the notion of intermediate produc .

Enough has been done to suggest that the postulates formulated so far can be expressed as lineality properties of matrices derived from the technology matrix Γ.
If this is true, there has emerged so far only one computational problem: to determine the lineality of a matrix, and if necessary the lineality space.

Simple examples show that the postulates A, B, and C_1 or C_2 or D_1 or D_2 are compatible.

[To be continued]