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A Mathematical Model of Production (final installment)

by

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Remarks on notation:

1. The solid cone is denoted $(-I \quad I)$ or briefly $(\pm I)$, where I is the unit matrix.
2. The subscripts P, I, F, used previously, will be replaced by subscripts

   pri for primary commodities

   int for intermediate products

   fin for (desired) final products
3. "$(A) \in (B)$", and "$(B) \ni (A)$" are synonymous with "(B) contains (A)".
4. Remember that

   ( ) denotes a cone

   [ ], or no brackets, a matrix or vector
5. $(A) \cap (B)$ denotes the intersection of (A) and (B), $(A) + (B)$ or $(A \quad B)$, the convex hull.
6. The terminology and notation used in LPC 410B (see p. 1) for the adjugate Γ^* of Γ is here retained for continuity. In a later, more definitive version of this study, a further change in terminology and notation is intended.

Contents: We first give a supplementary part to section 3 (see LPC 410) and then follow up with a new section 5. The title of section 3 is changed.

Correction: Theorem 3.8, LPC 410, p. 32, is to be renumbered 3.7.1.
Theorem 3.5.1, LPC 410, p. 7, is to be renumbered 3.6.1.

3. Fundamental properties of the technology matrix, continued.

3.7 The possibility of production with intermediate products, continued.
We shall now explore the mathematical content of the postulate D_1, which says that positive production rates of all final products are possible even if zero net outputs of all intermediate products are prescribed.

Again, (see LPC 410, p. 30), the limitations (3.10) on primary product availabilities do not restrict the technology matrix Γ under this postulate. Hence we require that there exist a solution y_{fin}, y_{int}, x, of

$$(3.24) \qquad y_{fin} = \Gamma_{fin}\, x > 0, \qquad x \geqq 0$$

and

$$(3.25) \qquad y_{int} + \Gamma_{int}\, x = 0 \ .$$

In the notation of convex cones we require that there exist a vector y_{fin} such that

$$(3.28) \qquad \begin{pmatrix} \Gamma_{fin} \\ \Gamma_{int} \end{pmatrix} \ni \begin{pmatrix} y_{fin} \\ 0 \end{pmatrix}, \qquad y_{fin} > 0 \ .$$

If (3.28) is satisfied, we also have

$$(3.29) \qquad \begin{pmatrix} -I_{fin} & \Gamma_{fin} \\ 0 & \Gamma_{int} \end{pmatrix} \ni \begin{pmatrix} -I_{fin} & y_{fin} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -I_{fin} & I_{fin} \\ 0 & 0 \end{pmatrix} ,$$

the second relation being a consequence of $y_{fin} > 0$.
Condensing (3.29), for postulate D_1 to be satisfied, it is necessary that

$$(3.30) \qquad \begin{pmatrix} -I_{fin} & \Gamma_{fin} \\ 0 & \Gamma_{int} \end{pmatrix} \ni \begin{pmatrix} \pm I_{fin} \\ 0 \end{pmatrix} .$$

We shall show that this condition is also sufficient.
If (3.30) is true, any vector $\overset{o}{y}_{fin}$ can be represented by

$$(3.31) \qquad \begin{bmatrix} \overset{o}{y}_{fin} \\ 0 \end{bmatrix} = \begin{bmatrix} -I_{fin} & \Gamma_{fin} \\ 0 & \Gamma_{int} \end{bmatrix} \cdot \begin{bmatrix} z \\ x \end{bmatrix}, \quad z \geqq 0, \quad x \geqq 0 \ .$$

which is equivalent to

(3.32) $$y^o_{fin} = -z + \Gamma_{fin}\, x, \qquad z \geqq 0, \quad x \geqq 0,$$

with x satisfying (3.28). Taking $y^o_{fin} > 0$, we conclude that

(3.33) $$y_{fin} \equiv y^o_{fin} + z = \Gamma_{fin}\, x > 0$$

satisfies both (3.24) and (3.25). This establishes:

Theorem 3.7.2. A Necessary and sufficient condition for the postulate D_1 of the possibility of production with intermediate products to be satisfied is that the cone

(3.34) $$\begin{pmatrix} -I_{fin} & \Gamma_{fin} \\ 0 & \Gamma_{int} \end{pmatrix}$$

shall contain the linear space of N_{fin} dimensions

(3.35) $$\begin{pmatrix} \pm I_{fin} \\ 0_{int} \end{pmatrix} .$$

3.8 Reduction of the technology matrix. The requirement that net output of intermediate products shall vanish is equivalent to intersecting the technologically achievable cone (Γ) with the linear space $y_{int} = 0$. Considered as a cone, this linear space is denoted by

(3.36) $$\begin{pmatrix} \pm I_{fin} & 0 \\ 0 & 0 \\ 0 & \pm I_{pri} \end{pmatrix} .$$

By property of , the intersection is again a cone, which we denote, omitting the "int"-coordinates, by

(3.37) $$(\bar{\Gamma}) \equiv \begin{pmatrix} \bar{\Gamma}_{fin} \\ \bar{\Gamma}_{pri} \end{pmatrix}$$

We shall say that $\bar{\Gamma}$ is obtained by a reduction of Γ which gives effect, once and for all, to the restriction $y_{int} = 0$. To indicate how $\bar{\Gamma}$ is obtained from Γ, we make use of certain properties* of the mapping of the space of cones (Λ) onto the space of their adjugate $(\Lambda)^*$. We have, interchanging the order of "int"- and "pri"-coordinates,

$$\begin{aligned}
\begin{pmatrix}\bar{\Gamma}\\ 0\end{pmatrix} &\equiv \begin{pmatrix}\bar{\Gamma}_{fin}\\ \bar{\Gamma}_{pri}\\ 0_{int}\end{pmatrix} \equiv \begin{pmatrix}\Gamma_{fin}\\ \Gamma_{pri}\\ \Gamma_{int}\end{pmatrix} \cap \begin{pmatrix}\pm I_{fin} & 0\\ 0 & \pm I_{pri}\\ 0 & 0\end{pmatrix} = \\
&= \begin{pmatrix}\Gamma^*_{fin} & 0\\ \Gamma^*_{pri} & 0\\ \Gamma^*_{int} & \pm I_{int}\end{pmatrix}^* = \begin{pmatrix}\Gamma^*_{fin} & 0\\ \Gamma^*_{pri} & 0\\ 0 & \pm I_{int}\end{pmatrix}^* = \\
&= \begin{pmatrix}\Gamma^*_{fin}\\ \Gamma^*_{pri}\\ 0\end{pmatrix}^* \cap \begin{pmatrix}\pm I_{fin} & 0\\ 0 & \pm I_{pri}\\ 0 & 0\end{pmatrix} = \\
&= \begin{pmatrix}\begin{pmatrix}\Gamma^*_{fin}\\ \Gamma^*_{pri}\end{pmatrix}^* & 0\\ & 0\\ 0 & \pm I_{int}\end{pmatrix} \cap \begin{pmatrix}\pm I_{fin} & 0\\ 0 & \pm I_{pri}\\ 0 & 0\end{pmatrix} .
\end{aligned} \tag{3.38}$$

Here Γ^*_{fin} is to be understood as the submatrix of a frame Γ^* of $(\Gamma)^*$ corresponding to the "fin"-coordinates, etc. From (3.38) we conclude (property)

$$(\bar{\Gamma}) = \begin{pmatrix}\Gamma^*_{fin}\\ \Gamma^*_{pri}\end{pmatrix}^* . \tag{3.39}$$

3.10* <u>Equivalence of the production possibility postulate D_1 before reduction with postulate C_1 after reduction.</u> We can now reason from the reduced technology matrix $\bar{\Gamma}$ as if no intermediate products had ever existed, and any results obtained must be true for the original technology matrix Γ under

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* These properties are given in an Appendix.

the restriction $y_{int} = 0$. In particular, the condition for the possibility of production with intermediate products (postulate D_1) obtained in Theorem 3.7.2, as applied to the matrix Γ, must be equivalent to the condition for the possibility of production without intermediate products (postulate C_1) obtained in Theorem 3.6.1, (erroneously numbered 3.5.1 in LPC 410), as applied to the matrix $\bar{\bar{\Gamma}}$. It may be useful to give an explicit proof of this equivalence. The condition of Theorem 3.7.2 is

$$(3.40) \qquad \begin{pmatrix} -I_{fin} \\ 0 \end{pmatrix} + \begin{pmatrix} \Gamma_{fin} \\ \Gamma_{int} \end{pmatrix} \ni \begin{pmatrix} \pm I_{fin} \\ 0 \end{pmatrix} .$$

This implies, and is implied in,

$$(3.41) \qquad \begin{pmatrix} -I_{fin} & 0 \\ 0 & 0 \\ 0 & \pm I_{pri} \end{pmatrix} + \begin{pmatrix} \Gamma_{fin} \\ \Gamma_{int} \\ \Gamma_{pri} \end{pmatrix} \ni \begin{pmatrix} \pm I_{fin} & 0 \\ 0 & 0 \\ 0 & \pm I_{pri} \end{pmatrix} ,$$

with which the following equivalent condition in the space of adjugate cones is associated:

$$(3.42) \qquad \begin{pmatrix} I_{fin} & 0 \\ 0 & \pm I_{int} \\ 0 & 0 \end{pmatrix} \cap \begin{pmatrix} \Gamma^*_{fin} \\ \Gamma^*_{int} \\ \Gamma^*_{pri} \end{pmatrix} \in \begin{pmatrix} 0 \\ \pm I_{int} \\ 0 \end{pmatrix} .$$

No restriction on Γ is expressed by the middle row of (3.42), which can therefore be omitted.

$$(3.43) \qquad \begin{pmatrix} I_{fin} & 0 \\ 0 & 0 \end{pmatrix} \cap \begin{pmatrix} \Gamma^*_{fin} \\ \Gamma^*_{pri} \end{pmatrix} = 0$$

Taking adjugate cones on both sides, and using (3.39), this is again equivalent to

$$(3.44) \qquad \begin{pmatrix} -I_{fin} & 0 \\ 0 & \pm I_{pri} \end{pmatrix} + \begin{pmatrix} \bar{\bar{\Gamma}}_{fin} \\ \bar{\bar{\Gamma}}_{pri} \end{pmatrix} = \begin{pmatrix} \pm I_{fin} & 0 \\ 0 & \pm I_{pri} \end{pmatrix} .$$

The last row does not restrict $\bar{\bar{\Gamma}}$, whereas the first row expresses the condition of Theorem 3.6.1 as applied to $\bar{\bar{\Gamma}}$.

5. The efficient point set under availability restrictions on primary factors.

5.1 Redefinition of the efficient point concept. We shall now assume that primary commodities are available in rates of flow limited by (1.6), which we restate here

(5.1) $$y_{pri} \geqq \eta_{pri}, \quad \eta_{pri} < 0 .$$

These limits cannot, it is assumed, be exceeded at any cost, but within these limits, an increase in the input (a decrease in the negative net output y_n) of any primary commodity is not regarded as in any way undesirable or costly.

We define as an (economically) achievable point y any point satisfying the conditions (5.1) and

(5.2) $$y = \begin{pmatrix} y_{fin} \\ y_{int} \\ y_{pri} \end{pmatrix} = \Gamma x, \; x \geqq 0, \; y_{fin} \geqq 0, \; y_{int} = 0 .$$

The restriction (5.2) on y_{int} is equivalent to a restriction

(5.3) $$y_{int} \geqq 0,$$

expressing that intermediate goods are not available in nature, whenever Γ contains a submatrix

(5.4) $$\begin{bmatrix} 0_{fin} \\ -I_{int} \\ 0_{pri} \end{bmatrix}$$

introducing costless disposal activities on all intermediate commodities. If such activities are not or not all present in the technology matrix, the restriction (5.2) on y_{int} is stronger than (5.3), and should be used to express the necessity of disposal of waste products.+

+It would be easily possible to add to (5.2) a condition $y_{pri} \leqq 0$ that the net effect of all production cannot be to add a positive amount to the availability flow of any (alternatively: of some) primary commodities without creating a need for disposal activities that consume resources--if such a condition were thought desirable.

An (economically) achievable point y is called <u>efficient</u> whenever an increase in one of its final commodity coordinates (in the net output of one final good), within the availability limitations on primary commodities and the zero-net-output restriction on intermediate commodities, can be achieved only at the cost of a decrease in some other final coordinate (the net output of another final good). Expressed mathematically, y is efficient if for any $\bar{\bar{y}}$ that satisfies (5.1) and (5.2) we have

$$(5.5) \qquad \bar{\bar{y}}_{fin} - y_{fin} \not\geq 0$$

As explained in section (3.8), the condition (5.2) can be simplified by replacing by the frame of the cone

$$(5.6) \qquad (\bar{\Gamma}) \equiv \begin{pmatrix} \tilde{\Gamma}^*_{fin} \\ \tilde{\Gamma}^*_{pri} \end{pmatrix}^*, \quad \text{with } (\tilde{\Gamma}) \equiv (\Gamma) \cap \begin{pmatrix} I_{fin} & 0 & 0 \\ 0 & \pm I_{int} & 0 \\ 0 & 0 & \pm I_{pri} \end{pmatrix}$$

being obtained from (Γ) by the restriction $y_{fin} \geqq 0$. We shall from now assume that this <u>reduction</u> of Γ has already been carried out, and omit the bar from $\bar{\Gamma}$, except in those sections where the implications of our results for the intermediate commodity space are explored. With that understanding, the definition of an efficient point is simplified as follows. An achievable point, i.e., a point

$$(5.7) \qquad y = \begin{pmatrix} y_{fin} \\ y_{pri} \end{pmatrix}$$

satisfying the conditions

$$(5.8) \qquad y = \Gamma x = \begin{pmatrix} \Gamma_{fin} \\ \Gamma_{pri} \end{pmatrix} x, \quad x \geqq 0, \quad y_{fin} \geqq 0, \quad y_{pri} \geqq \eta_{pri},$$

is now called efficient if, for any $\bar{\bar{y}}$ satisfying the same restrictions, the relation (5.5) is satisfied.

5.2 <u>Reformulation of the necessary and sufficient condition for efficiency.</u>

We shall now prove a theorem which is a counterpart to theorem 4.1 of section 4.3. Let us assume that y is an efficient point. Consider the cone (C) consisting of all vectors

$$(5.9)\qquad \bar{y} - y, \quad \bar{y} \in (\Gamma),$$

and their positive scalar multiples. Geometrically, this is the cone "projecting" (Γ) from the vertex y, translated so that its vertex falls in the origin. We partition the coordinates corresponding to primary commodities into two sets according to whether or not the availability limit on each commodity is or is not reached in the point y,

$$(5.10)\qquad y_{pri} \equiv \begin{pmatrix} y_{pri=} \\ y_{pri>} \end{pmatrix}, \quad y_{pri=} = \eta_{pri=}, \qquad y_{pri>} > \eta_{pri>}.$$

We shall first show that the efficiency of y is expressed in the condition

$$(5.11)\qquad (\bar{C}) \equiv (C) \cap \begin{pmatrix} I_{fin} & 0 & 0 \\ 0 & I_{pri=} & 0 \\ 0 & 0 & \pm I_{pri>} \end{pmatrix} \in \begin{pmatrix} 0 & 0 \\ \pm I_{pri=} & 0 \\ 0 & \pm I_{pri>} \end{pmatrix}.$$

Each vector $\bar{c}$ of $(\bar{C})$, selected from (C) by the intersection in the middle member of (5.11), is a positive scalar multiple

$$(5.12)\qquad \bar{c} = \lambda \cdot (\bar{y} - y)$$

of a vector $(\bar{y} - y)$ in which $\bar{y}$ has the following properties: since $\bar{c} \in (C)$, this value of $\bar{y}$ can be made to satisfy (5.8) by suitable choice of λ; in addition, since $\bar{c}_{fin} \in (I_{fin})$, and therefore

$$(5.13)\qquad (\bar{y}_{fin} - y_{fin}) \in (I_{fin}),$$

this value of $\bar{y}$ satisfies either

$$(5.14)\qquad \bar{y}_{fin} - y_{fin} = 0$$

or the negation

$$(5.15)\qquad \bar{y}_{fin} - y_{fin} \geq 0$$

of (5.5). Now the condition (5.11) requires y to be such that the restriction, contained in (5.11),

(5.16) $\quad \bar{y}_{pri=} - y_{pri=} \in (I_{pri})$

or, in view of (5.10), the equivalent restriction,

(5.17) $\quad \bar{y}_{pri=} \geqq \eta_{pri=}$

shall imply that (5.14) is satisfied by all, (5.15) by none, of the vectors $(\bar{y} - y)$ of $(\bar{C})$. This condition is obviously equivalent to the definition of efficiency of y, because the restriction

(5.18) $\quad \bar{y}_{pri>} \geqq \eta_{pri>}$

also required in that definition can always be satisfied, in view of (5.10), by proper choice of λ. For that reason, (5.11) imposes no restriction on $\bar{y}_{pri>}$.

The condition, equivalent to (5.11), in the space of adjugate cones, is, by the rules and ,

(5.19) $$(\bar{C})^* = (C)^* + \begin{pmatrix} -I_{fin} & 0 \\ 0 & -I_{pri=} \\ 0 & 0 \end{pmatrix} \ni \begin{pmatrix} \pm I_{fin} \\ 0 \\ 0 \end{pmatrix}$$

The first conclusion from this is that $(C)^*$ cannot be empty, hence (C) cannot be solid, hence y cannot be an internal point of the achievable cone (Γ) even before its truncation by (5.1). It follows, in view of the definition (5.9) of (C), that all vectors p of $(C)^*$ have the direction of normals to (Γ) in y.

Secondly, (5.19) implies that for each vector z satisfying

(5.21) $\quad z_{pri} = 0$

there exist vectors p, w, satisfying

(5.22) $\quad p \in (C)^*, \quad w_{fin} \gtrless 0, \quad w_{pri=} \leqq 0, \quad w_{pri>} = 0$

such that

(5.23) $\quad p + w = z.$

By taking $z_{fin} > 0$ we read from this that $(C)^*$ contains some vector p satisfying

(5.24) $\quad p_{fin} > 0, \quad p_{pri=} \geqq 0, \quad p_{pri>} = 0.$

Conversely, if $(C)^*$ contains a vector p satisfying (5.24), any vector z satisfying (5.21) can be obtained from

(5.25) $\quad \lambda p + w = z, \qquad \lambda$ a positive scalar,

by proper choice of λ and a vector w satisfying (5.22). Hence our condition (5.19) is satisfied. This establishes the following

Theorem 5.2. A necessary and sufficient condition that an (economically) achievable point y, as defined in (5.7), be efficient, as defined in the sentence following (5.8), is that y is a boundary point of the (untruncated) achievable cone (Γ) possessing a normal p which satisfies (5.24), i.e., which has positive components for all final commodities, nonnegative components for all primary commodities whose availability limit is reached in y, and zero components for all primary commodities whose availability limit is not reached in y.

5.3 Efficient points and maximization of a linear function of final outputs.

We shall now demonstrate that efficient points are points obtained by maximizing a given linear function of final outputs under the availability restrictions on primary commodities. Let

(5.26) $\quad p_{fin} > 0$

be a positive vector designated in advance, and suppose that the linear function

(5.27) $\quad L \equiv p'_{fin} \, \bar{y}_{fin}$

reaches a maximum in $\bar{y} = y$ under the restrictions

(5.28) $\quad \bar{y} \in (\Gamma), \qquad \bar{y}_{pri} \geqq \eta_{pri} \quad .$

Then (5.28) implies,

(5.29) $\quad p'_{fin} \, \bar{y}_{fin} \leqq p'_{fin} \, y_{fin} \, ,$

where (5.28) is also satisfied by y.

In view of (5.26), this is incompatible with

(5.30) $\qquad \bar{y}_{fin} \gtrsim y_{fin}$

It follows that y is an efficient point as defined in the sentence following (5.8).

We shall show also that, in particular, p_{fin} can be selected as part of a normal p to (Γ) in y, which satisfies the requirements (5.24) of Theorem 5.2. The fact that the linear function $p'_{fin}\ \bar{y}_{fin}$ is maximized by a vector y with which the partitioning (5.20) of "pri"-coordinates is associated, can be expressed in the condition

$$(5.31) \qquad (C) \ \cap \begin{pmatrix} \pm I_{fin} & 0 & 0 \\ 0 & I_{pri=} & 0 \\ 0 & 0 & \pm I_{pri>} \end{pmatrix} \in \begin{pmatrix} p_{fin} \\ 0 \\ 0 \end{pmatrix}^{*}$$

In words: Of each vector $\bar{y} - y$ of (C) which satisfies (5.17), (and hence, in a neighborhood of $\bar{y} - y = 0$, all availability restrictions (5.1) on primary commodities), the subvector $\bar{y}_{fin} - y_{fin}$ in the "fin"-space must satisfy $p'_{fin}\ (\bar{y}_{fin} - y_{fin}) \leqq 0$. By the rules of taking adjugates of sums and intersections, (5.31) is equivalent to

$$(5.32) \qquad \begin{pmatrix} (C^{*})_{fin} & 0 \\ (C^{*})_{pri=} & -I_{pri=} \\ (C^{*})_{pri>} & 0 \end{pmatrix} \ni \begin{pmatrix} p_{fin} \\ 0 \\ 0 \end{pmatrix}$$

In words: Of each normal $\bar{p}$ to (Γ) in y satisfying the second and third conditions (5.24), take the subvector $\bar{p}_{fin}$. The set of subvectors so obtained contains p_{fin}. This establishes the contention.

Conversely, let y be an efficient point. Then, by Theorem 5.2, (Γ) contains a normal p in y satisfying (5.24). It follows that $\bar{y} \in (\Gamma)$ implies

(5.31) $\qquad p'\ \bar{y} \leqq p'\ y ,$

or, in view of the third relation in (5.24),

(5.32) $\qquad p'_{fin}\ \bar{y}_{fin} + p'_{pri=}\ \bar{y}_{pri=} \leqq p'_{fin}\ y_{fin} + p'_{pri=}\ y_{pri=}$

From the second relation (5.24) we derive, remembering that (5.10) determines the partitioning of the "pri"-coordinates into "pri=" and "pri>", that $\bar{y}_{pri} \geqq \eta_{pri}$ implies

$$(5.33) \qquad p'_{pri=}\,\bar{y}_{pri=} \geqq p'_{pri=}\,\eta_{pri=} = p'_{pri=}\,y_{pri=} \;.$$

From (5.32) and (5.33) it follows that $\bar{y} \in (\Gamma)$, $\bar{y}_{pri} \geqq \eta_{pri}$ imply

$$(5.34) \qquad p'_{fin}\,\bar{y}_{fin} \leqq p'_{fin}\,y_{fin} \,,$$

which means that y is a restricted maximum of the kind defined above. These results can be summarized as follows.

Theorem 5.3. A necessary and sufficient condition that a net output point y be an efficient point is that it be a maximum of a linear function $L \equiv p'_{fin}\,\bar{y}_{fin}$ of $\bar{y}$ with a coefficient vector $[p'_{fin}\;\; 0'_{pri}]$ in which $p_{fin} > 0$, where $\bar{y}$ is restricted to that part of the achievable cone (Γ) in which the primary-commodity availability restrictions (5.1) are satisfied. The vector p_{fin} is a subvector of a normal p to (Γ) in y which satisfies the conditions (5.24) of Theorem 5.2, if y defines the partitioning (5.10) of the primary commodities underlying those conditions.

5.4 Existence of an efficient point. We shall make use of Theorem 5.3 to derive conditions for the existence of an efficient point y outside the origin. We are again interested only in economically achievable efficient points as defined by (5.2). This can be expressed by letting the economically achievable cone (Γ) be obtained from the technologically achievable cone $(\tilde{\Gamma})$, say, by the intersection

$$(5.35) \qquad (\Gamma) \equiv (\tilde{\Gamma}) \cap \begin{pmatrix} I_{fin} & 0 \\ 0 & \pm I_{pri} \end{pmatrix} \,.$$

Since the availability restrictions (5.1) on primary commodities are ineffective in a neighborhood of the origin, the question whether or not the origin trivially is an efficient point can be settled in disregard of those restrictions. The origin is efficient if and only if

(5.36) $\qquad (\Gamma_{fin}) \equiv (\tilde{\Gamma}_{fin}) \cap (I_{fin}) = (0)$

For, as soon as $(\tilde{\Gamma}_{fin})$ contains a vector $\gamma_{fin} \geq 0$, the function $L \equiv p'_{fin}\,\bar{y}_{fin}$, for any $p_{fin} > 0$, takes on positive values on the halfline

$$\bar{y}_{fin} = \lambda \;\gamma_{fin}, \qquad \lambda \text{ a positive scalar},$$

in any neighborhood of the origin, in excess of the value 0 assumed in the origin. On the other hand, if (5.36) is satisfied, the function L is constant in (Γ), and therefore assumes a maximum in the origin. The reader will recognize the equivalence between the negation of condition (5.36) and the weak postulate C_2 of the possibility of production, given in section 3.6. We shall henceforth assume that that postulate is satisfied by (Γ).

A maximum of a linear function $L = p'_{fin}\,\bar{y}_{fin}$ on the set of points satisfying (5.28) will be reached in a finite point y whenever the set (5.28) is bounded. This set is the intersection of two cones, one (Γ) with its vertex in the origin and given as the convex hull of halflines out of the origin, the other, to be called (H, η_{pri}), with its vertex in

(5.37) $\qquad \begin{bmatrix} 0 \\ \eta_{pri} \end{bmatrix}$,

given as the intersection of halfspaces having that point in their boundary. Since by theorem , , (Γ) is also an intersection of halfspaces (through another vertex), the set

(5.38) $\qquad D \equiv (\Gamma) \cap (H, y_{pri})$

to which $\bar{y}$ is restricted is an intersection of halfspaces, and as such is a closed convex body. Now suppose that D contains an entire halfline

(5.39) $\qquad \bar{y} = \lambda\,\gamma, \; \gamma \neq 0, \; \lambda \text{ scalar}, \qquad 0 \leq \lambda \leq \infty$

out of the origin. Then we must have

(5.40) $\qquad \gamma_{pri} \geq 0$,

because otherwise the second relation in (5.28) would prevent λ from ranging to ∞. However, (5.40) taken together with the inequality, implied in (5.35),

(5.41) $\quad \gamma_{fin} \geqq 0$

of section 3.5,
contradicts postulate B /since (5.39) precludes $\gamma = 0$. Hence, if that postulate holds, D does not contain a halfline out of the origin.

It has been proved by Gerstenhaber (see Appendix) that a closed convex body D containing a point O such that S contains no halfline out of O is bounded. It follows that D contains an efficient point. The following theorem summarizes our results:

<u>Theorem 5.4.</u> <u>If postulate</u> B <u>of section</u> 3.5 <u>(the impossibility of Cockaigne)</u> <u>is satisfied+ by the technology matrix</u> Γ, <u>and if availability restrictions</u> (5.1) <u>are imposed in the primary commodity space, then there exists at least one</u> <u>efficient point</u> y_{fin} <u>in the final commodity space. The origin is not efficient</u> <u>whenever postulate</u> C_2 <u>of the possibility of production is satisfied.</u>

+It has been kindly pointed out to me by A.W. Tucker that while postulate B is sufficient for the existence of an efficient point, the following weaker postulate $\bar{B}$ is necessary and sufficient: there is no $x \gtrless 0$ such that $y_{fin} = \Gamma_{fin}\, x \geq 0$, $y_{pri} = \Gamma_{pri}\, x \gtreqless 0$. Under this postulate, the availability restrictions $y_{fin} \geqq 0$, $y_{pri} \geqq \eta_{pri} < 0$ exclude a bonanza in final commodities, but do not necessarily exclude one in primary commodities only (since $y_{fin} = 0$ might permit $y_{pri} \geq 0$). Because there seems to be little economic meaning in a technology matrix which satisfies $\bar{B}$ but not B, we have not used postulate $\bar{B}$.

5.5 Properties of the efficient point set. We shall not attempt a topological analysis of the efficient point set under availability restrictions of primary commodities. A few remarks may be made regarding the nature of the facets of the efficient point set. An (open) facet of the efficient point set (briefly: an efficient facet) may be defined as a set of efficient points y which (a) lie on the same (open) facet $)\Gamma_+($ of the achievable cone (Γ) and (b) have the same partitioning (5.10) of y_{pri}.

In the analysis of section 4, the most general case as regards Γ allowed the dimensionality of an efficient facet to be as high as N - 1. It will be clear that this cannot be the case, except "by accident", that is, for specially chosen matrices Γ, if availability restrictions are introduced. A (N-1)-facet of (Γ) possesses one unique normal (p). However, in order to be eligible for containing efficient points with a given partitioning (5.10) of y_{pri}, the possession of a normal p with $N_{pri>}$ vanishing components is required. Counting only restrictions that have the form of equalities, we find that on any efficient facet the number of restrictions $p_{pri>} = 0$ on p plus the number of restrictions $y_{pri=} = \eta_{pri=}$ on y equals the number N_{pri} of primary commodities. This would lead one to expect $N-1-N_{pri}$ to be the highest dimensionality of an efficient facet, except possibly for "special" choices of Γ.

The interpretation of the components of p as technological substitution ratios within the efficient point set is therefore no longer available for all commodities. For instance, from an internal point y of an efficient facet, such that for all its normals $p_{pri=} > 0$ holds, the values of the "net outputs" comprised in $y_{pri=}$ can be neither increased nor decreased within the efficient facet of which y is an internal point. They cannot be increased without violating the conditions for efficiency, and they cannot be decreased without violating the availability restrictions. Nevertheless, the positive "prices" $p_{pri=}$ are found to apply to these commodities in the analysis. In section 5.7 below we shall give another interpretation of the "price vector" p, which applies independently of whatever restrictions the efficient facet may place on the components of y. Since this interpretation applies also to intermediate commodities, we shall first study the extension of a normal p to Γ in the space of final and primary commodities to that of intermediate commodities.

5.6 The prices of intermediate commodities. We resume the notation $\bar{\Gamma}$ of Section 3.8 (LPC 410C) for a technology matrix in the space of final and primary commodities, obtained from an original technology matrix Γ in the space of final, primary and intermediate commodities (partitioned in that order), by a reduction based on the restrictions

$$(5.42) \qquad y_{int} = 0$$

Similarly we use the notation $\bar{y}$, $\bar{p}$ for a boundary point of $(\bar{\Gamma})$ and a normal to it, respectively. The present analysis applies to any boundary point $\bar{y}$ and any normal $\bar{p}$ to $(\bar{\Gamma})$ in $\bar{y}$, whether or not a normal satisfying the conditions (4.d) or (5.24) for efficiency, according to some appropriate definition, exists.

The condition that $\bar{p}$ is a normal to $(\bar{\Gamma})$ in $\bar{y}$ is expressed by

$$(5.43) \qquad \bar{p} \in (-\bar{y} \quad \Gamma)^* .$$

We shall prove:

Theorem 5.6. Any normal $\bar{p}$ to the reduced achievable cone $(\bar{\Gamma})$ in a point $\bar{y}$ can be supplemented by a vector p_{int} to a normal

$$(5.44) \qquad \begin{pmatrix} \bar{p} \\ p_{int} \end{pmatrix} = p \in (-y \quad \Gamma)^*$$

to the original achievable cone (Γ), from which $(\bar{\Gamma})$ is derived by (3.38), in the point

$$(5.45) \qquad y \equiv \begin{bmatrix} y \\ 0 \end{bmatrix} \equiv \begin{bmatrix} y_{fin} \\ y_{pri} \\ 0_{int} \end{bmatrix} .$$

Conversely, if p is such a normal to (Γ^*), its subvector $\bar{p}$ is a normal to ($\bar{\Gamma}$) in $\bar{y}$. To prove the first contention, let $\bar{p}$ satisfy (5.43). Then

$$(5.46) \qquad \begin{pmatrix} p_{fin} \\ p_{pri} \end{pmatrix} \equiv \bar{p} \in (-\bar{y})^* \cap (\bar{\Gamma})^* = \begin{pmatrix} -y_{fin} \\ -y_{pri} \end{pmatrix}^* \cap \begin{pmatrix} \Gamma^*_{fin} \\ \Gamma^*_{pri} \end{pmatrix} \in \begin{pmatrix} \Gamma^*_{fin} \\ \Gamma^*_{pri} \end{pmatrix}$$

by (3.38). Hence, there exists a vector $q \geqq 0$ such that

$$(5.47) \qquad \begin{bmatrix} p_{fin} \\ p_{pri} \end{bmatrix} = \begin{bmatrix} \Gamma^*_{fin} \\ \Gamma^*_{pri} \end{bmatrix} q .$$

We define

$$(5.48) \qquad p_{int} \equiv \Gamma^*_{int} q,$$

and show that

$$(5.49) \qquad p \equiv \begin{bmatrix} p_{fin} \\ p_{pri} \\ p_{int} \end{bmatrix} = \Gamma^* q$$

satisfies (5.44), as follows. The condition (5.44) can be written as

$$(5.50) \qquad p \equiv \begin{pmatrix} \bar{p} \\ p_{int} \end{pmatrix} \in \begin{pmatrix} -\bar{y} \\ 0 \end{pmatrix}^* \cap (\Gamma)^* = \left(\begin{pmatrix} -y_{fin} \\ -y_{pri} \end{pmatrix}^* \begin{matrix} 0 \\ 0 \end{matrix} \atop \begin{matrix} 0 & \pm I_{int} \end{matrix} \right) \cap \begin{pmatrix} \Gamma^*_{fin} \\ \Gamma^*_{pri} \\ \Gamma^*_{int} \end{pmatrix}$$

This condition is satisfied on account of (5.46), because the addition of the "int"-rows in (5.50) only requires p_{int} to be derivable from Γ^*_{int} by the same weight vector q by which $\bar{p}$ is derived from $\begin{bmatrix} \Gamma^*_{fin} \\ \Gamma^*_{pri} \end{bmatrix}$, as specified in (5.48).

To prove the second contention, let p satisfy (5.50). Then $\bar{p}$ satisfies (5.43).

It should be emphasized that there is no restriction on the sign of the components of p_{int}. Negative prices adhere to by-products of which "too much" is produced in the process of producing other positively priced things, while the disposal of the excess consumes positively priced commodities. Negative values can occur only for those components of p_{int} corresponding to commodities for which no costless disposal activities are present in the technology matrix Γ.

5.7 Interpretation of the price vector when net output variations are restricted. It has already been remarked (section 5.5), that when the availability restrictions on primary factors and the efficiency requirements limit the dimensionality of the efficient point set in the neighborhood of an efficient point y, the interpretation of the price vector p as indicating substitution ratios in efficient production is no longer applicable to all commodities. However, it remains possible to attach a "price" interpretation to a normal p to (Γ) in an efficient point y. Let us imagine that, through communication with an economy outside that described by the technology matrix Γ, a possibility is provided to trade any commodity against any other at constant relative prices given by a price vector $\tilde{p}$. It is natural to interpret such prices as "efficiency prices" at the point y whenever the net output vector y cannot be improved upon (by increasing one component without decreasing any others) through the use of this trading opportunity in combination with productive activities. We shall show that such improvement is not possible if and only if $\tilde{p}$ is a normal to (Γ) in y satisfying the requirements (5.24) of Theorem 5.2.

The possibility to trade with an outside world at prices $\tilde{p}$ can be introduced formally by adding to (Γ) a set of "exchange activities." Let Π represent the matrix

$$(5.51) \qquad \Pi = \begin{bmatrix} -\tilde{p}_2 & -\tilde{p}_3 & \cdots & -\tilde{p}_N \\ \tilde{p}_1 & 0 & \cdots & 0 \\ 0 & \tilde{p}_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \tilde{p}_1 \end{bmatrix}, \text{ where } \tilde{p}_1 > 0,$$

and let the technology matrix, extended by exchange activities, be

(5.52) $\quad \tilde{\Gamma} \equiv [-\Pi \;\; \Pi \;\; \Gamma]$.

The point y is efficient in the original technology Γ if and only if the cone

(5.53) $\quad (P) \equiv (C)^* = (-y \;\; \Gamma)^*$

of normals to (Γ) in y contains a vector p satisfying the requirements (5.24) of Theorem 5.2. Assume this to be the case, and consider the question whether y is efficient in the new technology (5.52). To answer this, the same criterion must be applied to the cone

(5.54) $$(\tilde{P}) \equiv (-y \;\; \tilde{\Gamma})^* = (-y \;\; -\Pi \;\; \Pi \;\; \Gamma)^* = (-y \;\; \Gamma)^* \cap (-\Pi \;\; \Pi)^*$$
$$= (P) \cap (-\Pi \;\; \Pi)^*$$

of normals to $(\tilde{\Gamma})$ in y. However, since Π by (5.51) has the rank N-1 and satisfies

(5.55) $\quad \tilde{p}' \;\; \Pi = 0,$

we have

(5.56) $\quad (-\Pi \;\; \Pi)^* = (-\tilde{p} \quad \tilde{p})$.

Hence the cone $(\tilde{P})$ in (5.54) is one of the following four cones

(5.57) $\quad (0), \;\; (-\tilde{p}), \;\; (\tilde{p}), \;\; (-\tilde{p} \quad \tilde{p})$.

We recall that the criterion of efficiency of y in $(\tilde{\Gamma})$ is whether or not $(\tilde{P})$ contains a vector p satisfying (5.24). Since $-\tilde{p}$ does not satisfy (5.24), y is efficient in $(\tilde{\Gamma})$ if and only if $\tilde{p}$ is in (P) (is a normal to (Γ) in y) and satisfies (5.24). This completes the proof of

<u>Theorem 5.7.</u> <u>A necessary and sufficient condition that an efficient point y shall remain efficient after the addition, by (5.52), to the technology, of exchange activities (5.51) at constant relative prices $\tilde{p}$, is that $\tilde{p}$ be a normal to (Γ) in y satisfying the requirements (5.24) of Theorem 5.2.</u>

5.8 <u>The attainment of efficiency under a regime of decentralized decisions.</u>

An important use of the efficiency prices p arises if decisions about the levels x of the activities are made separately for each activity. So far the conditions for efficiency have been discussed without reference to the institutional arrangements under which decisions about the components of x are arrived at. One possible use of our results would be for a centralized decision-making agency to possess all the information that goes into the technology matrix Γ, and to choose an activity vector x such that by the proper mathematical criteria the output vector $y = \Gamma x$ is an efficient point. An opposite extreme is a situation in which knowledge of each column γ_k of Γ is available only to the individual who determines the level x_k of that activity. Even in this extreme case of decentralization, efficiency is still attainable if we assume that information about the price vector p is available to all managers. We shall consider an allocation model in which the various decisions which together determine the activity vector x are parcelled out to a number of individuals or administrative organs, each of which makes these decisions according to definite rules of behavior. In defining the rules of behavior, we shall use the concept of the profitability of the k-th activity with reference to the price vector p. This is defined as the vector product

(5.58) $\pi_k = p' \gamma_k$,

and represents the "shadow revenue" secured from carrying out the k-th activity in the amount $x_k = 1$.

Let the players in our allocation game be called the helmsman (or central planning board), a custodian for each commodity, and a manager for each activity. Consider the following rules of behavior:

I <u>For the helmsman</u>: Choose a vector p_{fin} of positive prices on all final commodities, and inform the custodian of each such commodity of its price.

II <u>For all custodians</u>: Buy and sell your commodity from and to managers at one price only, which you announce to all managers. Buy all that is offered at that price. Sell up to the limit of availability.

III For all custodians of final commodities: Announce to managers the price set on your commodity by the helmsman.

IV For all custodians of intermediate commodities: Announce a tentative price on your commodity. If demand by managers falls short of supply by managers, lower your price. If demand exceeds supply, raise it.

V For all custodians of primary commodities: Regard the available inflow from nature as a part of the supply of your commodity. Then follow the rule on custodians of intermediate commodities, with the following exception: Do not announce a price lower than zero, but accept a demand below supply at a zero price if necessary.

VI For all managers: Do not engage in activities that have negative profitability. Maintain activities of zero profitability at a constant level. Attempt to expand activities of positive profitability by increasing orders for the necessary inputs with the custodians of those commodities.

The dynamic aspects of these rules have on purpose been left vague. It is not specified by how much managers of profitable activities should increase their orders, or by how much custodians of commodities in short or excess supply should change the price. Neither is it indicated how during a temporary disequilibrium a commodity in short supply is apportioned to managers. These questions would be highly relevant if our purpose were to design an allocation model which automatically seeks and finds an efficient point from some initial non-optimal situation. However, our present purpose is only to demonstrate that an efficient point once achieved is maintained if all players follow the rules stated.

More precisely

Theorem 5.8. Let $p_{fin} > 0$ be the vector of positive prices of final commodities announced by the helmsman under rule I. Then, a necessary and sufficient condition that the vectors $x \geqq 0$, p_{int}, $p_{pri} \geqq 0$ will remain constant under the rules II-VI is that the point

$$y = \Gamma x$$

is an efficient point in which the function $L = p'_{fin}\,\bar{y}_{fin}$ is maximized under the restrictions on $\bar{y}$ stated in Theorem 5.3, (supplemented by $\bar{y}_{int} = 0$), and that

$$p = \begin{pmatrix} p_{fin} \\ p_{int} \\ p_{pri} \end{pmatrix}$$

is a normal to (Γ) in y satisfying the conditions(5.24) of Theorem 5.2.

To prove this theorem, we read from rules IV, V and VI, respectively, the following necessary and sufficient conditions for the constancy of p_{int}, p_{pri} and x, respectively, where of course $x \geqq 0$:

(5.61) $\qquad y_{int} = \Gamma_{int}\, x = 0,$

(5.62) $\qquad \begin{cases} p_{pri} \geqq 0, \quad y_{pri} = \Gamma_{pri}\, x \geqq \eta_{pri} \;, \\ \qquad y_n = \gamma^{\prime(n)} \cdot x = \eta_n \text{ if } p_n > 0 \text{ and } n \text{ refers to a primary commodity} \end{cases}$

(5.63) $\qquad p'\,\Gamma \leqq 0, \qquad p'\gamma_k = 0 \;\text{ if } x_k > 0.$

Condition (5.63) expresses that p is a normal to (Γ) in the boundary point $y = \Gamma x$. The conditions (5.61) and (5.62) express, in view of Theorem (5.6), that y is an efficient point. This completes the proof of Theorem 5.8.

The reader will have noticed that the behavior prescribed for individuals by the rules I-VI is similar to that which results from the operation of competitive markets. The rules on the custodians are only personalizations of the properties of competitive markets. The vector p_{fin} which ultimately gives direction to the allocation of resources in production, instead of being set by a helmsman, could equally well be the result of competitive bidding by many consumers, each of which maximizes his individual utility. The behavior attributed to each manager could also come about as the result of each activity being carried out independently by many entrepreneurs bidding competitively for the input commodities of that activity and selling its output commodities competitively.

However, the "personal" formulation also has relevance to problems of economic organization. The rules suggest methods whereby a planned economy can strive for efficient allocation of resources in production. With respect to an economy in which entrepreneurs individually make production decisions, the rules help in the appraisal of alternative forms of economic organization or of market behavior, from the point of view of efficiency. Finally, the analysis may be applied to production decisions within the firm, which can be regarded as a planned economy on a smaller scale.

It may be useful to explore some connections between the present analysis and the discussions by Meade[1)], Lange[2)], Lerner[3)], Reder[4)], and others of allocation problems in a welfare economy. The managers postulated in these studies are in control of plants, in which many activities (as here considered) are carried out in supposedly efficient combinations. The problem how to achieve efficient production within the plant is presumed solved in the discussions referred to, but is here analysed on the basis of a particular model of technology. This model is, however, somewhat narrower than the type of technology admitted by the authors mentioned, in that we have ruled out indivisibilities and increasing or decreasing returns to scale.

Accepting the narrower assumptions regarding technology made in the present study, the "value of marginal product" concept used in the discussions referred to can be identified with our "efficiency price". The main result of these discussions can be summarized in the statement that allocation of each commodity in the various productive processes, in such a manner as to equate the value of its marginal product in all its uses, is a necessary condition for efficient allocation of resources. The present analysis implies that, in addition, in a technology as assumed, observance of this rule also forms a sufficient condition for efficient allocation of resources.

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1) An introduction to economic analysis and policy, J. E. Meade and C. Hitch.
2) On the economic theory of socialism, O. Lange and F. Taylor.
3) The economics of control, by A. P. Lerner.
4) Studies in the theory of welfare economics, by M. Reder.

APPENDIX: A theorem on convex sets

by

Murray Gerstenhaber

Theorem: Let C be a closed convex body in Euclidean n-space E^n and let O be a point in C. Suppose C contains no halfline through O. Then C is bounded.

Proof: Consider the surface of the unit sphere about O; call it S. C is closed and S closed and compact, so $C \cap S$ is closed and compact. If C were not bounded there would be a sequence of points x_n such that the distance from O to x_n is greater than n. Consider the projections of these points are S projected from O: call them y_i. Since C is convex, the y_i are in C, hence in $C \cap S$. They therefore have a limit point y on S.

Let the line segment from O to x_i be denoted by Ox_i, the half line beginning at O and going through y be denoted by $\overrightarrow{Oy}$. Then it must be that $\overrightarrow{Oy}$ is in C; in fact every point on $\overrightarrow{Oy}$ is a limit point of points on the line segments Ox_i and C was assumed closed. To see this, assume x a point on $\overrightarrow{Oy}$. Let the distance of x from O be ℓ. Then there is an integer n_0 such that $\ell \leq n_0$. Consider the sphere about O with radius ℓ; call it S'. x is in $C \cap S'$. Each of the segments Ox_i with $i > n_0$ intersects S' in a point y_i', the y_i' are in C and x is a limit of the points y_i'.