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A Mathematical Model of Production (Continued)

by

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Changes in notation and terminology

(Γ) denotes the closed cone $y = \Gamma x$, $x \geqq 0$

$)\Gamma($ " " open " $y = \Gamma x$, $x > 0$

Y " " efficient point set

Γ^* adjugate of Γ, i.e., $y^* \in \Gamma^*$ implies $y' y^* \leqq 0$ for all $y \in \Gamma$.

${}_1\Gamma$ matrix obtained from Γ by deleting the first column

Sections marked * are technical in character, and can be passed over in first reading.

4. The Efficient Point Set.

4.1. Case where the achievable point set is a polyhedral cone. We shall now apply the allocative objective, introduced in section 1.8, to the achievable point set. We shall first study the case in which this set consists of all points

$$(4.1) \qquad y = \Gamma x \qquad x \geqq 0$$

of a polyhedral cone (Γ) satisfying postulates A, B and C of section 3. Our remarks will be equally applicable whether we think of (Γ) as the technologically achievable cone, or as an economically achievable cone obtained by imposing on the technologically achievable cone the appropriate restrictions on the signs of y_D and y_P. Since Γ itself thus incorporates any such restrictions we may wish to impose, it is unnecessary to distinguish notationally between y_D and y_P. Finally, we may also think of (Γ) as obtained through intersection of a cone

$$(4.2) \qquad \begin{bmatrix} y \\ y_I \end{bmatrix} = \begin{bmatrix} \Gamma \\ \Gamma_I \end{bmatrix} x, \qquad x \geqq 0$$

in a larger-dimensional space with the point set

$$(4.3) \qquad y_I = \Gamma_I x = 0$$

expressing both the inavailability in nature of intermediate goods and the fact that they are not directly consumed.

However, we exclude from consideration any nonhomogeneous availability restrictions on primary commodities. Instead, we include primary commodities among the desired commodities, with the interpretation that the objective of the economy is served by the algebraic increase of their net output, i.e., by the decrease of their input. With regard to labor or land this interpretation is justifiable by the existence of an alternative use of that factor for leisure or recreation. With regard to some other primary factor, a justification may be found in a desire for conservation of resources, although such a consideration really fits better into a dynamic model. However, the question of justification of the objective is not important at this stage, because the present case is considered mainly for its mathematical simplicity, as a step toward cases which are both more complicated and more realistic.

4.2 Definition of an efficient point. An achievable point (4.1) in the commodity space is called efficient whenever an increase in one of its coordinates (the net output of one good) can be achieved only at the cost of a decrease in some other coordinate (the net output of another good). Expressed mathematically, y is efficient whenever

(4.4) $\bar{y} = \Gamma \bar{x}, \quad \bar{x} \geqq 0,$

implies

(4.5) $\bar{y} - y \ngeq 0.$

4.3 A necessary and sufficient condition for efficiency. We must expect to find more then one point y satisfying this definition. Application of the criterion of efficiency thus serves only to eliminate a set of clearly wasteful modes of production, leaving us with a set of efficient points from which further choice by other criteria is to be made. These further criteria fall outside of the scope of this article. We are only studying the properties of the efficient point set Y, and the conditions under which it can be regarded as defining a noninformation function.

Theorem 4.1: An internal point y of (Γ) cannot be efficient.

[Proof to be provided after mathematical introduction is available.]

It follows that the efficient point set Y must be the whole or a part of the boundary of (Γ). If $y = \Gamma x$ is an efficient point, there must be a partitioning of x into positive and zero elements, which we combine in x^+ and x^0 respectively, such that, for the corresponding partitioning of Γ into Γ^+ and Γ^0,

(4.a) $\rho(\Gamma^+) \leqq N - 1.$

There is a one-to-one correspondence between the points $\bar{x}$ satisfying

(4.b) $\bar{x}^+ > 0 \qquad \bar{x}^0 = 0$

(where the partitioning of $\bar{x}$ is that defined above on the basis of the elements of x) and the points of the open facet

(4.c) $\bar{y} = \Gamma\bar{x} = \Gamma^+ \bar{x}^+$, $\bar{x}$ satisfies (4.b).

The following theorem implies that if one point y of a facet is efficient, all points of that facet are efficient.

<u>Theorem 4.2:</u> <u>A necessary and sufficient condition that a point y of a closed facet (Γ^+) be efficient is that there exists a normal to the cone (Γ) in y with positive direction coefficients</u>

(4.d) $p > 0.$

4.4* <u>Proof of Theorem 4.2.</u> Proof of sufficiency: If such a normal exists,

(4.e) $p'(\bar{y} - y) \leqq 0$

for all achievable points $\bar{y}$, as defined in (4.4). Now if for any such point we had

(4.f) $\bar{y} - y \geq 0,$

(4.d) would make (4.e) impossible. Therefore, the negation (4.5) of (4.f) holds, and $\bar{y}$ is efficient.

Proof of necessity: Let y be efficient, and take y as the vertex of a cone (C) containing all points of (Γ). It is worth noting for what follows, though not necessary for the present proof, that (C) is again a polyhedral cone, and that its lineality equals

(4.g) $\ell(C) = \rho(\Gamma^+).$

The important points are that, (a) since (C) contains (Γ), and has its vertex y on the boundary of (Γ), any normal to (C) on the vertex is a normal to (Γ) and that, (b) if $\bar{\bar{y}} \neq y$ is in (C), then there is a $\lambda > 0$ such that the point

(4.h) $\quad \bar{y} = y + \lambda(\bar{\bar{y}} - y)$

on the halfline from the vertex y through $\bar{\bar{y}}$ is in both (C) and (Γ). Through (b) it follows from the efficiency condition (4.5) on y that the direction coefficient vector c of any halfline out of the vertex in (C) is non-semi-positive

(4.i) $\quad c \not\geqq 0.$

It follows from a theorem by Gale[ ] that the same applies to a cone $\bar{C}$ obtained from C by translation such that the vertex of $\bar{C}$ falls in the origin.

It was proved by Gale[ ] that as a consequence of (4.i), (C) possesses a normal with positive direction coefficients p on the vertex. It follows from (a) that this is also a normal to (Γ).

The following proof of Gale's result is likely to be equivalent to his proof (a question I have not had time to explore). By Theorem ...[ ], the convex hull ([A, B]) of two convex cones (A), (B) has as its adjugate the intersection of the adjugates (A^*) = (C) and (B^*) = (I) (the positive orthant). Then, since (C) and (I) do not intersect, it follows that (C^*) and (I^*) = (-I) have the whole space as their convex hull. Since (-I) is pointed, it follows from theorem ...[ ] that (C^*) contains an internal vector p of (+I). This completes the proof.

For use below we note that if $p \neq 0$ is a normal to (Γ) in an efficient point, or in any boundary point, $y = \Gamma^+ x^+$, $x^+ > 0$, then

(4.j) $\quad p'\Gamma \leqq 0,$

and

(4.k) $\quad p'\Gamma^+ = 0.$

These equations express that the halfspace

(4.1) $\quad p' \bar{y} \leqq o$

in the commodity space $\bar{y}$, which contains both the origin and the point y in its boundary (i.e., in the hyperplane $p' \bar{y} = o$), contains the cone (Γ), and has the facet (Γ^+) in its boundary. Conversely, any p for which (4.j) and (4.k) hold indicates a direction normal to (Γ) in any point y of the closed facet (Γ^+).

4.5 Economic interpretation of the efficiency conditions. An interesting economic interpretation can be given to the vector p. We shall call it a vector of prices p_n of the commodities $n = 1, \ldots, N$ in the point y. There is in this term no necessary implication of a market in which exchange of commodities between different parties takes place. The terms "shadow prices" or "accounting prices" have been used in various contexts to express this reservation. For the moment, we shall use the term "prices" as the most general one, capable of different interpretations in different uses of the model.

To see the meaning of this interpretation we rewrite (4.j) and (4.k), having regard to the definition of Γ^+, as separate conditions on each column vector $\gamma^{(k)}$ of Γ, as follows

$$(4.m) \qquad \begin{cases} p' \gamma^{(k)} = 0 & \text{if} \quad x_k > 0, \\ p' \gamma^{(k)} \leqq 0 & \text{if} \quad x_k = 0. \end{cases}$$

The expression $p' \gamma^{(k)}$ is interpreted as the net (shadow) profit on the unit of the k-th activity, computed on the basis of the price vector p. Then (4.m) says that no activity in the technology yields a positive profit, while each activity carried out at a positive level to achieve the boundary point y yields a zero profit.

We have found in theorem 4.2 that, for the boundary point y to be efficient, it must possess a positive normal. This leads to the following equivalent economic formulation of that theorem.

Theorem 4.3: A necessary and sufficient condition that the activity vector x shall lead to an efficient point $y = \Gamma x$ in the commodity space is that there exist a vector p of positive prices such that no activity in the technology permit a positive profit, and such that the profit on all activities carried out at a positive level be zero.

It may be pointed out that the necessary condition (4.a) for y to be a boundary point is fulfilled as a consequence of (4.k). This leads to the following corollary to theorem 4.1:

Theorem 4.4: In an efficient point on an achievable cone (Γ) in an N-dimensional commodity space, at most N-1 linearly independent activities can be carried out simultaneously at positive levels.

4.6 The elements of p as defining marginal rates of substitution. The price vector p is uniquely determined, but for a scalar factor, by the facet $)\Gamma'^{+}($ if and only if

(4.n) $\rho(\Gamma'^{+}) = N-1,$

that is, if in fact exactly N-1 linearly independent activities are involved in achieving the efficient point y. For this most general case, the reader can easily prove Theorem 4.3 without recourse to the Theorems quoted in sections 4.4.

If (4.n) holds, the prices p_n can be regarded as defining marginal rates of substitution between all commodities in a neighborhood of the efficient point y. If $\bar{y}$ is another point on the same facet, and hence also efficient, we have

(4.oh) $p'\bar{y} = o = p'y$, or $p'(\bar{y} - y) = o$.

Within the limits of the closed facet (Γ'^{+}), therefore, choice between different modes of production $y, \bar{y}, \ldots$ opens the same alternatives as would trading at the constant prices p. To take an example, if

(4.p) $\quad \bar{y}_1 > y_1, \ \bar{y}_2 < y_2, \ \bar{y}_n = y_n, \ n = 3, \ldots, N,$

then (4.oh) implies

(4.q) $\quad p_1 (\bar{y}_1 - y_1) = p_2 (y_2 - \bar{y}_2).$

An amount $(y_2 - \bar{y}_2)$ of the commodity "2" is "traded" for an amount $(\bar{y}_1 - y_1)$ of commodity "1", at the price

(4.r) $\quad p_{12} = p_1/p_2$

of the unit of "1" expressed in terms of units of "2".

It is important to emphasize that these relative prices p_{nm} apply only to a change from one efficient point y to another $\bar{y}$. That is, commodities are substituted for each other in these ratios only after efficiency has been reached, and provided efficiency is maintained in the change in production.

Secondly, the set of substitution ratios belonging to an efficient point on an (N-1)-facet applies only to changes to points on the same facet, including its boundary. Upon entering an adjoining (N-1)-facet, a different set of substitution ratios becomes applicable to changes within that facet.

4.7 An equivalent characterization of the efficient point set Y. So far we have not proved the existence of a single (N-1)-facet, and it is not difficult to construct a technology matrix of rank N, which satisfies postulates A, B, C of section3, such that none of its (N-1)-facets has a positive normal. The three 2-facets of the nonsingular technology matrix

$$(4.s) \quad \Gamma = \begin{bmatrix} 1 & 1 & 0.6 \\ 1 & 0.5 & 0.8 \\ -1 & -1 & -1 \end{bmatrix}$$

have as normals the column vectors of

$$(4.t) \qquad P = \left[p^{(1)} \ p^{(2)} \ p^{(3)}\right] = \begin{bmatrix} -3 & -1 & 1 \\ -4 & 2 & 0 \\ -5 & 1 & 1 \end{bmatrix},$$

none of which is positive. The example is illustrated in Figure 4.7, which indicates the intersection

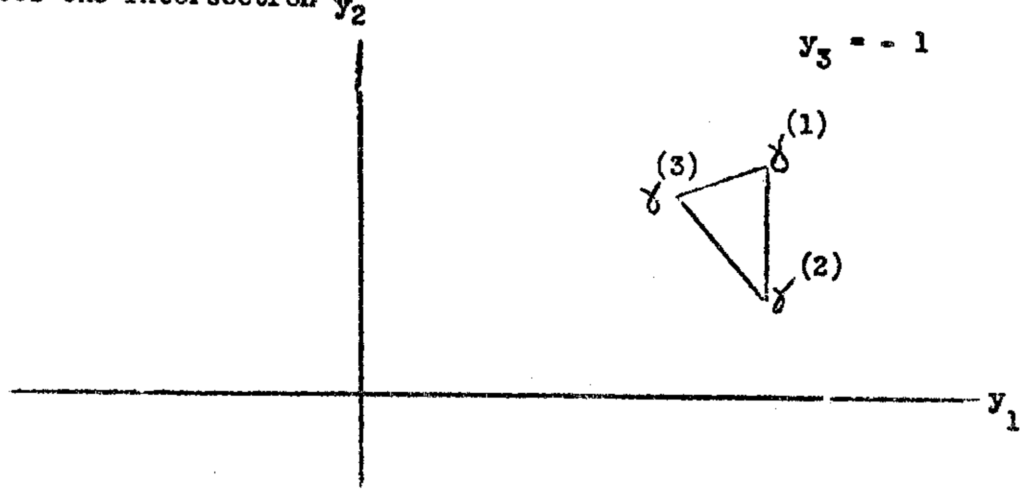

Figure 4.7

of (Γ) with the plane $y_3 = -1$. Thus (Γ) is the cone projecting the triangle $\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}$ out of the origin $y_1 = y_2 = y_3 = 0$, which may be though of as above the paper in which the figure is drawn.

The following illustration may help to visualize the nature of the efficient point set. Attach a source of light at each coordinate axis at the locations* $y_n = +\infty$, $n = 1,\ldots,N$, respectively and let (Γ) be opaque. By postulate B of section 3, all sources are located outside (Γ). Any facet which receives light from all sources consists of efficient points only. Any open facet which is in the shade of (Γ) with respect to at least one source does not contain any efficient points, although its boundary may contain efficient points (in addition to the origin $y = 0$, which is an efficient point on the boundary of all facets because of postulate B). A facet containing a straight line segment parallel to a coordinate axis is to be regarded as in the shade of one of its boundary facets with respect to the corresponding source of light.

* The illustration remains good, but is less easily grasped, if a finite positive location is selected for each source.

If this construction is applied to the example of figure 4.7, facet $)\gamma^{(2)}\ \gamma^{(3)}($ is in the shade from sources 1, 2 and 3, facet $)\gamma^{(1)}\ \gamma^{(3)}($ is in the shade from source 1, while facet $)\gamma^{(1)}\ \gamma^{(2)}($ first falls in the shade from source 2, thrown by its boundary facet $)\gamma^{(1)}($. Similarly, of the 1-facets, only $)\gamma^{(1)}($ receives light from all three sources.

These considerations suggest a method of constructing the efficient point set, expressed by the following theorem.

Theorem 4.5: Let

$$(4.u)\qquad \tilde{\Gamma} = \left[-I \ \ \Gamma\right]$$

be a technology matrix obtained from Γ by adjoining costless disposal activities for all commodities. The efficient point set Y in the technology Γ consists of all facets of $(\tilde{\Gamma})$ which do not contain any of the disposal vectors of $-I$.

4.8* Proof of Theorem 4.5: Without loss of generality we can assume that Γ is its own frame. By Theorem of [ ], the facets of (Γ) which are also facets of $(\tilde{\Gamma})$ are all those facets based on frame vectors of Γ, which are also frame vectors of $\tilde{\Gamma}$. We shall first show that all facets of (Γ) with positive normals are also facets of $(\tilde{\Gamma})$, which satisfy the requirements of the theorem.

Let (Γ^{+}) be a facet of (Γ) with positive normal p. Now suppose $\gamma^{(1)}$ were a frame vector of Γ in the boundary of (Γ^{+}) which is not a frame vector of $\tilde{\Gamma}$. Then there would exist a vector $\begin{bmatrix} z \\ {}_1x \end{bmatrix}$ such that

$$(4.w)\qquad \gamma^{(1)} = \left[-I \ \ {}_1\Gamma\right]\begin{bmatrix} z \\ {}_1x \end{bmatrix}, \qquad \begin{bmatrix} z \\ {}_1x \end{bmatrix} \geqq 0,$$

but no vector ${}_1x$ such that

$$(4.x)\qquad \gamma^{(1)} = {}_1\Gamma\ {}_1x, \qquad {}_1x \geqq 0.$$

Hence, there would exist vectors z, ${}_1x$ such that

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* A proceeding subscript indicates the ommission of the corresponding row or column.

(4.y) $\quad z = -\gamma^{(1)} + {}_1\Gamma_1 x \geq 0, \qquad x \geq 0.$

Premultiplying (4.y) by p' we obtain

(4.z) $\quad p'z = -p'\gamma^{(1)} + p'\Gamma_1 x > 0, \qquad x \geq 0,$

which contradicts (4.j) and (4.k) because $\gamma^{(1)}$ is a column vector of Γ^+. Therefore all boundary vectors of (Γ^+) are in the frame of $\tilde{\Gamma}$, and (Γ^+) is a facet of $(\tilde{\Gamma})$. Moreover, neither (Γ^+) nor its linear extension contains a vector of - I, because that would contradict $p > 0$.

We observe next that a facet (Γ^+) of (Γ) with a nonsemipostive normal cannot be a facet of $(\tilde{\Gamma})$. For, if p contains a negative element, p_1, say, (4.j) could not be satisfied in the first column if $\tilde{\Gamma}$ were substituted for Γ.

This leaves for exploration those facets $)\Gamma^+($ of (Γ) that possess a semipositive normal with a zero element, and therefore are not in the efficient point set. It is easily seen that these are also facets of $(\tilde{\Gamma})$, or parts thereof of the same dimensionality. For the purposes of this proof it is sufficient to remark that, if such a facet $)\Gamma^+($ is in the boundary of $(\tilde{\Gamma})$, the closed facet $(\tilde{\Gamma}^+)$ of $(\tilde{\Gamma})$ of equal dimensionality containing it would also contain a corresponding column vector of -I.

4.9 Topological Classification of efficient point sets. We shall give a brief classification of topologically different cases with regard to the efficient point set. We shall give graphical examples in a three-dimensional commodity space. We shall visualize vectors and cones in that space by their intersection with the plane

(4.aa) $\quad \theta' y = -1,$

where θ is the positive vector which, according to postulates A and B of section 3, satisfies

(4.ab) $\quad \theta'\tilde{\Gamma} = \theta'\left[-I\ \Gamma\right] < 0.$

It follows that every vector of $(\tilde{\Gamma})$ intersects the plane (4.aa) in a finite point. By proper choice of the units of measurement for the commodities, we can make all components of θ equal to 1. Thereby the intersection of (4.aa) with (-I) becomes an equilateral triangle (for $N > 3$ a regular simplex), as shown in Figure 4.9.1. The intersection of (4.aa) with the various octants is shown in Figure 4.9.2, where the origin can be thought of as located above the paper.

The reader is invited to construct corresponding examples for $N = 2$, and to imagine them for $N > 3$.

The simplest case, illustrated in Figure 4.9.3, is that where (-I) consists entirely of internal vectors of (Γ). In that case (Γ) and $(\tilde{\Gamma})$ are identical, and no facet of (Γ), or its linear extension, contains a column vector of -I. The cone (Γ) is necessarily N-dimensional, and the efficient point set is its entire $(N-1)$-dimensional boundary, containing the vertex as an internal point.

The second case is that in which (Γ) and (-I) again have an internal vector in common, but (-I) contains vectors outside (Γ). This case is illustrated in Figure 4.9.4. The dotted lines show how (Γ) is extended to $(\tilde{\Gamma})$. The application of Theorem 4.5 shows that the efficient point set consists of the pair of adjoining closed facets $(\gamma^{(1)}\ \gamma^{(2)})$ and $(\gamma^{(2)}\ \gamma^{(3)})$, and the separate closed facet $(\gamma^{(6)}\ \gamma^{(7)})$. These two 2-dimensional parts of the efficient point set are joined only by their common boundary point in the origin, so that exlusion of the origin would destroy the connectedness of Y. For $N = 3$ at most three so separated sections can arise in this way, some or all of which may degenerate to a 1-facet.

A third conceivable case, that in which (Γ) is contained in (-I), is excluded by postulate C of section 3. Therefore, the frame of $\tilde{\Gamma}$ contains at least

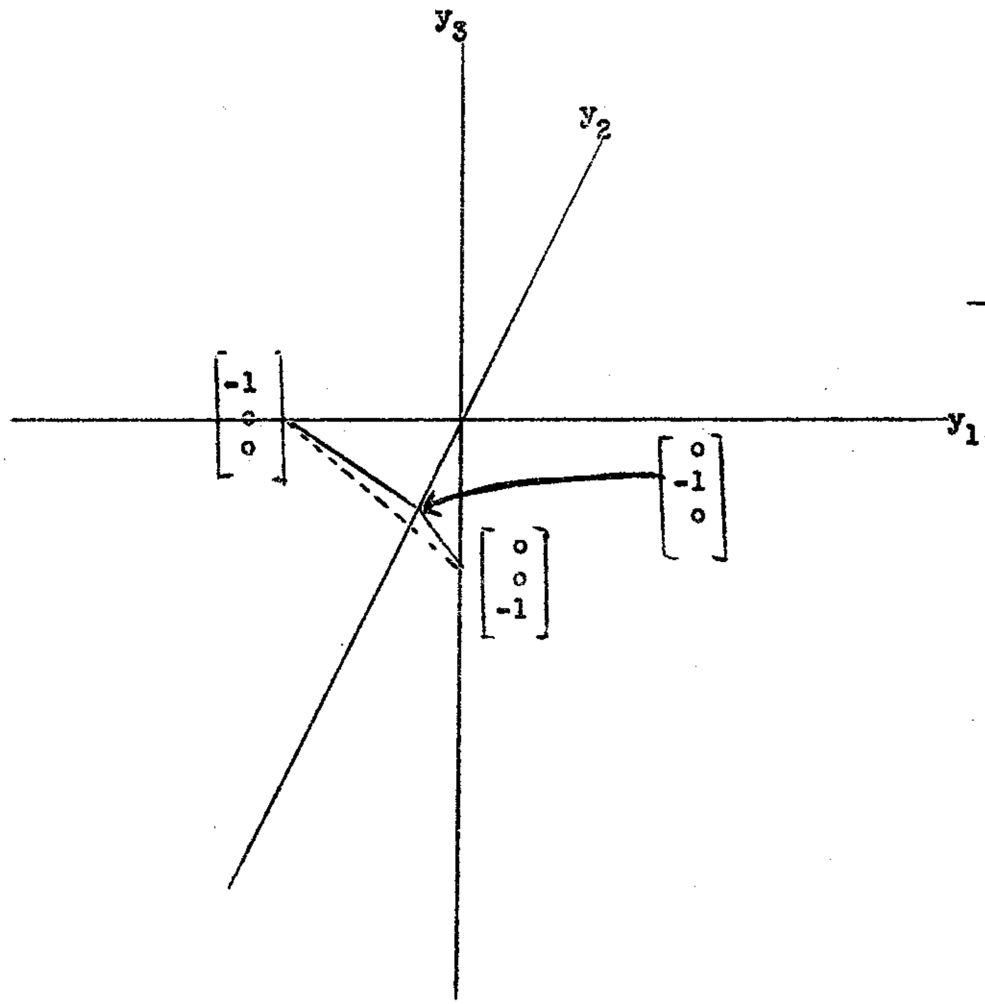

Figure 4.9.1

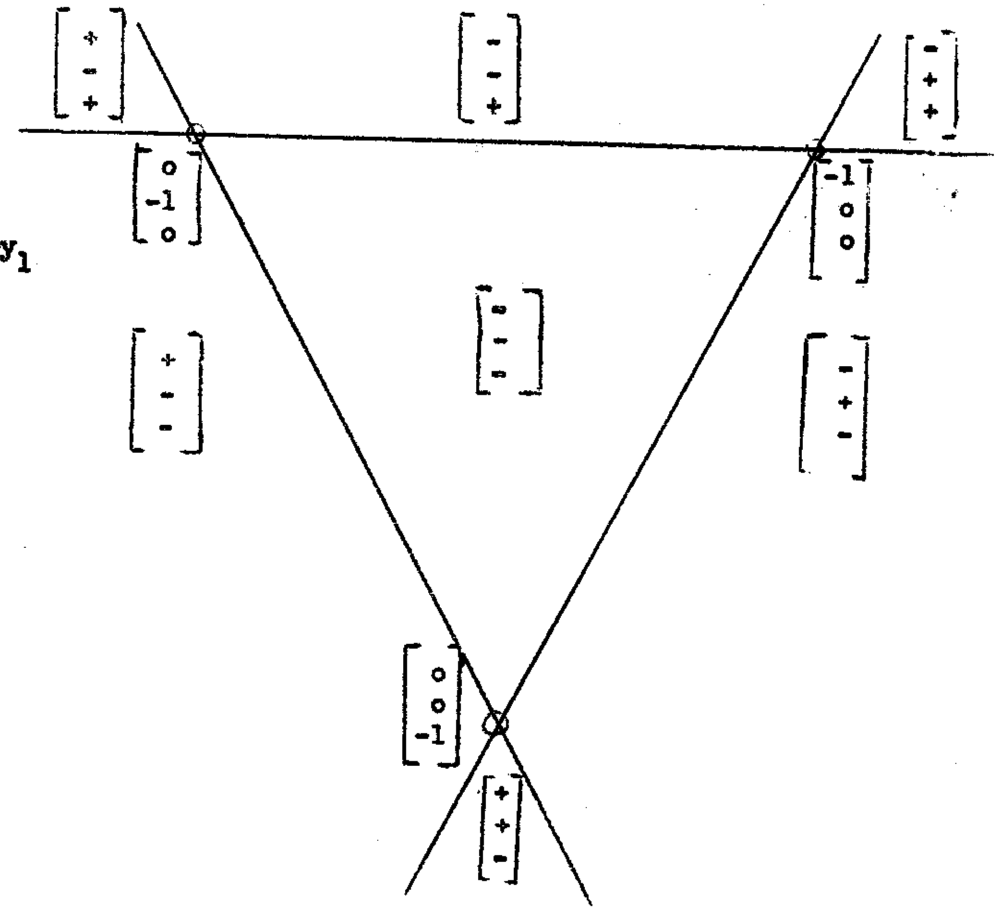

Figure 4.9.2

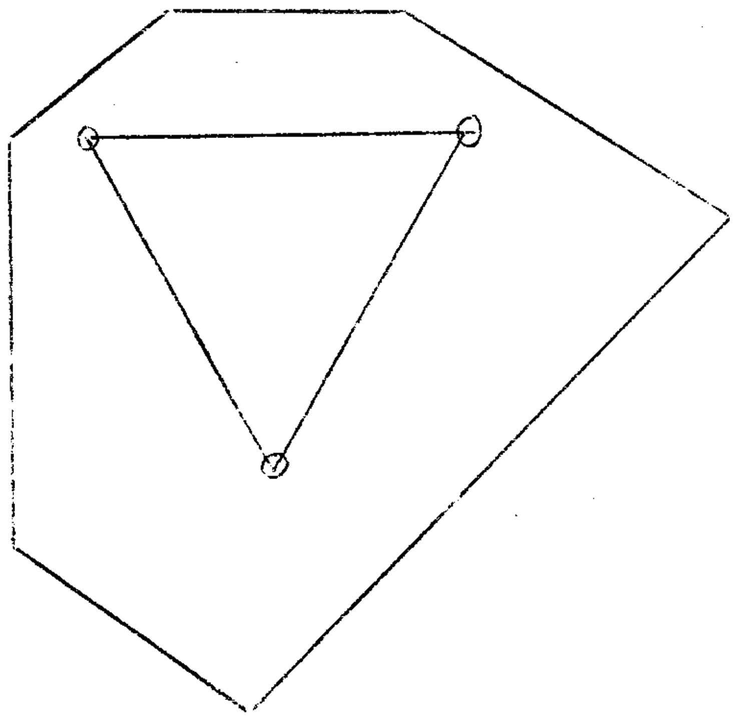

Figure 4.9.3

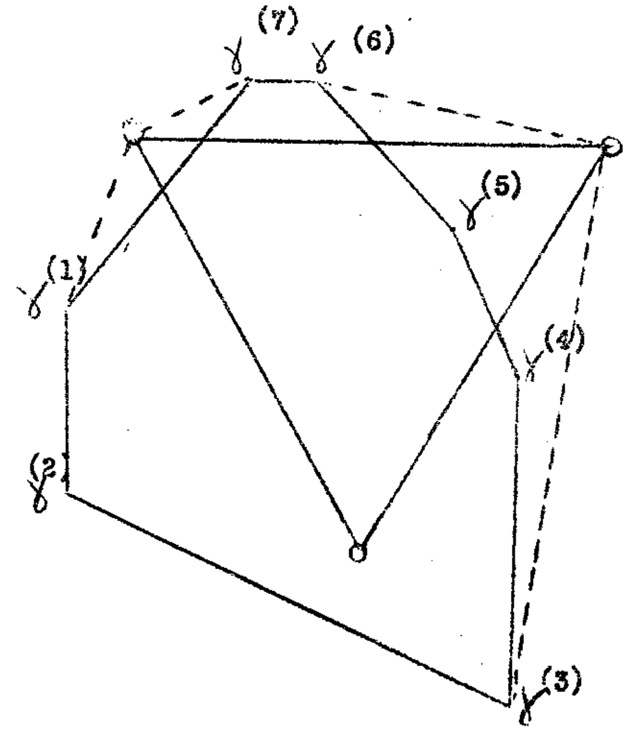

Figure 4.9.4

one column γ from Γ, which, regarded as a 1-facet (γ) must be part of the efficient point set. This establishes

Theorem 4.6: If the technology matrix Γ satisfies postulates A, B, C of section 3, the efficient point set contains at least one 1-facet γ based on a column γ of Γ.

The fourth case is that in which $(-I)$ and (Γ) have no internal vector in common. Topologically, this case is not different from that sub-case of the second case, in which the part of (Γ) outside $(-I)$ is singly connected. Because of its economic importance, this case will be explored further in the next section.

4.10 Single connectedness of the efficient point set in the economically achievable point set. The splitting up of the efficient point set into separate sections joined only by the origin can occur only if the technology matrix Γ permits each commodity to appear as a negative output (net inflow) for some activity vector $x \geqq 0$. This is a rather unrealistic situation, because in most applications it is known in advance that certain desired commodities are not given by nature. We have previously introduced the notion of the economically achievable point set to give effect to sign restrictions on elements of y, and have agreed to incorporate such restrictions in Γ. We shall roughly outline a proof of :

Theorem 4.7: Whenever one commodity, "1" say, is restricted by Γ to nonnegative net outputs,

(4.ac) $\quad y = \Gamma x,\ x \geqq 0$ implies $y_1 \geqq 0$,

the efficient point set Y remains singly connected after the removal of the vertex $y = 0$.

4.11[*] Proof of Theorem 4.7: If (Γ) is contained in the halfspace $y_1 \geq 0$, which can also be written as the cone[*] $(-_1I\ _1I\ i_1)$, then the adjugate (Γ^*) of (Γ) contains the adjugate $(-i_1)$ of $(-_1I\ _1I\ i_1)$, by theorem ...[ ]. We may exclude that (Γ^*) also contains (i_1), because in that case, (Γ) would be confined to the space $y_1 = 0$, and there would be no point in including the commodity "1".

Since (Γ') is the convex hull of $(-I)$ and (Γ), the adjugate (Γ'^*) of (Γ') is the intersection of (Γ^*) and $(-I^*) = (I)$, by theorem...[ ]. It should be noted that (Γ^*) and therefore (Γ'^*) contains at least one internal vector θ of (I), because of postulate B of section 3.

There is a one-to-one duality relationship between the open facets of (Γ') and of (Γ'^*), such that the sum of the dimensionalities of corresponding facets is N. We shall except the vertices from this relationship, because the corresponding "facets" would be the entire adjugate cones, which we do not wish to include.[**] The relationship conserves contiguity,[#] and therefore conserves connectedness characteristics of sets of facets on (Γ'^*) when going to corresponding sets of facets on (Γ'). The efficient point set Y is the set of all facets of $(\Gamma') = (-I\ \Gamma)$ not containing vectors of $(-I)$. In the duality relationship, this corresponds to the set Y^* of all facets of Γ'^*, or parts thereof, that fall inside (I), i.e., belong to $)I($. Therefore, the theorem is proved if one can establish the single connectedness of Y^*. Y^* is that part of the boundary of the intersection (Γ'^*) of (Γ^*) and (I) that is not boundary of (I), while (Γ^*) contains $-i_1$ but not i_1.

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* i_1 is the first column of I, $_1I$ the remaining columns.

** When (Γ') is less then N-dimensional, it is included as one of its own n-facets, $n < N$.

\# Two facets are called contiguous if one is in the boundary of the other.

To each point y of $(\tilde{\Gamma}^*)$ we associate the point ${}_1y$ of the half-space $y_1 = 0$. Since this mapping can be represented by

(4.ad) $\quad {}_1y = y + y_1 (-i)$,

${}_1y$ is in $(\ ^*)$ if y is. Also, ${}_1y$ is in (I) if y is. Therefore, ${}_1y$ is in $(\tilde{\Gamma}^*)$ if y is.

The set of points ${}_1y$ on which the points y of $(\tilde{\Gamma}^*)$ are so mapped is the closed convex cone $({}_1\tilde{\Gamma}^*)$. Consider the (possibly open) subset ${}_1Y^*$ of all points of $({}_1\tilde{\Gamma}^*)$ which are not on the boundary of $({}_1I)$. We shall show that (4.ad) establishes a one-to-one continuous relationship between the points ${}_1y$ of ${}_1Y^*$ and the points y of the relevant subset of Y^*, viz., the set $\bar{\bar{Y}}^*$ of all open facets of $(\tilde{\Gamma}^*)$ in Y^*, of which the linear extension does not contain (-i). Since each $y \in \bar{\bar{Y}}^*$ belongs to $(\tilde{\Gamma}^*)$, (4.ad) associates a unique ${}_1y$ of $({}_1\tilde{\Gamma}^*)$ with it. Conversely, let ${}_1y$ be a point of $({}_1\tilde{\Gamma}^*)$, and therefore of (Γ^*). Determine the highest value of y_1 for which

(4.ae) $\quad y = {}_1y - y_1 (-i) = ({}_1y + y_1\ i) \in (\Gamma^*)$.

This value is finite because otherwise i_1 would be in (Γ^*). It is non-negative because ${}_1y$ is in (Γ'^*). Hence the point y so determined is in both (Γ^*) and (I), and on the boundary of (Γ'^*), but not on the boundary of (I) unless $y_1 = o$, because ${}_1y$ is not on the boundary of $({}_1I)$. Hence y is in Y^* whenever $y_1 > o$. Also, y is in $\bar{\bar{Y}}^*$ whenever $y_1 > o$, because, if y were on an open facet of $(\tilde{\Gamma}^*)$ in Y^* of which the linear extension contains (-i), a larger value of y_1 could be found to satisfy (4.ae).

We shall further show that any points

(4.af) $\quad \bar{\bar{y}} = {}_1y + \bar{\bar{y}}_1 i, \quad 0 \leq \bar{\bar{y}}_1 < y_1$,

are not in $\bar{\bar{Y}}^*$. If $y_1 = 0$ no such points exist. If $y_1 > o$, any such point $\bar{\bar{y}}$ is either in the interior of $(\tilde{\Gamma}^*)$ or on its boundary. If it is an interior point, it cannot belong to $\bar{\bar{Y}}^*$. If it is on the boundary of $(\tilde{\Gamma}^*)$, and therefore in Y^*, it is on a facet of $(\tilde{\Gamma}^*)$ of which the linear extension contains (-i), and therefore not in $\bar{Y}^*$.

Finally, it is noted that y as defined by (4.ae) is a continuous function of ${}_1y$, with the property that, if $y^{(1)}$, $y^{(2)}$, y correspond to ${}_1y^{(1)}$, ${}_1y^{(2)}$ and

(4.ag) $\quad {}_1y = \lambda\, {}_1y^{(1)} + (1-\lambda)\, {}_1y^{(2)}, \qquad 0 \leqq \lambda \leqq 1,$

respectively, we have because of the convexity of (Γ^*), that

(4.ah) $\quad y_1 \geqq \lambda\, y_1^{(1)} + (1-\lambda)\, y_1^{(2)}.$

Hence we have either $y_1 > 0$ in all internal points of ${}_1Y^*$(internal in an (N-1)-dimensional sense), or $y_1 = 0$ identically in ${}_1Y^*$. The latter case is excluded, however, by the fact noted above that $(\tilde{\Gamma}^*)$ contains at least one internal vector θ of (I). A continuous one-to-one mapping of ${}_1\tilde{Y}^*$ has thus been established by (4.ae).

It follows that $\tilde{Y}^*$ has the same connectedness properties as the set ${}_1Y^*$, which is an open cone augmented by some or all of its boundary facets. Hence, $\tilde{Y}^*$ is singly connected. It remains to trace the effect, on the efficient point set Y, of the deletion from its dual Y^* of all facets of which the linear extension contains (-i). Since (-i) is outside Y^*, such facets have a dimensionality of at least 2. The corresponding facets of $(\tilde{\Gamma})$ in Y have a dimensionality at most N-2, and are located in the hyperplane $y_1 = 0$. It is easily seen that these make up the "boundary" of Y (in an (N-1)-dimensional sense of the term), and that their deletion or restoration does not affect the connectedness properties of Y.

[To be continued]