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ON THE SAMUELSON THEOREM FOR LEONTIEF MODELS

K. J. Arrow

A recent theorem stated by Samuelson concerning the relation between the Leontief model of production and models involving continuous substitution has been shown by Koopmans to be equivalent to the following theorem on convex sets in finite-dimensional vector spaces[1]:

<u>Theorem</u>: Let $C_1, \ldots, C_n$ be n closed convex sets in n-space such that if $a_{(j)} = (a_{1j}, \ldots, a_{nj})$ belongs to C_j, then $a_{jj} > 0$, $a_{ij} \leqq 0$ for $i \neq j$. Suppose there exists for each j a point $a'_j \in C_j$ and a number $x'_j > 0$ such that

$$y'_i = \sum_{j=1}^{n} a'_{ij} x'_j > 0 \quad (i = 1, \ldots, n), \quad \sum_{j=1}^{n} x'_j = 1.$$

Then the intersection of the boundary of the convex hull of the sets $C_1, \ldots, C_n$ with the non-negative orthant $y_i \geqq 0$ is the intersection of that orthant with an (n-1)-dimensional hyperplane whose outward normal has positive direction coefficients, provided that the convex hull of $C_1, \ldots, C_n$ intersects the nonnegative orthant in a compact set.

Koopmans has proved this theorem for the case n = 3. The present note will show that his proof extends readily to the general case with the aid of a simple lemma.

<u>Lemma1:</u> Suppose $a_{ii} > 0$, $a_{ij} \leqq 0$ for $i \neq j$, $\sum_{j=1}^{n} a_{ij} x_j > 0$ for all i, $x_j > 0$ for all j, and $\sum_{j=1}^{n} a_{ij} x'_j \geqq 0$ for all i. Then $x'_j \geqq 0$ for all j.

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1. T. C. Koopmans, "Proof of Samuelson's Theorem Regarding the Ineffectiveness of Substitution in the Leontief Model," C.C. 221, LPC: 408.

Proof: Since $x_j \neq 0$ for all j, the ratios x'_j/x_j are defined for all j. Let this ratio be a minimum for $j = k$; by relabeling coordinates, we may assume $k = n$.

$$x'_j/x_j \geq x'_n/x_n \quad \text{for all } j. \tag{1}$$

By assumption,

$$0 \leq \sum_{j=1}^{n} a_{nj}\, x'_j = \sum_{j=1}^{n-1} a_{nj}\, x_j\, (x'_j/x_j) + a_{nn}\, x_n\, (x'_n/x_n). \tag{2}$$

Since $a_{nj} \leq 0$ for $j < n$, $x_j > 0$ by assumption, it follows from (1) and (2) that

$$0 \leq \sum_{j=1}^{n-1} a_{nj}\, x_j\, (x'_n/x_n) + a_{nn}\, x_n (x'_n/x_n) = (x'_n/x_n) \sum_{j=1}^{n} a_{nj}\, x_j.$$

Since by hypothesis, $\sum_{j=1}^{n} a_{nj}\, x_j > 0$, it follows that $x'_n/x_n \geq 0$. From (1), then,

$$x'_j \geq 0, \text{ since } x_j > 0, \text{ for all } j.$$

Lemma 2: Suppose $a_{ii} > 0$, $a_{ij} \leq 0$ for $i \neq j$, $\sum_{j=1}^{n} a_{ij}\, x_j > 0$ for all i, and $x_j > 0$ for all j. Then the matrix (a_{ij}) is nonsingular.

Proof: Suppose the matrix were singular. Then there would exist a set of numbers $x'_1, \ldots, x'_n$, not all 0, such that

$$\sum_{j=1}^{n} a_{ij}\, x'_j = 0 \text{ for all } i. \tag{3}$$

If $x'_1, \ldots, x'_n$ is one such set of numbers $-x'_1, \ldots, -x'_n$ is another. Hence, there must exist a set of numbers satisfying (3) at least one of which is negative. Let there be m negative numbers in the set; by relabeling coordinates, we may assume that $x'_1 < 0, \ldots, x'_m < 0$, $x'_j \geq 0$ for $j > m$. For $i \leq m$,

$$\sum_{j=1}^{m} a_{ij}\, x_j > -\sum_{j=m+1}^{n} a_{ij}\, x_j \geq 0, \tag{4}$$

since $x_j > 0$, $a_{ij} \leq 0$ for $i \leq m < j$, while

$$\sum_{j=1}^{m} a_{ij}\, x'_j = -\sum_{j=m+1}^{n} a_{ij}\, x'_j \geq 0, \tag{5}$$

since $x'_j \geqq 0$ for $j > m$, $a_{ij} \leqq 0$ for $i < j$. But, by Lemma 1, (4) and (5) imply that $x'_j \geqq 0$ for $j \leqq m$, which is a contradiction.

Note that the numbers $x'_1, \ldots, x'_n$ on page 1 of C.C. 221 must be positive, not merely nonnegative. For example, from the condition that $\sum_{j=1}^{n} a'_{nj} \; x'_j > 0$, it follows that $x'_n > - \sum_{j=1}^{n-1} (a'_{nj}/a'_{nn}) \; x'_j \geqq 0$, since $a'_{nj} \leqq 0$ for $j < n$, $a'_{nn} \geqq 0$, $x'_j > 0$. Hence, all the conditions of Lemma 2 are fulfilled, with a_{ij} replaced by a'_{ij}, and x_j by x'_j, so that the matrix (a'_{ij}) is nonsingular.

Inspection of Koopmans' proof shows that no difficulty arises in extension to n dimensions until page 5 is reached. It is important that for each k, there exist a point $y''_{(k)}$ which is a convex combination of $a''_{(1)}, \ldots, a''_{(n)}$ and whose coordinates are all equal to zero except for the k^{th} which is positive.

The matrix (a''_{ij}) has the same properties as the matrix (a'_{ij}) and hence is nonsingular. Let its inverse be (A''_{ij}). Then,

$$\sum_{j=1}^{n} a''_{ij} \; A''_{jk} = \delta_{ik} \quad (i = 1, \ldots, n), \tag{6}$$

where δ_{ik} is the Kronecker delta, which is always nonnegative. For any fixed k, the conditions of Lemma 1 are fulfilled, with a''_{ij} replacing a_{ij}, x''_j replacing x_j, and A''_{jk} replacing x'_j. Therefore, $A''_{jk} \geqq 0$. Also, from (6) with $i = k$, we cannot have $A''_{jk} = 0$ for all j, so that $\sum_{j=1}^{n} A''_{jk} > 0$. Let

$$x''_{jk} = A''_{jk} / (\sum_{j=1}^{n} A''_{jk}),$$

$$y''_{kk} = 1/(\sum_{j=1}^{n} A''_{jk}).$$

Then,

$$\sum_{j=1}^{n} a''_{ij} \; x''_{jk} = 0 \quad (i \neq k),$$

$$\sum_{j=1}^{n} a''_{kj} \; x''_{jk} = y''_{kk} .$$

$$x''_{jk} \geq 0 \text{ for all } j, \ \sum_{j=1}^{n} x''_{jk} = 1, \ y''_{kk} > 0,$$

so that the point $y''_{(k)}$ with the desired properties exists.

It may be observed that the proviso at the end of the main theorem is not contained in Koopmans' paper, though the theorem is not true without it. It would be desirable to replace the proviso by conditions on the sets C_i which have more definite economic significance.