COWLES COMMISSION DISCUSSION PAPER: ECONOMICS: NO. 257

The Determination of Many-Commodity Preference Scales by Two-Commodity Comparisons[1]

Kenneth J. Arrow
April 25, 1949

Note: This paper is to be submitted for publication. Your comments are appreciated.

1. Introduction

If, as is customary in economics, it is assumed that the behavior of individuals is governed by a consistent pattern of preferences with respect to varying bundles of goods, it is natural to suggest that this pattern can be approximated by the answers to a questionnaire as to the preferences between pairs of such bundles. Against this it has been argued that while such a procedure might be satisfactory if only two or three commodities are involved, the procedure will be useless if there are several hundred commodities involved, since the questionee will be unable to truly apprehend the situation, and his answers may be only distantly related to the choice he would actually make if the situation arose. It is the purpose of the present paper to show that, under the assumptions usually made in economic literature, the determination of a preference scale relating to many commodities can always be accomplished by means of comparisons involving the variation of only two commodities at a time. In the present paper, the procedure will be illustrated by showing how to pass from a knowledge of the preferences among bundles in which the quantities of only two commodities vary to a knowledge of preferences among three-commodity bundles. The procedure for more commodities is sketched. Some points of mathematical rigor are discussed in the Appendix.

---

1. The research for this paper was carried out at Project RAND, a project of the United States Air Force. The author wishes to express his gratitude to J.W.T. Youngs, University of Indiana, who conjectured the main results and gave valuable suggestions for their proof.

This result suggests that there is no true gain in generality in considering more than two commodities. In principle, all properties of multi-commodity indifference maps are deducible from those of two-commodity maps. Of course, in any given situation, it may be simpler to consider the problem in full generality directly.

2. Assumptions and Definitions

A commodity bundle will be denoted by a point X (or Y or Z), with coordinates $(x_1, \ldots, x_n)$, where x_i is the quantity of the i^{th} commodity. It will be assumed that $x_i \geqq 0$ for all i; that is, we only consider the relations of preference and indifference as between two bundles both of which satisfy this restriction on their coordinates. For such commodity bundles, the relations of preference and indifference are assumed to have their usual properties, as listed in the following statement:

I. (a) For all pairs of bundles X and Y, either X is preferred to Y or Y is preferred to X or X and Y are indifferent.

(b) If X is indifferent to Y, then Y is indifferent to X; if X is indifferent to Y and Y is indifferent to Z, then X is indifferent to Z.

(c) X is preferred to Z if any of the following conditions hold: X is preferred to Y and Y is preferred to Z; X is preferred to Y and Y is indifferent to Z; X is indifferent to Y and Y is preferred to Z.

In addition, we shall make the following customary assumption.

II. If $X_i \geqq Y_i$ for all i and $X_j > Y_j$ for some j, then X is preferred to Y. I.e., other things being equal, an individual will prefer more of a good to less. This is equivalent to the usual, though sometimes only implicit, assumption in demand theory, that the marginal utility of each commodity be positive. If some of the commodities entering into the bundles among which choice is to be made are irksome, one can always take their negatives and then change the origin to make them non-negative; e.g., replace the item, "work," by , "leisure," defined as the difference between the length of the day and the number of hours worked. Assumption II is only essentially contradicted when the preferences for varying quantities of one commodity (given the quantities of all other commodities) are neither always in the same nor always in the opposite direction to those quantities. This situation can arise, for example, if there is a point of saturation on the indifference map, i.e., if there is one bundle of goods which is preferred to all others, even to bundles having more of each commodity.[2] This case has not yet been adequately formalized in the literature on demand theory and will not be treated here.[3]

Under Assumption II, the preference scale in the one-commodity case is completely defined, being simply that more of that commodity is preferred to less. Henceforth, it will be assumed that the number of commodities is at least two.

To state the next assumption, we shall define a linear combination of two commodity bundles. If X and Y are two commodity bundles, with X_i the amount of the i[th] commodity in bundle X and Y_i the amount of the i[th] commodity

---

2. The existence of a point of saturation in an indifference map was apparently first suggested by I. Fisher, "Mathematical Investigations in the Theory of Value and Prices," Transactions of the Connecticut Academy of Arts and Sciences, IX (1894, pp. 68, 70-1. The idea is implicit in some of the Austrian discussion of diminishing marginal utility of a single commodity; this marginal utility, it was held, might become zero as the quantity of the commodity became indefinitely large. See for example C. Menger, Grundsatze der Volkwirtschaftslehre, Vienna, 1871, pp. 57-60.

3. For example, points of saturation are not handled by H. Wold in his classic restatement of the field: "A Synthesis of Pure Demand Analysis," Skandinavisk Aktuarietidskrift, 1943, pp. 85-118, 220-263; 1944, pp. 69-120.

in bundle Y, we will use the symbol aX + bY to denote the commodity bundle in which the amount of the i^{th} commodity is $aX_i + bY_i$ for each commodity i. If each bundle is represented by a point in n-dimensional space whose coordinates are the quantities of the different commodities in the bundle, and if we consider the particular case where a = t, b = 1-t, $0<t<1$, then tX + (1-t)Y is represented by a point between X and Y on the straight line segment joining them.

III. If X is preferred to Y and Y is preferred to Z, then there is a number t such that $0 < t < 1$ and tX + (1-t) Z is indifferent to Y.

Assumptions I-III are essentially equivalent to Professor Wold's Axioms I-V.[4] Assumption I is exactly equivalent to his Axioms I-III; Assumption II is slightly stronger than his Axiom IV; Assumption III is somewhat weaker than his Axiom V.

The essential role of Assumption III is to establish the continuity of the indifference map. More precisely, an indifference map satisfying Assumptions I-III admits of a continuous utility indicator.[5] From this point on, we shall assume without special discussion that our indifference maps have all the usual continuity properties.

For the purposes of the present paper it is unnecessary to make the usual assumptions of differentiability and convexity.

By a rationed collection of commodity bundles, we shall mean a collection consisting of all bundles in which certain commodities occur in fixed quantities but all other commodities can occur in any non-negative amount; e.g., the set of all commodity bundles containing just five units of the first commodity, or the collection of all bundles containing exactly three

---

4. Wold, op. cit., pp. 221-3.

5. Wold, op. cit., p. 226.

units of the first commodity and exactly six units of the third. Particular interest centers on rationed collections in which all but two of the commodities are fixed in amount; such collections will be termed ration planes. Among the bundles in a given rationed collection, there will still hold relations of preference and indifference, and these relations will still satisfy Assumptions I - III in the restricted space, Assumption II holding with respect to the coordinates which are not restricted. This statement is obvious for Assumptions I and II. To see that III still holds within a rationed collection, suppose that collection defined, for example, by prescribing $x = c$. If X, Y and Z are bundles in the collection such that X is preferred to Y and Y to Z, then there is a number such that $tX + (1-t)Y$ is indifferent to Y and $0<t<1$. But the first coordinate of $tX + (1-t)Y$ is $tX_1 + (1-t)Y_1 = tc + (1-t)c = c$, since X and Z both have c as a first coordinate. Hence, all the usual properties of indifference maps hold on the rationed collections as well as on unrestricted collections of bundles.

3. The Case of Three Commodities.

In this section, we will assume that there are only three commodities altogether. In this case, a ration plane is defined by fixing the amount of one commodity. We assume that for each ration plane we know the relations of preference or indifference as between any pair of bundles on that plane. That is, we consider that between any two bundles which have the same quantity of one commodity, we can ask the individual whether he is indifferent between the two or prefers one to the other, and, in the latter case, which he prefers. We are not permitted, let us say, to ask these questions as between two bundles in which the quantities of each commodity differ. It will turn out that we can infer the preferences in the latter case from the preferences as between pairs of bundles having the same quantity of some commodity.

Let X and Y be any two bundles. If $X_3 = Y_3 = a$, say, both X and Y lie on the ration plane $x_3 = a$, and from a knowledge of the indifference curves on every ration plane we can certainly infer the proper preference or indifference relation as between X and Y. Hence, we need only consider the case $X_3 \neq Y_3$; since it does not matter which bundle we call X and which Y, we may assume that $X_3 > Y_3$. Let $X_3 = a$, $Y_3 = b$.

Our procedure will be to derive, for each of the propositions, X is preferred to Y, X is indifferent to Y, and X is disfavored to Y, the respective necessary implications concerning the indifference maps on each of various ration planes; then, it will be easily seen that the conditions in question are also respectively sufficient. Let I_1 be the indifference curve through X on the ration plane $x_3 = a$; by assumption this curve is known. We will derive the conditions in question separately for the two cases where the projection of I_1 on the x_1-axis is unbounded and where it is bounded.

Case I: <u>The projection of I_1 on the x_1 -axis is unbounded.</u> Consider the bundles represented by points on the ration plane $x_3 = b$ which are indifferent to X; these points, if there are any, form an indifference curve I_2. Since $b < a$, the curve I_2, if it exists at all, must lie further away from the origin than the curve I_1, if we imagine them plotted together (see Fig. 1). In particular, I_2 must also be unbounded.

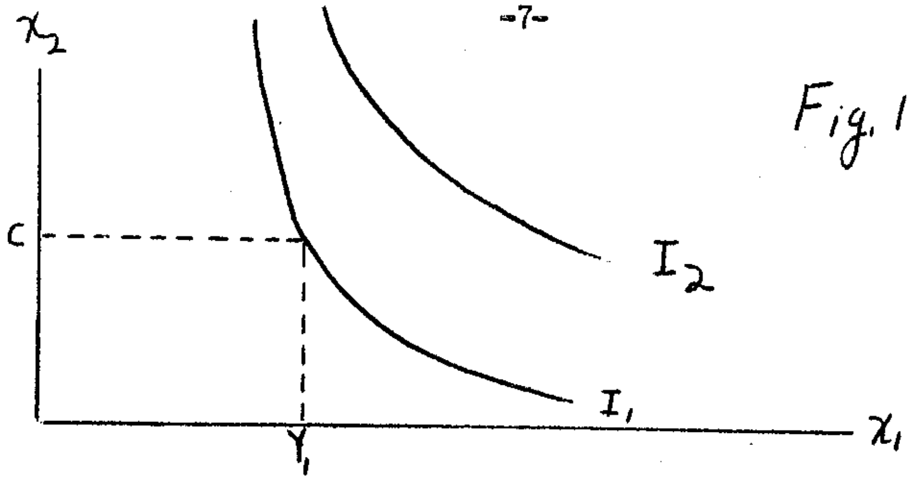

Fig. 1

Suppose X is, in fact, preferred to Y. Consider the intersection, if any, of the line $x_1 = Y_1$ with I_1 on the plane $x_3 = a$; let the x_2-coordinate of the point of intersection be c. On the ration plane $x_1 = Y_1$, this point of intersection has the coordinates $x_2 = c$, $x_3 = a$; let I_3 be the indifference curve on this plane through the indicated point. (See Fig. 2).

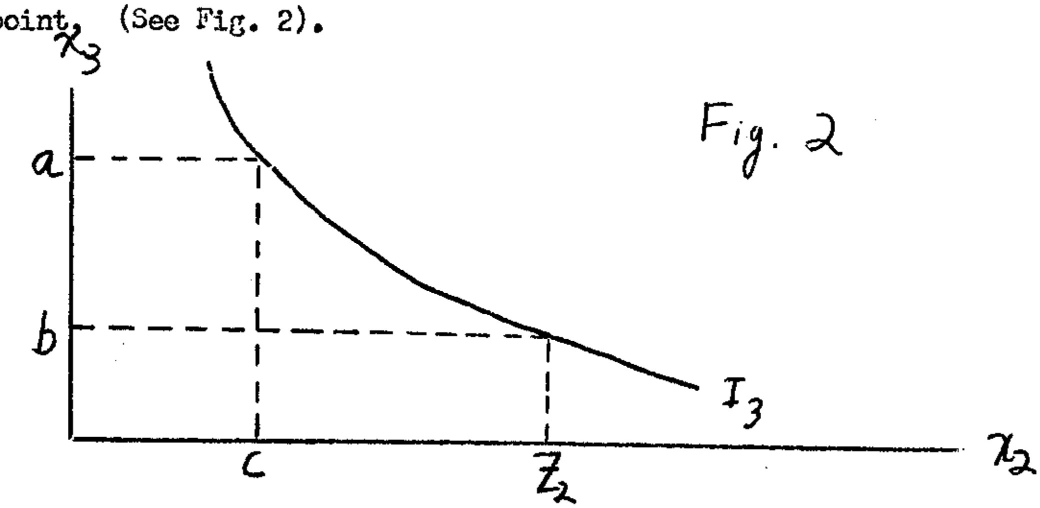

Fig. 2

Consider the intersection, if any, of I_3 with the line $x_3 = b$; let Z be bundle designated by this point of intersection. By construction, $Z_1 = Y_1$, since Z is on the ration plane $x_1 = Y_1$, and $Z_3 = b = Y_3$; also, Z is indifferent to the bundle (Y_1, c, a), which in turn is indifferent to X, since X, and hence Z, is preferred to Y, $Z_2 > Y_2$. That is, <u>if X is</u>

preferred to Y, then either Z is not constructible by the above procedure or $Z_2 > Y_2$.

Suppose Z is, in fact, not constructible. This means that either the line $x_1 = Y_1$ does not cut I_1 on the plane $x_3 = a$ or that the curve I_3 does not intersect the line $x_3 = b$ on the plane $x_1 = Y_1$. Since I_1 is unbounded in the x_1-direction by assumption, the first contingency can only arise if I_1 is separated from the origin by the line $x_1 = Y_1$ (see Fig. 3). But then

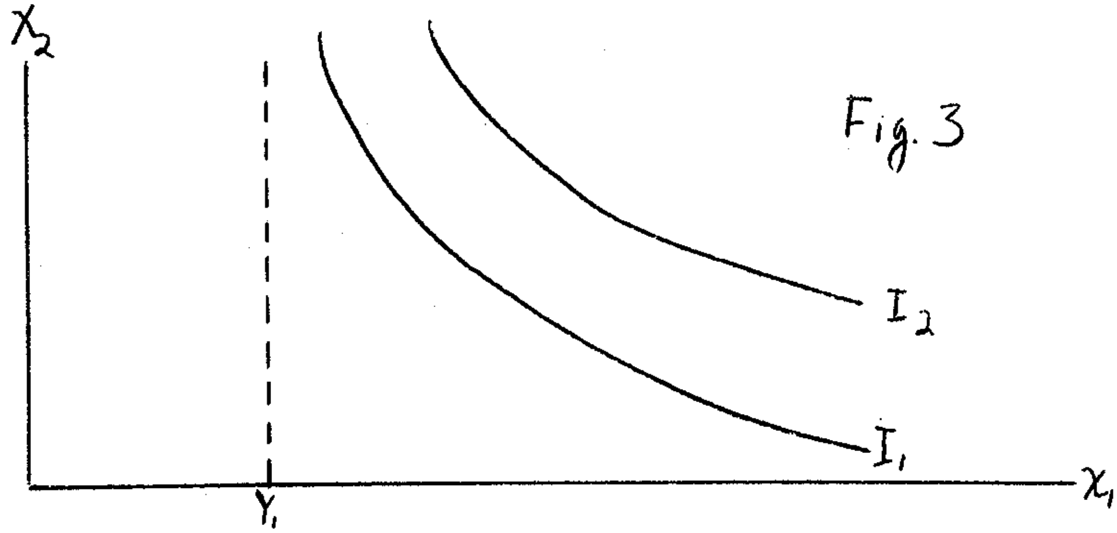

Fig. 3

either I_2 does not exist at all or I_2 must also be separated from the origin by the line $x_1 = Y_1$, since it must lie still further away from the origin than I_1. If I_2 does not exist at all, there are no points on the plane $x_3 = b$ which are indifferent to X; since $X_3 > b$, this means that X is preferred to all the points on the plane $x_3 = b$, and hence in particular to Y. If I_2 is separated from the origin by the line $x_1 = Y_1$, on which Y lies, the points of the plane $x_3 = b$ which are indifferent to

X are preferred to Y, and again X must be preferred to Y.

Now suppose that the line $x_1 = Y_1$ does intersect I_1, but that I_3 does not intersect the line $x_3 = b$. Since any point of I_2 for which $x_1 = Y_1$ would lie on the intersection of I_3 with the line $x_3 = b$, there can be no such points of I_2. Either I_2 does not exist, in which case X is preferred to Y as before, or, since I_2 is unbounded, it must be separated from the origin by the line $x_1 = Y_1$, so that again X is preferred to Y.

If Z is not constructible, then X is preferred to Y. Therefore, if X is indifferent to Y, Z must be constructible and must be indifferent to Y; since $Z_1 = Y_1$, $Z_3 = Y_3$, we must have $Z_2 = Y_2$. Similarly, if X is disfavored to Y, Z is constructible, and $Z_2 < Y_2$.

If I_1 is unbounded in the x_1-direction, then we may ascertain the preference or indifference relation between X and Y, where $X_3 \quad Y_3$, by the following criterion: Try to construct the point Z. If it is not defined, then X is preferred to Y. If Z is defined, then X is preferred, indifferent, or disfavored to Y according as $Z_2 > Y_2$, $Z_2 = Y_2$, or $Z_2 < Y_2$.

It is to be observed that the question of whether or not Z is defined and of finding the value of Z_2 if it is can be answered solely from a knowledge of the indifference maps on certain ration planes; in fact, not more than two such planes need be considered.

Case II: the projection of I_1 on the x_1-axis is bounded. In this case, I_1 cuts the x_1-axis; let the x_1-coordinate of the point of intersection be

c_1 (see Fig. 4). Suppose X is preferred to Y. On the plane $x_1 = c_1$, let I_3 be the indifference curve through the point with $x_2 = 0$, $x_3 = a$. This point represents the bundle $(c_1, 0, X)$, which is the point of intersection of I_1 with the x_1-axis on Fig. 4, and hence I_3 represents points indifferent to X. If I_3 intersects the line $x_3 = b$, let the point of

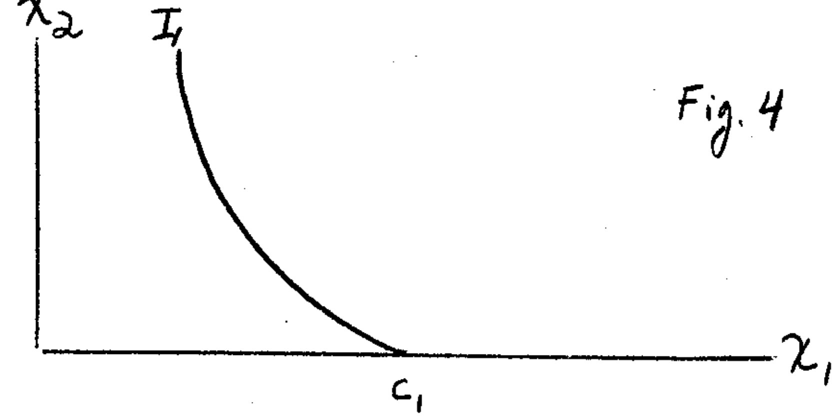

Fig. 4

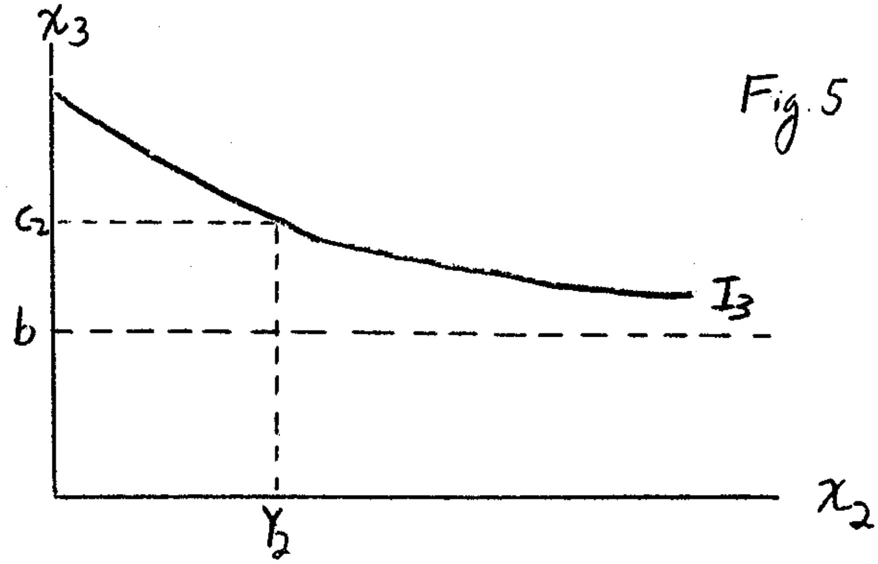

Fig. 5

intersection be Z. Then, $Z_3 = Y_3 = b$, so that Z and Y are on the same ration plane; since X is preferred to Y, so is Z. If Z is not defined, then I_3 must be unbounded in the x_2-direction (see Fig. 5). Since I_3

does cut the line $x_2 = 0$, there is a point on I_3 with any preassigned (non-negative) value of x_2 and in particular a point with $x_2 = Y_2$. Let the x_3 -coordinate of this point of intersection be c_2. On the ration plane $x_2 = Y_2$, the point of intersection just found has the coordinates $x_1 = c_1$, $x_3 = c_2$. Draw the indifference curve through this point on the plane $x_2 = Y_2$; let W be its intersection, if any, with the line $x_3 = b$. $W_2 = Y_2$, and $W_3 = Y_3$; also W is indifferent to X, by construction. Hence, W is preferred to Y since X is; therefore, $W_1 > Y_1$.

If X is preferred to Y, then either Z is preferred to Y or $W_1 > Y_1$ or neither Z nor W is defined.

Suppose neither Z nor W is defined. Since Z is not defined, it follows, as above, that there is a point on the plane $x_2 = Y_2$ which is indifferent to X. If there were a point on the plane $x_3 = b$ which was indifferent to X and for which $x_2 = Y_2$, that point would be W; since W is not defined, either I_2, which contains all points on the plane $x_3 = b$ indifferent to X, does not exist, or I_2 does not cross the line $x_2 = Y_2$. If I_2 does not exist, then, as before, X must be preferred to Y. If I_2 does not cross the line $x_2 = Y_2$, then either I_2 is separated from the origin by the line $x_2 = Y_2$, in which case, again as before, X is preferred to Y, or I_2 is on the same side of that line as the origin in which case every bundle for which $x_2 = Y_2$, $x_3 = b$ is preferred to X. This last case is impossible, however; for, in particular, the bundle (c_1, Y_2, b) would be preferred to X, but it is disfavored to the bundle (c_1, Y_2, c_2), by Assumption II, since $c_2 > b$, and this last bundle is indifferent to X by construction.

If neither Z nor W is defined, then X is preferred to Y.

If X is indifferent to Y, then either Z or W must be defined; in the first case, Z must be indifferent to Y, in the second $W_1 = Y_1$. Similarly, if X is disfavored to Y, either Z is defined and disfavored to Y or W is defined and $W_1 < Y_1$.

If I_1 is bounded in the x_1-direction, then we may ascertain the preference or indifference relation between X and Y, where $X_3 > Y_3$, by the following criterion: Try to construct the point Z as indicated; if it is not defined, try to construct W. If neither Z nor W is defined, then X is preferred to Y. If Z is defined, then X is preferred, indifferent or disfavored to Y according as Z is preferred, indifferent or disfavored to Y. If W is defined, then X is preferred, indifferent or disfavored to Y according as $W_1 > Y_1$, $W_1 = Y_1$, or $W_1 < Y_1$.

It is to be observed again that all the processes implied in the above statement can be carried out with a knowledge of the indifference maps on certain ration planes separately; in this case, the indifference maps on three ration planes are needed.

Thus we have shown that a comparison of any two bundles of three commodities can be effectuated by a knowledge of all comparisons of two bundles in which the same quantity of one commodity appears in the two bundles.

4. The Case of Four or More Commodities

The development of indifference maps for more than three commodities from the indifference maps on the various ration planes will be exemplified by the case of four commodities. The crucial point here is that each rationed collection of bundles obtained by fixing the quantity of just one commodity is essentially a world of three commodities, and the indifference surfaces in each such rationed collection can be derived from the indifference curves on the individual ration planes to a comparison of any two three-commodity bundles.

Let X and Y be any two bundles; as before, we may assume $X_4 > Y_4$. Let $X_4 = a$, $Y_4 = b$.

Let I_1 be the indifference surface in the ration hyperplane $x_4 = a$ through the point X; let I_2 be the indifference surface, if any, in the ration hyperplane $x_4 = b$ of points indifferent to X.

First, suppose I_1 to be unbounded in the x_1-direction. Find a point in I_1, if any, for which $x_1 = Y_1$; if there is one such point, there will in general be many. Find the indifference surface through this point on the ration hyperplane $x_1 = Y_1$ and then a point Z of this surface with $x_4 = b$. As before, if Z is not defined, X is preferred to Y. If Z is defined, then the preference or indifference relation of X to Y is the same as that of Z to Y. But the latter relation is known, since Z and Y are both in the ration hyperplane $x_4 = b$, and we have said that we may assume the indifference surfaces known on each ration hyperplane.

Next, suppose I_1 to be bounded in the x_1-direction. It must then contain a point with $x_2 = 0$, $x_3 = 0$, and, of course, $x_4 = a$. Let the x_1-coordinate of this point be c_1. This point is, then, on the hyperplane $x_1 = c_1$; consider the indifference surface through the given point on the indicated hyperplane. If there is a point on the indifference surface for which $x_4 = b$, call that point Z. If there is no such point Z on the indifference surface, there must be one point for which $x_2 = Y_2$. Consider the indifference surface through this point in the hyperplane $x_2 = Y_2$. If there is a point on this indifference surface for which $x_4 = b$, call it W. Then, if neither Z nor W is defined, X must be preferred to Y. If either Z or W is defined, then X bears the same preference or indifference relation to Y that Z or W does, respectively.

By this method, the indifference hypersurfaces in four-commodity spaces may be built up out of the indifference surfaces in three-commodity spaces and hence ultimately out of the indifference curves in two-commodity spaces. The same procedure can be used to pass from four- to five-commodity worlds and so on.

5. Appendix

~~Certain~~ propositions about the nature of indifference surfaces deducible from Assumptions I-III which have been used implicitly in the above analysis will here be brought out into the open. A repeatedly used theorem is the following:

Proposition I. The projection of an indifference curve or surface on the x_1-axis is an interval (finite or infinite).

Proof: Let I be the given indifference surface. It suffices to show that if $a_1 < a_2 < a_3$ and if there are points in I for which $x_1 = a_1$ and $x_1 = a_3$, respectively, then there is a point in I for which $x_1 = a_2$.

Let X be a point in I for which $X_1 = a_1$, Y a point in I for which $Y_1 = a_3$. Define points X' and Y' by their coordinates as follows: $X'_1 = a_2$, $X'_i = X_i$ for $i \neq 1$; $Y'_1 = a_2$, $Y'_i = Y_i$ for $i \neq 1$. Since $X'_i = X_i$ $(i \neq 1)$, $X'_1 > X_1$, X' is preferred to X, by Assumption II;

similarly, Y is preferred to Y', so that X, which is indifferent to Y, is preferred to Y'. Hence, by Assumption III, there is a number t such that $Z = tX' + (1-t)Y'$ is indifferent to X and hence belongs to I. But $Z_1 = tX'_1 + (1-t)Y'_1 = a_2$, so that there is a point of I which $x_1 = a_2$.

This proposition alone suffices to justify the procedure indicated for deciding the relation between X and Y, with $X_n = a > Y_n = b$, in the case where I_1, the indifference surface in the hyperplane $x_n = a$ through X, is unbounded in the x_1-direction. If the point Z is defined, then the procedure clearly determines the relation between X and Y. If Z is not defined, then either I_1 has no points for which $x_1 = Y_1$, or I_2, the indifference surface of points in the hyperplane $x_n = b$, has no points for which $x_1 = Y_1$, or I_2 does not exist. In the last case, certainly X is superior to some point for which $x_n = b$, since $X_n > b$, by Assumption II; if X were not preferred to Y, there would, by Assumption III, be a point for which $x_n = b$ which is indifferent to X, since Y_n also equals b. But there can be no point for which $x_n = b$ which is indifferent to X if I_2 does not exist.

Now suppose that I_1 has no points for which $x_1 = Y_1$. Since I_1 is unbounded in the x_1-direction, there are points for which $x_1 > Y_1$. By
for if there were, there would have to be a point on I, for which $x_1 = Y_1$.
Proposition I, then, there cannot be any points on I_1 for which $x_1 > Y_1$. Hence $x_1 > Y_1$ for every point of I_1; every point for which $x_n = a$, $x_1 = Y_1$ must be disfavored to the points of I_1 and hence to X. Since $b < a$, any point for which $x_n = b$, $x_1 = Y_1$ must be disfavored to X a fortiori, by Assumption II; in particular, X must be preferred to Y.

Finally, suppose that I_2 has no points for which $x_1 = Y_1$. Since I_1 is unbounded, there is a point of I_1 with x_1 Y_1. Consider the bundle obtained by replacing the x_n-coordinate of this point, which is a, by b. Since $b < a$, the new bundle must be disfavored to I_1 and hence X. That is, there is a point disfavored to X on the hyperplane $x_n = b$ for which $x_1 > Y_1$. The projection of I_2 on the x_1-axis is an interval not containing the number Y_1. Hence, either $x_1 < Y_1$ for all points of I_2 or $x_1 > Y_1$ for all such points. If the first alternative held, all points on the hyperplane $x_n = b$ for which $x_1 \geq Y_1$ would be preferred to points of I_2 and hence to X; but we know this to be false. Hence, $x_1 > Y_1$ for all points of I_2, so that any point for which $x_1 \leq Y_1$, $x_n = b$ must be disfavored to X. In particular, X is preferred to Y.

The case where I_1 is bounded in the x_1-direction requires another proposition.

Proposition II. If an indifference surface is bounded in the x_1-direction, it contains a point for which $x_i = 0$ for all $i \neq 1$. That is, the indifference surface must cut the x_1-axis.

Proof: Since the x_1-coordinates of the points on I are bounded from above, they must possess a least upper bound c; i.e., for all points on I, $x_1 \leqq c$, while for any number $c' < c$, there is a point on I such that $x_1 > c'$. Let X be the bundle $(c, 0, \ldots, 0)$; we will show that X belongs to I.

Suppose it did not. Then either X is preferred to the points of I or is disfavored to them. In the first case, let X' be any point in I, so that $X'_1 \leqq c$. Since $X'_i \geqq 0 = X_i$ for $i \neq 1$, if also $X'_1 = c$, we would have X' preferred or indifferent to X, contrary to assumption. Hence, $X'_1 < c$. Defined X" as follows: $X''_1 = X'_1$, $X''_i = 0$ for $i \neq 1$. Then X' is preferred or indifferent to X; since X is preferred to X' by assumption, there is a linear combination $Y = tX'' + (1-t)X$, $0 < t \leqq 1$, which is indifferent to X' and hence belongs to I. Clearly, Y_1 c, $Y_i = 0$ $(i \neq 1)$. As noted above there is a point Z in I for which $Z_1 > Y_1$; as $Z_i \geqq 0 = Y_i$ $(i \neq 1)$, Z is preferred to Y, which is a contradiction since both are in I. Hence, X cannot be preferred to the points of I.

Now suppose that X is disfavored to the points of I. Again let X' be any point of I; $X'_1 \leqq c$. Define X" as follows: $X''_1 = c$, $X''_i = X'_i$ $(i \neq 1)$. Then X" is preferred or indifferent to X', which, in turn, is preferred to X. There is a linear combination $Y = tX + (1-t)X''$ which is indifferent to X' and hence belongs to I; $Y_1 = c$. If $Y_i = 0$ for all $i \neq 1$, then $Y = X$, and X would belong to I, contrary to assumption. Hence $Y_j > 0$ for some $j \neq 1$. Choose numbers c', c" so that $0 < c' < Y_j$, $c'' > c$. Define Y', Y" as follows: $Y_j = c'$, $Y'_i = Y_i$ $(i \neq j)$; $Y''_1 = c''$, $Y''_i = Y_i$ $(i \neq 1)$. Then Y" is preferred to Y and Y to Y'; hence, there is a linear combination $Z = tY' + (1-t)Y''$, $0<t<1$, which is indifferent to Y and hence belongs to I. But $Z_1 > c$, which is impossible by definition of c. Hence, X cannot be disfavored to the points of I; it must belong to I. Q.E.D.

Proposition II establishes the validity of the procedure for deciding the preference as between X and Y, $X_n > Y_n$, when I_1 is bounded in the x_1-direction. If c is the least upper bound of the values of x_1 for points of I_1, then, by Proposition II applied to the ration plane or hyperplane $x_n = a$, the bundle $(c, 0, \ldots, 0, a)$ belongs to I_1 and hence is indifferent to

X. Therefore, I_3, the surface in the hyperplane $x_1 = c$ of points indifferent to $(c,0,\dots,a)$, consists of points indifferent to X. Hence, if Z is defined, X bears to the same relation to Y as Z does. If Z is not defined, there is no point of I_3 for which $x_n = b$. Since there is a point of I_3 for which $x_n = a > b$, there cannot be a point of I_3 for which $x_n = 0$, by Proposition I. On the hyperplane $x_1 = c$, the coordinates which vary are $x_2,\dots,x_n$. If I_3 is bounded in the x_2-direction, then by Proposition II there would have to be a point of I_3 for which $x_i = 0 (i = 3,\dots,n)$; but there is no point for which $x_n = 0$. Hence I_3 is unbounded in the x_2-direction; since there is a point of I_3 for which $x_2 = 0$, it follows from Proposition I that there is a point of I_3 with any assigned value of x_2 and and in particular with $x_2 = Y_2$. For this point $x_n > b$. Let I_4 be the indifference surface on the hyperplane $x_2 = Y_2$ of points indifferent to the one just found; I_4 consists of points indifferent to X. If W is defined, then the relation of X to Y is the same as that of W to Y. If W is not defined, then there is no point of I_4 with $x_n = b$; since there is a point of I_4 with $x_n > b$, we must have, by Proposition I, that $x_n > b$ for all points of I_4, and therefore any point for which $x_2 = Y_2$, $x_n = b$ must be disfavored to the points of I_4 and hence to X. In particular, X is preferred to Y.